[Pre-print of paper to appear in vol.11, no.1 of *The International Journal for Technology in Mathematics Education*]

Playing with Powers

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Abstract

This paper explores the wide range of pure mathematics that becomes accessible through the use of problems involving powers. In particular we stress the need to balance an applied and context based pedagogical and curricular approach to mathematics with the powerful pure mathematics beneath the simplicity of easily stated and understandable questions in pure mathematics. In doing so pupils realize the limitations of computational tools as well as gain an appreciation for the aesthetic beauty and power of mathematics in addition to its far-reaching applicability in the real world.

Introduction

In this paper we explore the curricular and pedagogical implications of the general area of "powers". By "powers" we mean the mathematics arising from the study of exponential functions. In the United States, the Algebra Standard recommends that students in grades 9-12 be introduced to the exponential function and classes of functions in general including polynomial, logarithmic and periodic functions (NCTM, 2000). Traditionally 9^{th} and 10^{th} graders (13-16 year olds) in the U.S are introduced to exponential functions by first extending the laws of exponents from integer powers to real powers. This is usually followed by graphing exponential functions such as 2^x , 3^x etc in order to convey the fact that for functions of the form a^x ($a > 1$), the domain is the set of real numbers whereas the range is $(0, \infty)$. The graph also reveals properties such as the increasing behavior of the function and the X-axis being the horizontal asymptote for this function as $x \to -\infty$. Students are then introduced to the case where $0 < a < 1$ to study properties that we are all familiar with. A capstone topic in this traditional treatment of exponential functions is introducing students to the irrational number 'e' by studying the behavior of $(1+ 1/n)^n$ as $n \to \infty$. Finally students are introduced to logarithms and the analogous laws of logarithms are derived from the laws of exponents.

In the mathematical experiences of the second author in Poland, as a high school student, exponential functions appeared a tiny bit later in the curriculum in comparison to the U.S., at the beginning of the 3^{rd} year of secondary school (for students at the age of 16 to 17). The sequence of events was similar to the one described in the previous paragraph. First, students learned how to define powers with rational exponents so that all laws that hold for natural exponents were still satisfied. Then, looking at graphs, they were prompted to note that exponential functions map arithmetic progressions to geometric ones (and to use this property for graphing exponentials on graph paper) etc. Finally, powers with arbitrary real exponents were defined in a more or less precise way, using monotonicity. It is hard to say precisely how many students were able to digest this section of the textbook and to retain the comprehension. Not too many, we are afraid. In our opinion, the aim of the textbook was clearly, to purify and to "bourbakize" ... alas, with no real success. The Bourbaki[1] were a group of mostly French mathematicians, who began meeting in the 1930s and aimed to write a thorough (formalized) and unified account of all mathematics. The "bourbakized" definition of $2^{\sqrt{2}}$ as the supremum of a suitable set of rational powers of 2 is not really entertaining, nor enlightening! Even mathematically oriented 16 years olds have lots of more interesting things to do! Pure mathematics should be, among other things, a source of fun – we shall come back to this later.

In the traditional curriculum students also encounter word problems associated with exponential functions. Functions obtained by running an exponential regression on data arising from the natural, social and financial sciences are given *a priori* to the students in the context of solving these word problems. Numerous examples of such functions abound. For instance $p = 760e^{-0.145h}$ relates the pressure p on a plane (in millimeters of mercury) to height h (in kilometers) above sea level. The spread of information via mass media (TV and magazines) is modeled by $d = P(1-e^{-kt})$ where P stands for a fixed population and t denotes time (Coleman, 1964). Classical examples are of course found in physics. For instance Newton's law of cooling is given by $f(t) = T + (f_0 - T)\, e^{kt}$, where $k<0$, f_0 is the initial temperature of the object that is heated, T is the temperature of the surrounding medium. Other classical examples are radioactive decay, growth of bacteria etc.

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[1] The Bourbaki essentially aimed to write a body of work based on a rigorous and formal foundation, which could be used by mathematicians in the future. For more information see Bourbaki, N. (1970). *Théorie des Ensembles dela collection elements de Mathématique*, Hermann, Paris. The internet savvy can also refer to the Bourbaki website located at http://www.bourbaki.ens.fr/

This approach has been reversed by reform-based curricula in the U.S, which emphasize modeling activities as a means to generate data. For instance bouncing balls with a known co-efficient of elasticity can be used. Students are asked to drop a ball (basketball, tennis ball etc) from a starting height and asked to keep track of the bounce height as the bounces progress until the ball is flat on the ground. Such data is made sense of by realizing that the plot of bounce height over bounce numbers shows an exponentially decaying pattern and therefore results in the invoking of an exponential regression. For instance the data for a bouncing basketball is modeled fairly accurately by $f(x) = A(0.5)^x$ where A is the initial height and x is the bounce number. These are specific examples of the wider range of model eliciting activities (Lesh & Doerr, 2003) that expose students to applied mathematics in a real world context very early in their schooling experiences. This is in stark contrast to the "good old" days when one had to take a course in differential equations to encounter such modeling problems in a "recipe driven" didactic environment.

