Symbols and Mediation in Mathematics Education

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Prelude In this paper we discuss topics that are relevant for designing a theory of mathematics education. More precisely, they are elements of a *pre-theory* of mathematics education and consist of a set of interdisciplinary ideas which may lead to understand what occurs in the *central nervous system*—our metaphor for the classroom, and eventually, in larger educational settings. In particular, we highlight the crucial role of representations, the mediation role of artifacts, symbols viewed from an evolutionary perspective, and mathematics as symbolic technology.

Introduction

We will describe some basic elements of a *pre-theory* of Mathematics Education. Our field is at the crossroad of a science, mathematics, and an institutional practice, education. The main interest of mathematics educators is the *people* whose learning takes place mainly at schools. This reminds us of Thurston (1994) who wrote this about mathematicians: That what they should do is finding ways for *people* to understand and think about mathematics.

As soon as we consider how to approach the problems of teaching, learning, more precisely, of mathematical cognition within the social and cultural environments provided by educational institutions, we become aware of the *semiotic dimension* of mathematics. This *semiotic dimension* introduces a deep problem for mathematics cognition and epistemology. As Otte (2006, p. 17) has written,

> A mathematical object, such as a number or a function does not exist independently of the totality of its possible representations, but it must not be confused with any particular representation, either.

This paper is dedicated to Jim Kaput (1942–2005), whose work on representations and technology is an inspiration to us all.

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Indeed, we never exhaust the set of semiotic representations for a mathematical object. The mathematical object is therefore, always, *under construction*.

The feeling of objectivity that we experience whilst dealing with mathematical objects (in fact, with *some* semiotic representation(s)) has been the matter of intense discussions since Plato's Ideal Forms. Recently, Connes et al. (2001, pp. 25–26) has written:

> I confess to be resolutely Platonist... I maintain that mathematics has an object that is just as real as that of the sciences... but this object is not material, and it is located in neither space nor time. Nevertheless, this object has an existence that is every bit as solid as external reality...

The problem with this approach is that we, creatures living in *this* space and time, cannot properly answer *how to cognitively access these objects*. Nevertheless there is no scarcity of answers. Plato's answer is that our immaterial souls acquired knowledge of abstract objects before we were born and that mathematical learning is really just a process of coming to remember what we knew before we were born. In Cantor's correspondence with Hermite, they discussed extensively about the epistemology of natural numbers (Dauben 1990, p. 228). Hermite wrote:

> The whole numbers seem to me to be constituted as a world of realities which exist outside of us with the same character of absolute necessity as the realities of Nature, of which knowledge is given to us by our senses.

Trying to answer the underlying question about the mode of existence of mathematical objects, Cantor replied that the natural numbers,

> Exist in the highest degree of reality as eternal ideas in the *Intellectus Divinus*.

And Goedel (1983), following the same Platonic tenor:

> Despite their remoteness from sense-experience, we do have something like a perception of the objects of [mathematics]... I don't see any reason why we should have less confidence in this kind of perception, i.e. in mathematical intuition, than in sense perception.

We could easily add more examples in this line, but we think it is clear that the aspiration to explain the pre-semiotic access to mathematical objects has to be abandoned. Mathematics is a human activity and an outcome of this activity is the feeling of objectivity that mathematical objects possess. Furthermore, the mode of existence of mathematical object is a semiotic mode. Let us give an example that reveals much with respect to the aforementioned feeling of objectivity. Speaking about the mathematical power coming from Maxwell's equations for electromagnetism (see Kline 1980, p. 338) Hertz wrote:

> One cannot escape the feeling that these mathematical formulas have an independent existence and intelligence of their own, [...] that we get more out of them than was originally put into them.

Facing the profound mathematical nature of Maxwell's theory, it seems that Hertz's understanding was that those semiotic representations—the equations—had outrun his intuition and sensuous perceptions. Furthermore, those equations were the semiotic *mediators* in a wonderful example of mathematical prediction in science. Ernest Mach expressed his views (Kline 1962, p. 542) with these words: *It* *must sometimes seem to the mathematician that it is not he but his pencil and paper which are the real possessors of intelligence*. Again, the mediating role of the symbolic representations (the equations) is present. The pencil, for Mach, is obviously the *writing* as a cognitive mirror. A vivid example of the understanding of semiotic mediation and its profound connections with human cognition, is narrated in a biography of R. Feynman (Gleick 1992). Gleick narrates that, during a conversation of Feynman with the historian Charles Weiner, the latter mentioned casually that Feynman's notes were "a record of the day-to-day work" and Feynman reacted sharply (Gleick's book, p. 409):

> "I actually did the work on paper", he said.
>
> "Well," Weiner said, "the work was done in your head, but the record of it is still here."
>
> No, it's not a *record*, not really. It's *working*. You have to work on paper, and this is the paper. Okay?

Computers has made it feasible a new way of looking at symbols, looking *through* them, and transform the resources of mathematical cognition. Besides, it has offered the potential to re-shape the goals of our research field. However, the daily urgency of teaching and learning, so distant from the pace of most research activities, has resulted in *practices without corresponding theories*. Again, we must make clear we are not dismissing the experience accumulated under such institutional state of affairs. We simply want to underline that institutional pressures can result most frequent than desirable, in the losing of track of research goals. Perhaps this is an incentive for re-considering the need to promote a more organized level of reflection in our community. Let us remind that mathematics education exists at the crossroad of mathematics and education but not as a simple aggregate but as a subtle blending. Nevertheless the tension between the local and the global comes to existence here. We have come to think that, by now, only local explanations are possible in our field, more global coherence is (also) under construction. *Local theories* might work as the organizing principles for the profusion of explanations we encounter around.

