The Steiner TME-program of 1987: Where are we today?

Günter Törner (Germany)
Bharath Sriraman (USA)

Abstract: In this contribution we discuss the six theses presented by Hans-Georg Steiner (1987), which were instrumental in the community becoming interested in theories and philosophies of mathematics education. We discuss overlooked aspects of this seminal paper particularly in light of recent developments in the field of mathematics education. Nearly twenty years later, we reflect on the development of Steiner's program for theory development and examine if any progress has been made at all on the open questions that Steiner (1987) posed to the community.

Introduction

Philosophy has been both an outstanding and an integral part of mathematics for a long time and was regarded as a subdiscipline. It was also implicitly accepted that philosophical positions of a bearer influence his or her view on mathematics and its teaching. However, this lasted until 1973 when the famous mathematician René Thom (1973) proclaimed in the Second International Congress on Mathematical Education that "whether one wishes it or not, all mathematical pedagogy even if scarcely coherent, rests on a philosophy of mathematics" (p. 204) Simply put, philosophy of mathematics is the important framework for teaching (and learning) mathematics.

It was Hans-Georg Steiner who had become aware of the importance of this correlation long before Ernest formulated his research questions regarding the impact on the teaching of mathematics in 1989. Thus, the Steiner-paper (1987) titled *Philosophical and epistemological aspects of mathematics and their interaction with theory and practice in mathematics education* can be regarded as a key for the development of theories on mathematics learning and - as well as what is called today - beliefs theory. Thus, nearly twenty years later, it is time to reflect on the development of the Steiner programm 1987 and examine if any progress has been made at all on the open questions that Steiner (1987) posed to the community.

The development of mathematics didactics in Germany is in a sense inseparably connected to the name of Hans Georg Steiner. The question is what among his many contributions should be most appreciated by the community. The authors of this article have chosen to focus on an overlooked and visionary contribution the field, namely his 1987 paper which was based on earlier work. Steiner's approaches to theories of mathematics education were concerned both with the philsophy of mathematics, the processes surrounding its instruction, and the need to internationalize the basic problems of the field. And so we would like to appreciate this exemplary individual in the context of this particular FLM article, whose original drafts date back to 1984. As stated earlier, many of today's theories which address beliefs, psychological and epistemological issues can be viewed in light of the theses presented in his article.

Background of the 1987 article

The 1987 article is based on lectures, which Steiner held at the 5th ICME in Adelaide in 1984, PME 8 in Sydney, and at the University of Georgia in 1985. Based on these lectures, Steiner initiated the formation of TME = Theory of Mathematics Education Group, which was concerned with basic problems confronting the field, namely to systematize the foundations, theories and methodologies for mathematics education research, development and practice (see Steiner, 1984, 1985 and 1992).

If one carefully examines Steiner's articles which led up to this particular 1987 publication, the work comes across as an axiomatic approach to delineating the foundational problems of the field. We believe this approach was shaped predominantly by Steiner's career trajectory with formative experiences in the 60's, at which time axiomatic drafts were en vogue. The evidence of this lies in our examination of the following works (Steiner, 1966a, 1966b, 1974). Steiner essentially dedicated himself to the development of a meta-theory which took into consideration the philosophical and epistemological aspects of mathematics education. We pose the question as to why epistemological and philosophical considerations are important for theories of mathematics education? Epistemology is a branch of philosophy concerned with the nature of knowledge and justification of belief. Sierpinska and Lerman (1996) state:

> *Epistemology as a branch of philosophy concerned with scientific knowledge poses fundamental questions such as: 'What are the origins of scientific knowledge?' (Empirical? Rational?); 'What are the criteria of validity of sci-*

> *entific knowledge?' (Able to predict actual events? Logical consistency?); 'What is the character of the process of development of scientific knowledge?' (Accumulation and continuity? Periods of normal science, scientific revolutions and discontinuity? Shifts and refinement in scientific programs?). (p. 828)*

The question: what is mathematics?, for teaching and learning considerations brings into relevance the need to develop a philosophy of mathematics compatible with mathematics education. In order to answer this question for mathematics education, we think it is necessary to examine the writings of mathematicians who are sympathetic to the issues of teaching and learning mathematics, as opposed to simply doing original research. Hersh (1979) defined the “philosophy of mathematics” accordingly to the working philosophy of the professional mathematician, a philosophical attitude to his work that is assumed by the researcher, teacher, or user of mathematics and especially the central issue – the analysis of truth and meaning in mathematical discourse. Much later, Hersh (1991), wrote

> *Compared to "backstage" mathematics, "front" mathematics is formal, precise, ordered and abstract. It is separated clearly into definitions, theorems, and remarks. To every question there is an answer or at least, a conspicuous label: "open question". The goal is stated at the beginning of each chapter, and attained at the end. Compared to "front" mathematics, mathematics "in back is fragmentary, informal, intuitive, tentative. We try this or that, we say "maybe" or "it looks like". (p. 128)*

So, it seems to us that Hersh is not concerned with dry ontological statements about the nature of mathematics and mathematical objects, but is more concerned with the methodology of doing mathematics, which makes it a human activity.