Exposure to the applied aspects of mathematics conveys to students only one aspect of the spectrum of mathematics. We claim that pure math activities can also be accomplished by problems involving powers, which – despite their surprisingly elementary statements – can lead students to deep insights into the behavior of numbers and the exciting possibilities in pure mathematics. Our goal in this paper is to demonstrate these alternative possibilities in pure mathematics by playing with powers, thus broadening the spectrum of the students.

Power Problems and Remainders: Lead-ins to Number theory basics

Students encounter basic number theory concepts such as prime and composite numbers, the division algorithm, divisibility tests, notions of least common multiple and greatest common divisor and the fundamental theorem of arithmetic in the middle school grades [10-13 year olds].

Unfortunately there is little or no follow up to these topics in the higher grades. One of the curricular flaws of viewing Calculus as the pinnacle of the students' secondary schooling experience is working pre-dominantly over the set of real numbers with functions of continuous variables. The traditional sequence of Algebra-Geometry-Trigonometry and Analytic Geometry offers little opportunity to further develop elementary number theory notions that students have previously encountered. Problems involving powers can be very useful to remedy this unfortunate situation. Most of the extant books that include a treatment of problems involving powers and number theory are classified as "competition" books, thereby creating an "elitist" aura around such problems. We think otherwise and invite teachers to make appropriate use of "power" tasks to convey to students the power of pure mathematics and the limitations of computing tools. An illustration of such a problem follows.

One can pose the question: What is the remainder when we divide 6^{131} by 215? Most calculators would not be able to handle 6^{131}, which immediately necessitates us to examine the problem with different conceptual tools. We can reflect on what remainders mean? We can go back to notions from the early grades and start basic computations such as:

What is the remainder when we divide 5 by 3? It's 2. That was easy.

What is the remainder when we divide $5 \times 8 = 40$ by 3? It's 1. Still easy!

Now what is the remainder when we divide $5 \times 8 \times 7 = 280$ by 3? We could laboriously perform the division by 3 and find that the remainder is 1.

Is there a lesson to be learned through this empirical work? Yes! The important mathematical phenomenon to communicate is that if we calculate the remainders of 5, 8 and 7 (which are 2,2, and 1 respectively) and multiply these individual remainders $2 \times 2 \times 1 = 4$ and divide this by 3, we get the same remainder when dividing 280 by 3. This phenomenon is in general called the Remainder theorem, which can simply be stated as: the remainder of a product of numbers divided by a given number is equal to the remainder of the product of the remainders of the numbers (constituting the product) divided by this given number.

Now getting back to the original problem. We can employ the laws of exponents that our students so faithfully memorize and write 6^{131} as a product of numbers that leave a “nice” remainder when divided by 215. The best candidate is 6^3 since $6^3 = 216$ and 216 divided by 215 leaves a remainder of 1, and we can easily multiply 1’s.

So $6^{131} = 6^3 \times 6^3 \times 6^3 \times 6^3 \ldots$(43 times) $\times\, 6^2$ divided by 215 leaves a product of remainders $1 \times 1 \times 1 \times 1 \ldots$(43 times) $\times$ 36, which is 36 and 36 divided by 215 leaves a remainder of 36. Done! This problem not only allowed us to make use of laws of exponents but also led to some insights into the theory of numbers.

Another nice problem involving powers is calculating the last digit of a given power, typically a huge number, such as 777^{777} or 2004^{2004}. Once again this allows us to make a nice foray into the basic behavior of numbers. If we consider a simpler problem such as 1^{12345} we observe that the last digit is obviously 1 since we are simply multiplying a product of 1’s. What if we had 11^{12345}, we can empirically verify that 11^1, 11^2, 11^3, 11^4…always end in 1. Students can be led to observe that the last digit is 1, and in general the last digit of any huge number such as 777^{777} is the product of the last digit multiplied by itself the given number of times. So the last digit of 777^{777} is the same as the last digit of 7^{777}. Now if we write out the powers of 7, we find a periodicity phenomenon in the last digits related to its powers. This is observed by listing out the sequence of the powers of 7: 7^1, 7^2, 7^3, 7^4, 7^5, 7^6, 7^7, 7^8, 7^9… which gives last digits 7,9,3,1,7,9,3,1,7…. So we can once again employ laws of exponents to rewrite 7^{777} as:

$7^{777} = (7^4) \times (7^4) \times (7^4) \times (7^4) \times \ldots$(194 times) $\times\, 7^1$, so the last digit is 7.