We need to understand better our symbolic nature. Consequently, we take an interdisciplinary approach. In the previous discussion, we succinctly explained some aspects of the semiotic dimension of mathematics and its mediational role whilst trying to understand the external world by means of mathematical models. *Being guided* by the mathematical model is a way to disclose unexpected facts; conversely, *guiding* the model is the way towards deepening our understanding of the world. This process (being guided-and-guiding a cognitive artifact like writing) can be generalized and extended to our *co-action* with every artifact. This dual process (guiding/being guided) entails that human activities, intentional activities, develop within a framework in which it is no longer possible to separate the person and the environment (including the artifacts, especially those that "make us smarter", to borrow D. Norman's expression) because they are not mutually independent: They are co-extensive.

Now, we are going to offer an abridged narrative—taking a long view perspective though—to explain why our minds would be essentially incomplete in the absence of co-development with material and symbolic technologies. We must distinguish between the diversity of dimensions of development—phylogenesis, ontogenesis, and historical-cultural developments.

In the natural world, an organism cannot *communicate* its experiences, that is, the organism lives confined within its own body. However during their evolution, hominids became able to overcome the solipsism of their ancestors, extending the reach of their cognition and developing a communal space for life. This is an important stage in the process that transformed our ancestors into *symbolic beings*. Our symbolic and mediated modern nature comes to the front as soon as we try to characterize our intellectual nature. This is different from *implicit* (or analogue) cognition, in that this latter kind of cognition is imprinted in the nervous system. Birds "knows" how to build a nest; eagles "know" how to fly, fishes how to swim. As Donald (2001) has nicely written:

> Animal brains intuit the mysteries of the world directly, allowing the universe to carve out its own image in the mind.

Sometimes, some people decide to forget our biological heritage when reflecting on cognition or culture. There are certainly, extreme points of view which may be opposed, nevertheless, as Donald (2001, p. 106) continues to explain with substantial evidence, we have a deep mental structure that speaks for our evolutionary history. A key idea to understand our cognitive nature is offered by Donald (2001, p. 157) where he explains:

> Humans thus bridge two worlds. We are hybrids, half analogizers, with direct experience of the world, and half symbolizers, embedded in a cultural web. During our evolution we somehow supplemented the analogue capacities built into our brains over hundreds of millions of years with a symbolic loop through culture.

Even in higher mammals, implicit cognition is a survival cognition. However this implicit knowledge of the environment does not elicit an *intentional* response for transforming the environment. Only humans possess, besides implicit cognition, what can be termed *explicit cognition* that allows us to go from learning to knowledge. Explicit cognition is symbolic cognition. The symbol refers to something that although arbitrary*, is shared and agreed by a community*. But there is always room for personal interpretations: The experienced world inside each of us plays a considerable role in our quest for meaning but, for communication to work, we need to share a substantial part of the reference fields, the meanings, of our symbolic systems. Unfolding an intense communicative activity, something that is particularly clear when dealing with systems of mathematical symbols, properly does this.

At the beginning, when one is dealing with a symbol the reference field can be rather narrow. As time passes by and we become more proficient with its use, the corresponding reference field begins to amplify. Dealing with the complex nature of the *reference field* of a sign/symbol, Charles S. Pierce (1839–1914) explained that the difference between different modes of reference could be understood in terms of levels of interpretations.

> Reference is hierarchic in structure. More complex forms of reference are built up from simpler forms. (Deacon 1997, p. 73)

Take the example of a stone tool: The communal production of those tools implied that a shared conception of them was present. But eventually, somebody could discover a new use of the tool. This new experience becomes part of a personal reference field that re-defines the tool for the discoverer; eventually, that experience can be shared and the reference field becomes more complex as it unfolds a deeper level of reference.

The reference field lodged within a symbol can be greatly enhanced when that symbol is part of a network of symbols—in fact, it is the only way. Emergent meanings come to light because of the new links among symbols. This phenomenon can be termed the *semiotic capacity* of a symbol system. An obvious example is provided by natural languages wherein the meaning of a word can be found inside the net of relations that are established with other words in utterances or texts. Reading in a foreign language illustrates this situation very well. With a high frequency you can realize the meaning of an unknown word thanks to the context (sentences, paragraphs) in which that word appears. Miller et al. (2005) suggest that the apparent "universality" of mathematics in spite of the linguistics, symbolic and cultural variations make it particularly appealing for the study of cognitive development across cultures with reference to the effects of their particular symbolic systems. (p. 165)

As we become expert users of a symbolic system, we can work at the symbolic level without making a conscious effort to connect with the reference field. The system of symbols is transformed into a cognitive mirror in the sense that one's ideas about some field of knowledge can be externalize with the help of that system; then we can see our own thought reflected in that "text" and discover something new about our own thinking. Evolution and culture have left their traits in our cognition, in particular, in our capacity to duplicate the world at the level of symbols. The complexity of this field of inquiry is evident in the fact that the *Handbook of Mathematical Cognition* includes 21 sub-categories in their subject index for representations.

Diverging epistemological perspectives about the constitution of mathematical knowledge modulate multiple conceptions of learning and the theories of the agenda of mathematical education as a research field. Today, however, there is substantial evidence that the encounter of the conscious mind with distributed cultural systems has altered human cognition and has changed the tools with which we think. The origins of writing and how writing, *as a technology*, changed human cognition is key from this perspective (Ong 1998). These examples suggest the importance of studying the evolution of mathematical systems of representation as a vehicle to develop a proper epistemological perspective for mathematics education.