The Six theses

Given this background, we are now ready to examine the six theses presented by Steiner and our interpretation of the basic message in each of them.

Thesis 1: *Generally speaking, all more or less elaborated conceptions, epistemologies, methodologies, philosophies of mathematics (in the large or in part) contain - often in an implicit way - ideas, orientations or germs for theories on the teaching and learning of mathematics.*

Message 1: Philosophy and Epistemology are so closely interwined that it is difficult to separate the two.

Thesis 2: *Concepts for the teaching and learning of mathematics - more specifically: goals and objectives (taxonomies), syllabi, textbooks, curricula, teaching methodologies, didactical principles, learning theories, mathematics education research designs (models, paradigms, theories, etc.), but likewise teachers' conceptions of mathematics and mathematics teaching as well as students' perceptions of mathematics carry with them or even rest upon (often in an implicit way) particular philosophical and epistemological views of mathematics.*

Message 2: Non-philosophy is also a philosophical position.

Corollary 2: There is no neutrality in ones philosophical and epistemological views about mathematics.

We refer to Davis (1972); we also assume that Steiner considered at that time the current publications of Davis and Hersh. The papers of Hersh (1979) and Davis & Hersh (1980) also belong to this tradition as well as representing a spirited reaction of working mathematicians to various pompous aberrations of the New Math. Lastly, the book by Hersh, R. (1997), *What is Mathematics, really?* constitutes a plea for a deliberate epistemological positioning, with a particular view towards the importance of epistemological perspectives for teaching and learning. This message is best illustrated by Hersh’s (1997) critique of a formalist view towards mathematics. He writes:

> *The devastating effect of formalism on teaching has been described by others.* ***I haven’t seen the effect of Platonism on teaching described in print.*** *But at a teachers’ meeting I heard this: „Teacher thinks s/he perceive other wordly mathematics. Students is convinced teacher really does perceive other wordly mathematics. No way does student believe he’s about to perceive other wordly mathematics (p.238).*

If we turn the message of thesis 2 positively, then one can refer to Schoenfeld (1998), and his theory of teaching-in-context, in which one interprets teacher behavior rationally based on a teacher’s dominant goals and beliefs. Much earlier, Fenstermacher (1978) described the role and the associated problems with beliefs. Around the time Steiner published his work, Thompson (1982,1992) was laying the foundations of a theory to explain teacher’s actions in a mathematics classroom based on their beliefs about mathematics. What we are suggesting is that is an instance of the development of a local philosophy based on the particular problems in the domain of beliefs. Thompson (1992) wrote:

> *I think we will get further evidence on the role of teachers' views of mathematics when we go into more detail and investigate their understanding of different domains of mathematics, of specific components such as the meaning of mathematical concepts, proof, definition, theorem, conjecture, variable, symbols, rule, formula, axiom, problem, problem solving, application, model, computation, graphical representation, visualization, metaphor, etc., both with respect to the various sub-domains of mathematics as well as in a more general sense. page*

Today we usually speak of teachers' beliefs, which are generally formulated as „views about mathematics“ (see e.g. Pajares, 1992, Grigutsch, 1996). It is assumed that different beliefs about mathematics have different associated philosophies and/or epistemologies. Törner (2002) explores this question somewhat more in detail, unaware of the work of Steiner (1987) and that of Aguirre (2006). In contrast, in the literature from psychology the focus is placed on very domain specific epistemological beliefs (Paulsen et al. ,1998; Buehl et al. .2001).