Since we seem to be able to determine last digits of ridiculously large numbers, why not look at the problem of determining first digits.

Starting digit problems: "Excuses" into Combinatorics and Analysis

Let's start with an existence problem. Is there an integral power of 2 that begins with 1999…in its decimal expansion? In other words the question is asking us to prove the existence of an integer 'n' such that $2^n = 1999\ldots$ without explicitly asking exactly what this power is. We can clearly assume that $n > 0$. When we raise 2 to any integer power, there are 9 obvious choices for the first digit since we naturally exclude zero, and then there are 10 choices for each digit after that. So there are $9 \times 10 \times 10 \times 10 = 9000$ possible ways of listing the first four digits. Since $n > 0$, we have no restrictions on the number of values we can generate, and we can easily generate more than 9000 values for 2^n. By the pigeonhole principle[2], some powers of 2 have to begin with the same four-digit string but we are still not sure that one of those starting strings really equals 1999. This is a subtle question with an intriguing relation to irrationality and we postpone it for a second to mention something simpler.

A nice problem that is easily solved via the pigeonhole principle and makes use of divisibility properties is to prove that there exists some power of 3 that ends in 001. This can easily be shown as follows. Suppose 3^m and 3^n (where $m>n>1$) upon division by 1000 have the same remainder. (The existence of such m and n needs to be established by applying the pigeonhole principle and we will leave it up to the readers.) Then $3^m - 3^n = 3^n(3^{m-n} - 1)$ is divisible by 1000. Now 3^n and 1000 clearly have no common factors, which means 1000 has to divide the factor $(3^{m-n} - 1)$. This implies 3^{m-n} ends with 001.

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[2] The pigeonhole (or Dirichlet) principle states that if we have "m" pigeons and "n" pigeonholes, where $m > n$, then some pigeonhole contains more than one pigeon. This seemingly obvious principle has wide ranging applicability in mathematics.

Now let us come back to powers of 2, and to the question whether one of them begins, in decimal notation, with 1999. Maybe this string is too strange to appear at the beginning of the decimal notation of some power of 2? Let us try something simpler first. Consider a sequence a(n) consisting of the first digits of the consecutive powers of 2:

1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, 8, ... Will 7 ever appear in this sequence?

This problem in different versions is known in mathematical literature. The first mention is found in the famous book *Ordinary differential equations* (Arnold, 1978). It is usually accompanied by auxiliary facts or suggestions intended to make a solution more accessible. Nevertheless, we have not encountered any detailed solutions to this problem. We will remedy this unfortunate situation and in the process illustrate the rich mathematics that comes out of it. To start with - a desperate solution using a piece of paper and a pencil or another powerful computing tool will easily verify that

$$2^{46} = 70\ 368\ 744\ 177\ 664$$

Going further with this experiment we can see that 7 is the first digit of 56^{th}, 66^{th}, 76^{th}, 86^{th} and the 96^{th} power of 2 (but the first digit of 106^{th} power of 2 is 8, not 7). This empirical method, however, is clearly far from being mathematically elegant. We need a better solution, which would allow us to draw other conclusions. First of all, we must try to realize the meaning of the statement that 7 is the first digit of a number 2^n. The answer is simple: 7 is the first digit of 2^n if and only if for some natural k we have

$$7 \times 10^k < 2^n < 8 \times 10^k$$

We can get a simpler description of this condition if we take the decimal logarithms of both sides, something our students would be quite familiar with. This yields $k + \log(7) < n \log(2)$

< k + log(8). Since decimal logarithms of 7 and 8 lie between 0 and 1, we conclude that k is the integer part of the number n log (2), which leads to the following inequalities:

$$\log 7 < n \log 2 - [n \log 2] < \log 8$$

And now it suffices to bring together some known facts, which we invite the readers to verify.

Lemma 1. *The number log (2) is irrational.*

Lemma 2. *If a number x is irrational and c(n):=nx-[nx], then for any a and b belonging to [0,1] infinitely many members of the sequence c(n) lie in the interval (a,b).*

We hope that readers will encourage their students to prove lemma 1, which is very easily provable via contradiction. Assuming this done, let us have a look at a proof of the second lemma and then examine their consequences.