Having said that, any model of mathematical learning has to take into account the representations (internal and external) associated with the objects/phenomenon under investigation. In fact, Goldin (1998) has proposed a model for mathematical learning during problem solving which is based on the different types of representations one invokes when engaged with a problem. These are (a) verbal/linguistic-syntactic systems (the role of language); (b) imagistic systems, including visual/spatial, auditory and tactile representations; (c) formal notational systems of mathematics (internal systems). In addition any model must take into account systems invoked for planning, monitoring, and executive control that include heuristic processes and affective representations. Further Goldin and Kaput (1996) have commented on the important distinction between *abstract mathematical reasoning* using formal notations (that interact meaning-fully with other kinds of cognitive representation), and symbol manipulation that is merely *decontextualized* in the sense that it is detached from meaningful, interpretive representational contexts. Our evolution offers us insights into the necessity of distinguishing between the two.

Human evolution is coextensive with tool development. In a certain sense, human evolution has been an *artificial* process as tools were always designed with an explicit purpose that transformed the environment. And so, since about 1.5 millions years ago, our ancestor Homo Erectus designed the first stone tools and took profit from his/her voluntary memory and gesture capacities (Donald 2001) to evolve a pervasive technology used to consolidate her/his early social structures. The increasing complexity of tools demanded optimal coherence in the use of memory and in the transmission of the building techniques of tools by means of articulate gestures. We witness here what is perhaps the first example of deliberate teaching. Voluntary memory enabled our ancestors to engender a mental template of their tools. Templates *lived* in their minds as outcomes of former activity and that granted an objective existence as abstract objects to those templates even before they were *extracted* from the stone. Thus, tool production was not only important for plain survival, but also for broadening the mental world of our ancestors—and introducing a higher level of objectivity.

The actions of our ancestors were producing a *symbolic* version of the world: A world of intentions and anticipations they could imagine and *crystallize* in their tools. What their tools meant was tantamount to what they *intended* to do with them. They could *refer* to their tools to *indicate* their *shared* intentions and, after becoming familiar with those tools, they were looked as *crystallized* images of all the activity that was crystallized in them.

We suggest that the synchronic analysis of our relationship with technology hides profound meanings of this relationship that coheres with the co-evolution of man and his tools, no matter how we further this analysis. It is then unavoidable, from our viewpoint, to revisit our technological past if we want to have an understanding of the present. Let us present a substantial example. Another consideration for education is the difficulty of constructing a "shared" language in which intended meanings are co-ordinated with what the students are attending to, which Maturana (1980) has called the consensual domain.

Arithmetic: Ancient Counting Technologies

Evidence of the construction of one-to-one correspondences between arbitrary collections of concrete objects and a *model set* (a template) can already be found between 40,000 and 10,000 B.C. For instance, hunter-gatherers used bones with marks (tallies) as reckoning devices. In 1937, a wolf bone with these characteristics, dated 30000 B.C., was found in Moravia (Flegg 1983) This reckoning technique (using a one-to-one correspondence) reflects a deeply rooted trait of human cognition. Having a set of stone bits or the marks on a bone as a *modeling set* constitutes, up to our knowledge, the oldest counting technique humans have imagined. The modeling set plays, in all cases, an instrumental role for the whole process. In fact, something is crystallized by marking a bone: The *intentional* activity of finding the size of a set of hunted pieces for instance or, as some authors have argued, the intentional activity to compute time.

The modeling set of marks is similar in its role to that of a stone tool—as both mediate an activity, finding the size of a collection in one case, and producing a template in the other. Both *crystallize* the corresponding activity. In Mesopotamia, between 10000 B.C. and 8000, B.C., people used sets of pebbles (clay bits) as modeling sets. However, this technique was inherently limited. If for instance, we had a collection of twenty pebbles as a modeling set, then it would be possible to estimate the size of collections of twenty or less elements. Yet, to deal with larger collections (for instance, of a hundred or more elements), we would need increasingly larger model sets with evident problems of manipulation and maintenance. And so, the embodiment of the one-to-one technique in the set of pebbles inhibits the extension of it to larger realms of quantitative experience. It is plausible that being conscious of these shortcomings, humans looked for alternative strategies that eventually, led them to the brink of a new technique. The idea that emerged was to replace the elements of the model set with clay pieces of diverse shapes and sizes, *whose numerical value were conventional*. Each piece *compacted* the information of a whole former set of simple pebbles—according to its shape and size. The pieces of clay can be seen as embodiments of pre-mathematical symbols. Yet, they lacked rules of transformation and that inhibited those pieces to become a genuine mathematical system.

Much later, the consolidation of the urbanization process (about 4000 B.C.) demanded, accordingly, more complex symbol systems. In fact, the history of complex arithmetic signifiers is almost determined by the occurrence of bullae. These clay envelopes appeared around 3500–3200 B.C. The need to record commercial and astronomic data led to the creation of symbol systems among which mathematical systems seem to be one of the first. The counters that represented different amounts and sorts of commodities—according to shape, size, and number—were put into a bulla which was later sealed. To secure the information contained in a bulla, the shapes of the counters were printed on the outer side of the bulla. Along with the merchandise, producers sent to their business associate a bulla, with the counters inside, describing the goods they would receive. When receiving the shipment, the merchant could verify the integrity of it, comparing the received goods with the information contained in the corresponding bulla.

A counter in a bulla represents a *contextual* number—for example, the number of sheep in a herd was not an abstract number: there is five of something, but never *just five*. The shape of the counter is impressed on the outer side of the bulla. The mark on the surface of the bulla *indicates* the counter inside. That is, the mark on the surface keeps an *indexical relation* (in a Peircean sense) with the counter inside as its referent. And the counter inside has a *conventional* meaning with respect to amounts and commodities. It must have been evident, after a while, that *counters inside were no longer needed;* impressing them on the outside of the bulla was enough. That decision altered the semiotic status of those external inscriptions. Afterward, instead of impressing the counters against the clay, due perhaps to the increasing complexity of the shapes involved, scribes began *to draw* on the clay the shapes of former counters. The scribes used sharp styluses to draw on the clay. From this moment on, the *symbolic* expression of numerical quantities acquired an infra-structural support that, at its time, led to a new epistemological stage of society. Yet the semiotic contextual constraints, made evident by the simultaneous presence of diverse numerical systems, constituted an epistemological barrier for the mathematical evolution of the *numerical ideographs*. Eventually, the collection of numerical (and contextual) systems was replaced by one system (Goldstein 1999). That system was the sexagesimal system that also incorporated a new symbolic technique: numerical value according to position. In other words, it was a *positional* system. There is still an obstacle to have a complete numerical system: the presence of zero that is of primordial importance in a positional system to eliminate representational ambiguities. For instance, without zero, how can we distinguish between 12 and 102? We would still need to look for the help of context and this hinders the access to a truly symbolic system. The limitations of the Babylonian representational system only became obvious to modern historians when the cuneiform script was deciphered in the mid-nineteenth century by George Frederick Grotofend and Henry Rawlinson. Joseph (1992) points out that the mathematics of these representational systems was only seriously studied by math historians from the 1930's onwards.

Mathematical objects result from a sequence of crystallization processes with an ostensible social and cultural dimension. As the levels of reference are hierarchical the crystallization process is a kind of recursive process that allows us to state:

Mathematical symbols co-evolve with their mathematical referents and the induced semiotic objectivity makes possible for them to be taken as shared in a community of practice.

The development of symbol leads to symbolic technology: Symbolic technologies have produced, since prehistory, many devices that have a direct impact on thought and memory, for instance, marks on bones, calendars and more recently, pictorial representations (Lascaux, Altamira). This is the technology that embodied the transition from analog (implicit) cognition to digital (explicit) cognition. That is, from experiencing the world through holistic perception to experiencing the world as a decoding process. The digital or symbolic experience put us on new ground. Mainly it enabled us to "project the mind" to outer space. That is semiosis: projecting intentionality, sharing-and-building meaning, not remaining inside the brain box. Reading the other. Symbolic thought and language are in themselves, distributed phenomena. This again begs the question as to whether our evolution has resulted in domain specific language structures which are easily and objectively interpreted by others. Is mathematics uniquely different from other domains of inquiry? Dietrich (2004) writes:

> If all structures we perceive are only human-specific artifacts that can be defined only as invariants of cognitive operators, then this concept must apply also to our perception (or

> interpretation) of language structures, i.e., as a physical object cannot have objective properties that can be used for an objective description, neither can verbal texts have an objective interpretation. Then the question arises whether a text can carry an autonomous message and if not, what the notion of communication means? (p. 42)

In what follows, we should try to articulate some reflections regarding the presence of computational technologies in mathematical thinking. It is interesting to notice that even if the new technologies are not yet fully integrated within any mathematical universe, their presence will eventually erode the mathematical way of thinking. The blending of mathematical symbol and computers has given way to an *internal mathematical universe* that works as the reference field to the mathematical signifiers living in the screens of computers. This takes abstraction a large step further.

Mathematics from a Dynamic Viewpoint: The Future of Mathematics Education

We tend to believe that with the help of some software students can, for instance, achieve diverse representations, explore different cases, and find loci or trajectories of points. This belief is attractive in designing students' learning activities. But technology by itself, does not solve any educational problem. In the last years, research and practice have shown that the use of technology can play an important role in helping students represent, identify, and explore behaviors of diverse mathematical relationships. An important goal during the learning of mathematics is that students develop an appreciation and disposition to practice genuine mathematical inquiry. The idea that students should pose questions, search for diverse types of representations, and present different arguments during their interaction with mathematical tasks has become an important component in current curriculum proposals (NCTM 2000). Here, the role of students goes further than viewing mathematics as a fixed, static body of knowledge; instead, it includes that they need to conceptualize the study of mathematics as an activity in which they have to participate in order to identify, explore, and communicate ideas attached to mathematical situations.

A lack of perspective may give the impression that it is only in the last years that educators have come to consider the role of technology within our educational systems. What has changed in the last years, has been the understanding of the nature of the role of technology in the students' learning processes.

It is important then, to have a long term perspective to be able to gauge the role that *computational* technologies can play in contemporary education. Many researchers in Math Education have already taken a lead in this direction (see for instance, Guin and Trouche 1999), opening a window to newer research and understanding. To achieve our goals, we will first explain some key ideas as *cognitive tools* and *executable representations*. This is the purpose of the next section.

Computational and Cognitive Technologies

Cognitive technologies include a multiplicity of devices that have a direct impact on thought and memory. These technologies emerged since early times in human history. One of the most important steps in this direction has been the creation of external memory supports (Donald 2001). It is difficult to exagerate the importance of marking a bone in order to externally capture a bit of memory. Once this memory strategy is socially established, it becomes crucial to modify the workings of individual and collective memory. Tokens, pictorial representations, and simple measuring instruments, are also among the first examples of these technologies. More elaborate cognitive technologies appeared later, which included powerful systems of writing and numeration. In modern educational systems, devices such as tables of functions, slide rules, and scientific calculators have been used, mainly, as devices able to enhance computations. However, in more recent times, these devices have been used to help students graph functions, collect data and so on. But these activities have been developed *inside* a curriculum explicitly designed as pre-computational. That is, where the role of technology is conceived of as an amplifier of what could be done without that technology only that, now, with the technology at our service, we can do those tasks *better*. It does not mean the technology is fundamental for the realization of the task at hand, only that it *enhances* our actions without qualitatively *transforming* them.