Proof of Lemma 2. Observe first that all the members of the sequence c (n) are different. Indeed, if c(k)=c(m) for k > m, then (k-m)x = [kx]-[mx]. This is a contradiction, since the product of a non-zero integer (k-m) and an irrational number x cannot be an integer.

Take now a positive integer n such that 1/n < b-a. The numbers c(1), c(2), ..., c(n+1) being all different and belonging to the interval [0,1], we infer by the pigeon-hole principle that for some i and s such that i and (i+s) both lie between 1 and (n+1) the following inequality holds:

$$0 < \varepsilon = |c(i) - c(i+s)| \le 1/n < b - a \qquad (1)$$

Now wrap the real axis into a circumference **T** of perimeter 1 with a distinguished point 0. For two numbers a and b in [0,1] we denote by (a,b) the arc of **T** which corresponds to the interval (a,b) on the real axis. Let f**:** **T** $\to$ **T** be an anti-clockwise revolution by an angle of $2\pi x$ radians. Instead of watching the numbers c(n) in the interval [0,1] we shall look at the images of the distinguished point 0 under iterations of f on **T**. After a moment of reflection we note that the length of an arc (0,b(n)), where $b(n) = f^n(0) = f \bullet f \bullet \ldots \bullet f(0)$ [$\bullet$ = composition of mappings] is equal to c(n) Hence, due to (1) we know that the length of an arc between the points b(i) and b(i+s) is smaller than b-a. This means that the s^{th} iterate of f is a revolution by an angle of $2\pi\varepsilon$ radians; the direction of this revolution is of no importance to us. This obviously implies that infinitely many of the points b(s), b(2s), b(3s), ... belong to the arc (a,b). Indeed, if we start from the fixed point 0 and walk for an infinitely long time along the circumference **T** in one and the same direction making steps of length ε, then infinitely many times we shall step into the arc (a,b), since its length b-a is greater than the length of our step. This completes the proof.

Now, by applying Lemma 2 to x = log(2), a = log(7), b=log(8) we get that 7 is the first digit of infinitely many powers of 2. If we apply again Lemma 2 to numbers x = log (2), a = log (77)-1, b = log (78)-1, then because of the equalities 1 = [log 77] = [log 78], we conclude that the figure seven can even appear twice in the first two places of the decimal notation of a power of 2. Using an analogous argument we readily discover that any finite sequence of digits can appear at the beginning of a decimal notation of a power of 2, like 1234 or 567890, or (finally) 1999. If you really do not believe this last statement, we invite you to compute (say) 2^{9030} or 2^{11166} . At the end of this article we exhibit some important historical dates compared with corresponding powers of 2 - to satisfy the skeptics. We can further deduce the following corollary:

Corollary. *If an integer p > 1 is not an integer power of 10, then any sequence of digits can appear at the beginning of the decimal notation of n-th power of p for some n.*

So the question again is why does 7 not appear among the first members of the sequence we introduced at the very beginning? Why does this deceitful sequence pretend to be periodical? The reason is simple. The number log (2)=0.3010299956... can be very well approximated by the rational number 0.3, and for all rational x the sequence c(n) = nx - [nx] is periodical. In other words: 2^{10} = 1024, which is quite close to 1000. And multiplication by 1000 just adds zeroes at the end; front digits stay unchanged. This is why after seeing the first few members of the sequence a(n) we come to an erroneous conclusion that the sequence has period 10 and 7 is not a member of it, while 8 appears quite often. To see the first 7 one has to look at the first 6 in a(n) (i.e., the first digit of $64 = 2^6$), and then wait until the cumulative effects of small perturbation $24 = 2^{10}-1000$ do their job.

In 1910 Waclaw Sierpinski, Hermann Weyl and P.Bohl proved independently of one another that for every irrational x the sequence c(n)=nx - [nx] is equidistributed over the interval [0,1] (Arnold & Avez, 1968). More precisely, if we take arbitrary a and b (with a < b) from [0,1], and let k(n,a,b) denote the number of elements of the set $\{c(i) : 1 \leq i \leq n, c(i) \in (a, b)\}$ then we get:

$$\lim_{n \to \infty} k(n, a, b)/n = b - a$$

Speaking in more illustrative terms, this theorem states that if we walk along a circle of circumference 1 unit, taking steps of irrational length, then we step on each hole with a frequency which is directly proportional to the size of the hole! Let's translate this fact into our language of powers of 2. Let a(7,n) and a(8,n) be the number of sevens and eights, correspondingly, among the first n members of the sequence a(n). By the last formula, we have

$$\lim_{n \to \infty} a(7,n)/n = \log 8 - \log 7$$

$$\lim_{n \to \infty} a(8,n)/n = \log 9 - \log 8, \quad \text{so consequently}$$

$$\lim_{n \to \infty} a(7,n)/\, a(8,n) = (\log 8 - \log 7)/(\log 9 - \log 8) = 1.1337\ldots > 1$$