Nevertheless, the increasing complexification of tools and the now better understood nature of the symbiotic relation of a user-agent with a tool, suggest that amplification is not enough as a unit of analysis with respect to the use of technology by students. Let us first present some simple examples to make plausible the existence of a deeper layer of activity beyond amplification. Consider an artist, a violinist, for instance. The violin is like an extension of herself in the sense that while playing, the violin is *transparent.* The artist can *see* the music through the violin. We will say that the violin has become an *instrument* not just a tool for the artist. There is a process by means of which the violin, that at the beginning is *opaque*, is transformed into an instrument, almost invisible, that allows the artist to display her art. That process is, in fact, a double process: the agent-user (in the present example, the violinist) explores the possibilities of the tool (the violin) and dialectically, modifies her own approach to the tool and to the knowledge (in the present example, the music) generated by her activity. Her strategy to develop a playing technique are deeply linked to the workings of the tool. The cognition of the artist is transformed: her art is not the result of doing something better, that she could do without the tool, but something intrinsically linked to the new activity that results from the new dialectical interactions user-tool. When this finally happens, we say that the tool (violin) has been transformed into an instrument: it is one with the violinist.

This process of transforming a tool into an instrument, that we have just described through an example, has been the object of research in the last years. A seminal work in this direction is Rabardel's book *Les Hommes et les technologies* (1995) which presents a cognitive approach to contemporary tools.

At the basis of this tool-instrument complexities, lies the central idea of how a tool can *mediate* the cognitive processes of an agent-user. That any cognitive activity is a *mediated activity* has been aptly established by research in the field of cognition (Rabardel 1995). For research in math education, this thesis constitutes a starting point from which we would try, more especially, to understand the nature of the mediational role of computing tools in the learning and teaching of mathematics. We suggest that the research into the nature of this special form of tool mediation is a crucial goal for the future development of the discipline. Tool mediation has been developed over extended periods of time and, as a consequence, have become an integral part of human intellectual activity. Vygotsky developed this point of view in his theory of sociocultural cognition. In his theory, Vygotsky compared the role of material tools in labor, with the role of symbolic tools in mental activity. Moreover, Vygotsky (1981) stated that:

> The inclusion of a tool in the process of behavior, introduces several new functions connected with the use of the given tool and with its control... alters the course and individual features... of all the mental processes that enter into the composition of the instrumental act replacing some functions with others, i.e., [the inclusion of tools] re-creates and reorganises the whole structure of behavior just as a technical tool re-creates the whole structure of labor operations. (pp. 139–140)

In this context, the use of the term "behavior" might be misleading. In fact, it refers to *human action*. It is convenient to remind that Wertsch (1991), speaking of human action, aptly expressed his view with these words:

> The most central claim I wish to pursue is that human action typically employs mediational means such as tools and language and that these mediational means shape the action in essential ways. (p. 12)

Let us try to establish a distinction between material tools and symbolic tools. In general terms, a material tool like a labor tool, affects the nature of the (mediated) human activity; it can modify the goal of that activity. On the other hand, a symbolic tool like the written language, affects the knowledge, the cognition of the reader. But what happens with a tool like a computer? It is, simultaneously, a tool that can affect the human activity (writing) and the cognition of the agent-user (reorganization of her ideas). In other words, the computer is externally oriented and, at the same time, internally oriented. This distinction is, in certain sense, artificial, as the reader might have guessed. The mastery of a technological tool like the microscope, for instance, affects the research activity of the researcher and, at the same time, modifies her knowledge. The conclusion seems inexorable: cognitive tools blends, dialectically, the activity of the agent-user and transformation of her own cognition.

Computing environments provide a window for studying the evolving conceptions of students and teachers. Their conceptions will be dynamically linked while going from one system of representation to another and so capturing different features, behaviors, of the mathematical objects under consideration. Graphing tools, for instance, produce a shift of attention from symbolic expressions to graphic representations. Representations are tools for understanding and mediating the way in which knowledge is constructed.

Computational representations are *executable* representations, and there is an attribute of executable representations on which we want to cast light: They serve to *externalize* certain cognitive functions that formerly were executed, exclusively, by people without access to a technological device embodying the representation. That is the case, for instance, with the graphing of functions. Graphing with the help of a computer (including any graphing utility) is very different from graphing with paper and pencil as in traditional mathematics. Both are technologically assisted process, but totally different between them. With paper and pencil, even having recourse to the algorithms of calculus, the student has to manipulate the symbolic expression defining the function under consideration. This symbolic manipulation is necessary to assist the student in her figuring out how the graph goes on. To compute the values of the independent variable where a maximum is reached, for instance, requires a certain level of dexterity in algebraic manipulation.

Now, if the student is using a graphing utility, then after the utility has graphed the function, he has to "interpret the graph". To do this, the student must develop an understanding of the implications for the shape from the algebraic expression, for instance. The approaches are complementary, not equivalent. Ideally, the student must have the chance to transform the graph into an object of knowledge. Let us insist for a moment on the nature of *executability* of computational representations. If you are using, let's say, a word processor you can take advantage of the utility "Spelling and Grammar" that comes with the software, to check the correction of your own spelling. Formerly, when you had this kind of doubts you asked a friend to help you with the task at hand. He was contributing a cognitive service that now is done by the software itself. So a cognitive function has been installed in the software, externalized by the software, thanks to the executable nature of the representation. The conclusion in this example is that not only memory has an external support, but also a certain cognitive function. The computer will be transformed, gradually, into a cognitive mirror and a cognitive agent from which students' learning can take considerable profit. It is not anymore the simplistic idea "the computer is doing the task of the student" but something new and radical: providing students with a cognitive partner.

Domains of Abstractions

Mathematics is not only abstract but formal. These two components of the nature of mathematics raises formidable challenges to the teaching and learning of that discipline. How to deal with abstraction and formalism when we are, at the same time trying to incorporate the computer into the latter processes? There is a description of formalism that is very convenient to remind here. In their book, *Descartes' dream:the world according to mathematics*, Davis and Hersh (1986, p. 284) say:

> Formalism, in the sense of which I still use the term, is the condition wherein action has become separated from integrative meaning and takes place mindlessly along some preset direction.

What this quote remarks is precisely what happens to many students: they follow the rules without being aware that computing must have a sense. And when using a grapher, they should be aware that not all outputs are acceptable as graphs of a certain function. How to deal with these kind of problems when teaching a class?

Researchers, aware of this problem, have proposed alternative approaches to deal with the abstract nature of mathematics. The notion of *domain of abstraction* which can be understood as a setting in which students can make it possible for their informal ideas to start to coordinate with their more formalized ideas on a subject. Alternatively, such a domain can be conceived of as an environment where a general is embodied in a particular. Let us give a general example to illustrate the meaning of this idea.

A father, as a gift for his 15th birthday, presents to his son a beautiful table and explains to him:

> Whenever you act in an incorrect way, you should introduce a nail in your table. And when you correct your mistake you should extract the nail. Following this rule, you will have, at the end of your life, a fair idea of your ethical behavior in life.

The idea is clear: using a story as a support the father has introduced an abstract idea of ethic behavior. Trying to explain that abstraction to his son might be futile as the son, probably, does not have the necessary life experience to understand the abstract version. Now the key: having the experience is tantamount to having enough lived examples. Here lies the importance of having an abstraction domain: it is the environment where a general idea finds a dress that makes it visible at the eyes of the students.

A domain of abstraction supplies the tools so that exploration may be linked to formalization. Constructing bridges between students' mathematical activity and formalization links the mathematical thinking in the classroom with the official mathematical discourse. Computers enhances this possibility because they enhance the expressive capacity of students as they can profit from the computer's facilities (for instance, the language that comes with instructions) to communicate ideas that are impossible to communicate due to the lack of a sufficiently developed mathematical language. This happens for instance, when students are working within a dynamic geometry activity and one of them wants to explain her fellows how to build the perpendicular bisector of a given segment. In our work we have witnessed how students use expressions like "open F4 then press 4... then...". Of course, the idea is using the mathematical formalism embodied in this views and with the scaffolding supplied by the environment, orient the student towards the recognition of *the general living in the particular*.

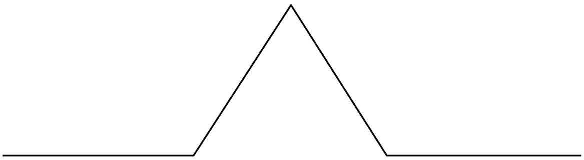
In general terms, this is the strategy that an adequate domain of abstraction helps to build in the classroom.

We can use the history of mathematics to illustrate how brilliant mathematicians have used this strategy, in particular, during those times when a new field or, more simply, a new way of looking at a particular mathematical phenomenology, was being understood.

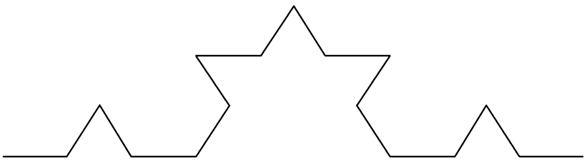
In 1904, the Swedish mathematician Helge Von Koch (1870–1924), published a paper in which he disapproved the exceedingly analytic approach led by Weierstrass.

> Until Weierstrass constructed a continuous function not differentiable at any value of its argument it was widely believed in the scientific community that every continuous curve had a well determined tangent... Even though the example of Weierstrass has corrected this misconception once and for all, it seems to me that his example is not satisfactory from the geometrical point of view since the function is defined by an analytic expression that hides the geometrical nature of the corresponding curve... This is why I have asked myself—and I believe that this question is of importance also as a didactic point in analysis and geometry—whether one could find a curve without tangents for which the geometrical aspect is in agreement with the facts.

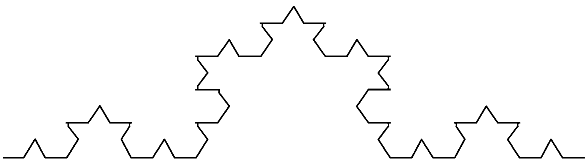
Von Koch geometrical approach to this problem was genuinely geometrical. In fact, geometry was used as a domain of abstraction to provide meaning to the new object that appeared as pathological in the mathematical landscape of his time. Today mathematical culture has evolved and those curves that were seen as non-objects, are emblematic in the world of fractals. It is simple to understand the construction process employed by Von Koch from the following figures:



Stage 1



Stage 2



Stage 3

This construction is easily made with a dynamic geometry software and following this construction the students learn to appreciate the recursive nature of a process. The object built on the screen is easily manipulable opening the door to what Balacheff and Kaput (1996) called a "new mathematical realism" due to the intense use of computers environments. We can propose activities relating the nature of the construction of Koch curve with the resolution of the screen. Compared with the paper and pencil description of the construction of the curve, the screen version has added a precision that enables us to play with the screen resolution. This establishes a link between paper and pencil reasoning with the one made possible by the new digital/dynamic construction of a mathematical object. The computer adds a new system of representation that, besides, has a virtue: being executable.