This means that by looking at sufficiently long initial fragments of the sequence a(n) we will see slightly more sevens than eights. This result of Bohl, Sierpinski and Weyl, which we previously mentioned and our little fact on 7's and 8's are in fact simple consequences of a very general and deep theorem in ergodic theory due to G.D.Birkhoff (Cornfeld, Fomin and Sinai, 1982), which the interested reader is urged to pursue[3]. We conclude this section with a little problem for our readers.

> Problem: For what n does the number 2^n have four consecutive sevens at the beginning? What about five sevens? How can we estimate from above the least n such that the decimal notation of 2^n begins with 2004 consecutive sevens?

Functional Problems: Getting Deeper

If we play with the laws of powers (exponents), then we observe that $a^{x+y} = a^x \cdot a^y$. Let us assume that a > 0. The question can be posed generally as: Suppose f: $R \rightarrow R$, what are all the other solutions to $f(x + y) = f(x) \cdot f(y)$? Another good problem arises from observing that $\log(xy) = \log x + \log y$. So what are all the other solutions to $f(x \cdot y) = f(x) + f(y)$? A classic related problem is of course to find all functions that satisfy the Cauchy functional equation: $f(x+y) = f(x) + f(y)$. In answering these problems one gets into a deep investigation of functional equations, something that teachers can use as an extended project for the motivated and bright students.

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[3] An online reference with the specifics of this theorem is found in *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/BirkhoffsErgodicTheorem.html

Some Unusual Powers of 2 and Conclusions

In order to satisfy the skeptics of the unusual property of powers of 2 we list some powers of 2 that connect with a (biased) sample of important historical dates.

The baptism of Poland	966	$2^{568} = 9.66... \times 10^{170}$
The battle of Hastings	1066	$2^{5561} = 1.066... \times 10^{1674}$
Columbus discovers America	1492	$2^{3761} = 1.492... \times 10^{1132}$
The founding of Harvard University	1636	$2^{9528} = 1.636... \times 10^{2868}$
Cromwell's death	1658	$2^{3223} = 1.658... \times 10^{970}$
The founding of Royal Society	1660	$2^{4874} = 1.660... \times 10^{1467}$
New Amsterdam changes name to New York	1664	$2^{6040} = 1.664... \times 10^{1818}$
First edition of Newton's "Principia"	1687	$2^{6143} = 1.687\ldots \times 10^{1849}$
Walpole becomes Britain's first Prime Minister	1721	$2^{10229} = 1.721\ldots \times 10^{3079}$
French Revolution	1789	$2^{9857} = 1.789\ldots \times 10^{2967}$
Waterloo	1815	$2^{931} = 1.815\ldots \times 10^{280}$
Beginning of World War II	1939	$2^{5522} = 1.939\ldots \times 10^{1662}$
End of World War II	1945	$2^{1931} = 1.945\ldots \times 10^{581}$

We hope to have conveyed to the readers the richness of pure mathematics present in problems involving powers. Playing with problems involving powers to make deep connections with topics in Number Theory, Combinatorics and Analysis complements the Applied mathematics and Statistics that students learn through the modeling approach that is presently gathering momentum. The first author used first and last digit problems similar to the ones mentioned in this paper with 14-year old pupils enrolled in an Algebra course. The pedagogical goal was to mediate "pure math" problem solving experiences and resulted in considerable student interest in the mysteries of the integers. Among other things, students realized the limitations of computing tools and understood the need to create/invent conceptual tools to tackle the problem. Other problems involving a particular phenomenon among the positive integers and resulting in the discovery of the pigeonhole principle also met with great success in the classroom (Sriraman, 2004a; 2004b).

In conclusion, our hope is that we never forget the aesthetic beauty inherent in pure math activities and convey to our students that such activities have sustained the imagination of mathematicians and contributed to its growth from the very onset of its history. We feel that the image of a pure mathematician lying under a tree (apparently doing "nothing" to the untrained eye!) complements and balances the image of the diligent applied mathematician and scientist engrossed in making sense of the hubris and chaos of the real world. After all, what would the second person do if the first one really did nothing?

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