Induction and Deduction: The Computer as a Mediating Tool[1]

Courant and Robbins, in their classic book *What is Mathematics?* Called attention towards the risks that mathematics can run if, inadvertently, the balance between inductive and deductive thinking is broken:

> There seems to be a great danger in the prevailing overemphasis on the deductive-postulational character of mathematics. True, the element of constructive invention, of directing and motivating intuition … remains the core of any mathematical achievement, even in the most abstract fields. If the crystallized deductive form is the goal, intuition and construction are at least the driving forces.

Reading the history of mathematics, one can observe that the mathematical pendulum has always gone from inductive approaches to deductive ones and viceversa. As if it were a natural law!

Gauss, used to say that: *I have the result but I do not yet know how to get it* (Bailey and Borwein 2001, p. 52). Besides, he considered that to obtain the result a period of *systematic experimentation* was necessary. There is no doubt then, that Gauss made a clear distinction between *mathematical experiment* and *proof*.

Nowadays, the computer (the tool that "speaks mathematics" in Lynn Steen apt expression) is responsible for the new face of this old tension. In 1976, when Appel and Haken proved the Four Color Theorem using a computer in a crucial, substantial, way they were far from imagining the irritated reaction of many members of the mathematical community. That was not a proof according to the classical definition, they said, adding that it was not the case of using the computer to help the mathematicians in their quest for truth. Cognition, in a certain sense, had been transferred to a machine. The computer appeared as a cognitive partner, on equal terms, with the humans. The challenge cast by this new partner could not be ignored: The Gauss' *mathematical experiments* evolved into a new kind of beings, thanks to the computer. Since then, the role of the computer in mathematics research has increased, but this does not mean that all accepts its role. This is a very delicate matter that has to be thought with the utmost care as it involves deep epistemological questions. To give a flavor of the tensions introduced into mathematics by the computer, let us remind some excerpts from the letter written by Archimedes and addressed to Erathostenes in order to introduce his newly invented *Mechanical Method* to obtain, among other results, his formula for the volume of the sphere (Peitgen et al. 1992):

> Certain things became clear to me by a *mechanical method*, although they had to be demonstrated by geometry afterwards because their investigation by the said mechanical method did not furnish an actual demonstration. But it is of course easier, when we have previously

[1]We have presented more examples that illustrate the ideas in this section in Moreno and Sriraman (2005).

> acquired, by *the method*, some knowledge of the questions, to supply the proof than it is without any previous knowledge.

If we replace the bold expressions with the word "computer" we obtain the modern viewpoint of many mathematicians with respect to the use of computers with the intention to validate mathematical results. That is, the computer is at most a tool for exploring, for guessing, never for justifying. Is this a mistake? That is, the decision to put the computer aside from the activity of justification. Of course it is not, but this must not lead us into the belief that this should be always so. In these days, numerical algorithms have been designed that allow the computation of a numerical answer with a precision beyond one hundred thousand decimal figures (Bailey and Borwein 2001, p. 56). Then one can ask oneself if we are not entering a new era in which the previous relationships between exploration and justification are profoundly changing—at least at school levels. To deal with this kind of question one must use extreme prudence. This is a guiding force for the enquiry we are trying to develop.

One of the aims of research in this field is to understand how technology implementation should be conducted. We know that the first stage could entail working within the framework of a pre-established curriculum. Successful innovations should be able to *erode* traditional curricula. At that point, though, it becomes crucial to understand the nature of students' knowledge that emerges from their co-actions with those mediating tools.

Algorithms, Representations and Mathematical Thinking

During the last decade and a half, in the U.S., there has been a considerable push for the inclusion of discrete mathematics in the school curricula. There exists a body of research on the benefits of including non-continuous mathematics for facilitating enumerative reasoning, abstraction and generalization. Some research has also been done on the mediation between algorithms, internal and external representations and computing environments, particularly in the content area of discrete mathematics and probabilistic thinking. Goldin (2004) points out the caveat of the high memory load associated with such tasks and the interplay between representational systems in the solution pathway. He uses the two-pail problem, namely how can one measure 4 liters of water if one has two pails which each measure 3 and 5 liters assuming an un-ending supply of water. The problem is discrete in the sense that one has to keep track of the previous steps in order to reach the solution.

After representing the problem schematically in the form of a connected vertex-edge graph (see Fig. 1), Goldin comments on the difficulties students encounter with this problem even after they have understood all the stipulations of the problem.

> Supposing these possibilities to be understood—i.e., adequately represented internally—by the problem solver, important potential impasses still remain. Many solvers begin to imagine pouring water from pail to pail, but after three or four steps come to feel they are making no progress—and repeatedly start over. Some are hesitant to construct an external

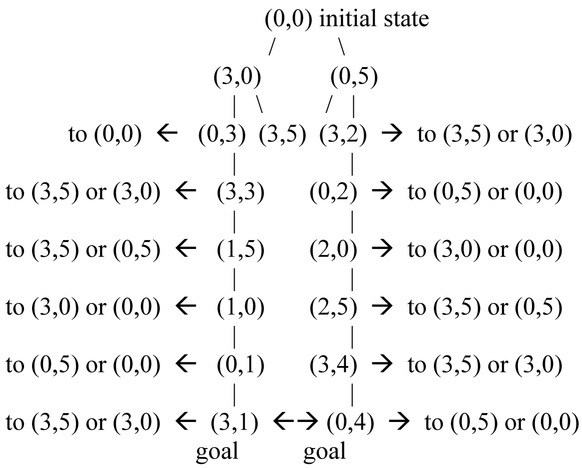


Fig. 1 Goldin's (2004) schematic representation of paths through the state-space for the problem of the two pails

> written record, or perhaps are not sure how to do it—but without some systematic external representation, the memory load is high.... Others overcome this impasse by keeping track systematically of the steps they have taken, or by persevering despite feelings of "getting nowhere." (p. 58)

On the other hand, there are cases when students begin to systematize their moves into algorithms which when efficiently applied generate the result that one is after. For instance in Sriraman (2004), ninth grade students (approx 14 years old) constructed an algorithm to efficiently generate Steiner Triplets. Could this be viewed as a natural pre-cursor to writing a computer program that efficiently generate triplets for large start values of n. In fact many of them later went on to do exactly this. Although Abramovich and Pieper (1996) recommend that teachers provide visual representations (manipulative and computer generated) to illustrate combinatorial concepts (arrangements, combinations etc.), an evolutionary perspective suggests that it is better to have students generate the algorithm based on their scratch-work (physical/mental) to produce the computer-generated representation. In a sense one can view the process of generating an efficient algorithm to produce a computer generated representation as the interface between the physical act of counting efficiently (or systematizing moves), translating this efficiency into an algorithm (symbolism) in order to get the computer generated representations.

Batanero and Godino (1998) analysis of the difficulties of children and adolescents to fully conceptualize and understand the phenomenon of randomness (p. 122) is compatible with the mathematician's general view of probability theory as enumerative combinatorial analysis applied to finite sets with considerable difficulties to generalize the theory when considering infinite sets of possible outcomes.

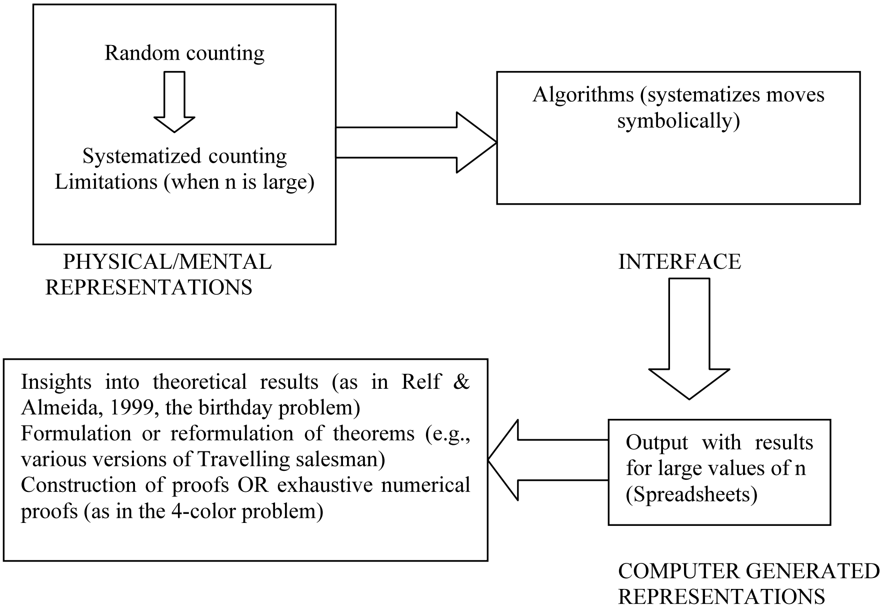


Fig. 2 A schematic of representational transits

Representational Fluidity in Dynamic Geometry

In Moreno and Sriraman (2005) we had proposed the notion of situated proofs for the learning on geometry mediated in a dynamic environment. That is at first, students might make some observations *situated* within the computational environment they are exploring, and they could be able to express their observations by means of the tools and activities devised in that environment. That is the case, for instance, when the students try to invalidate (e.g., by dragging) a property of a geometric figure and they are unable to do so. That property becomes a theorem expressed via the tools and facilitated by the environment. It is an example of *situated proof*. A question to consider is whether situated proof within a computational environment removes the dichotomy between the learner and what is learned because of the manipulators gestures that connect him/her to the environment? A situated proof is the result of a systematic exploration within an (computational) environment. It could be used to build a bridge between situated knowledge and some kind of formalization. Students purposely exploited the tools provided by the computing environment to explore mathematical relationships and to "prove" theorems (in the sense of situated proofs). As a new epistemology emerges from the lodging of these computational tools into the heart of today's mathematics, we will be able to take off the quotations marks from "prove" in the foregoing paragraph. Ruler and compass provided a mathematical technology that found its epistemological limits in the three classical Greek problems (trisection of an angle, duplication of the cube, and the nature of π). Ruler and compass embody certain normative criteria for validating mathematical knowledge. And more general, they are an example of how an expressive medium determines the ways to validate the propositions that can be stated there. Now we can ask: What kind of propositions and objects are embodied within dynamical mathematical environments? The way of looking at the problem of formal reasoning within a dynamical environment is of instrumental importance. What we propose as a *situated proof* is a way to deal with a transitional stage. We cannot close the eyes to the epistemological impact coming from the computational technologies, unless we are not willing to arrive at new knowledge but only at *new education*. For instance if one teaches abstract algebra and uses computational software such as MAPLE to compute subgroups, cosets centralizers etc, does this help students when they are trying to understand the proofs? Take for instance something like Lagrange's theorem? One certainly sees the orders of the subgroups and gets a "visual" representation of how one carves up a group into nicely divisible orders, but does this connect to the ultimate logical structure of the proof? Or does the dichotomy re-occur when the learner uses symbols denoting partitions, equivalence classes and functions to construct the proof? It would be a worthwhile goal for the community to further study this with more mathematical examples in computational environments.

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