Thesis 3. *There is no distinguished, constant, universal philosophy of mathematics. One should evaluate philosophies of mathematics according to their fruitfulness for particular goals and purposes and develop criteria for evaluation.*

Message 3: We accept the relativistic stance of this thesis and suggest that a relativistic stance towards a philosophy of mathematics education is necessary dependent on the particular problem context/ framework being addressed. Another implication that Steiner places in this thesis is that research mathematics is a different (higher) instance of school mathematics. This creates a co-existence of belief systems with those held by mathematicians. Hersh (2006) grants or certifies schizophrenia to the mathematicians. Thus are mathematicians in terms of their philosophical orientation sometimes chameleons?
For instance Hersh (1997) writes:

> *"The working mathematician is a Platonist on weekdays, a formalist on weekends. On weekdays, when doing mathematics, he's a Platonist, convinced he's dealing with an objective reality whose properties he's trying to determine. On weekends, if challenged to give a philosophical account of the reality, it's easiest to pretend he doesn't believe it. He plays formalist, and pretends mathematics is a meaningless game." (p. 39)*

Does this mean that the community should accept that no universal philosophy of mathematics education is therefore possible? We turn to the next thesis to further develop this line of inquiry.

Thesis 4: For mathematics education one should prefer and elaborate philosophies of mathematics which especially respect the following aspects: different forms and conditionalities of mathematical knowledge, means and modes of representation and activities, relations between subjective and objective developments of knowledge (complementantity, obstacles, dynamics), relation of mathematical knowledge to, other knowledge, special fields and applications; the personal, social and political dimension of mathematics.

Message 4: We think this thesis is a difficult one to directly interpret. Our interpretation is that focusing on how learning occurs may develop a more favorable philosophy of mathematics education.

The two dominant philosophies that arose in the 80’s and 90’s were radical constructivism (see von Glasersfeld, 1983) and social constructivism (Ernest; 1989,1991). With a very instrumental view of mathematics - understandably - the classical “Stoffdidaktik” tradition in Germany asserts the need to continually develop the pedagogy of mathematics. However there were some inherent problems in each of these philosophies as is pointed out by Goldin (2003)

> *Radical constructivism helped overthrow dismissive behaviorism, rendering not only legitimate but highly desirable the qualitative study of students' individual reasoning processes and discussions of their internal cognitions (but with the unfortunate provision that no 'objective validity' could be claimed for the conclusions of research). It led to many in-depth, observational studies that have been of value to those who have advocated meaningful, guided discovery-oriented mathematical learning. Social constructivism pointed to the importance of social and cultural contexts and processes in mathematics as well as mathematics education, and postmodernism highlighted functions of language and of social institutions as exercising power and control. And*

> *'mind-based mathematics' emphasized the ubiquity and dynamic nature of metaphor in human language, including the language of mathematics. Unfortunately, in emphasizing its own central idea, each of these has insisted on excluding and delegitimizing other phenomena and other constructs, even to the point of the words that describe them being forbidden - including central constructs of mathematics and science - or, alternatively, certain meanings being forbidden to these words. Yet the ideas summarized here as comprising the 'integrity of knowledge' from mathematics, science, and education are not only well-known, but have proven their utility in their respective fields. There are ample reasoned arguments and supporting evidence for them.*

Goldin then gives an interesting analogy between researchers working in mathematics education and those working in mathematics. He writes:

> *Educational researchers who might be characterized as 'behaviorists' or 'neo-behaviorists', as well as those who might be termed 'ultrarelativists', have performed groundbreaking work. Mathematicians who might be characterized as 'Platonists' or 'formalists', as well as those holding quite different views, have achieved important mathematical insights, and argued for attention to important educational priorities.*

Although Goldin's (2003) analysis of mathematics and mathematics education in their mutually dependent conditions is helpful, it should be noted that he does not go all the way in taking a completely relativistic stance. We argue that for Goldin, mathematics remains a last instance of a monolithic super structure. He writes:

> *My thesis here is that the chasm that has opened is in part attributable to the long fashionableness of certain epistemologies or theoretical 'paradigms' in mathematics education, that dismiss or deny the integrity of fundamental aspects of mathematical and scientific knowledge.*

The question then is does Goldin really acknowledge that paradigm shifts which occur even in mathematics? (see Stewart, 1995)
The message in the next two theses of Steiner's (1987) article are quite self evident:

Thesis 5: Such philosophies of mathematics should become an ingredient of a form of reflective mathematics teaching and learning, and contribute to the development of an adequate meta-knowledge not only for teachers but also for students.

Thesis 6: Mathematics education needs comprehensive approaches and meta-theories which should comprise an adequate philosophy of mathematics. For a metatheory which is built on a systems approach based on human activity and social interaction, an adequate philosophy of mathematics should view mathematics itself as a system from the point of view of human object-related cooperative activities.

Where are we today? Where should we go?

The question now is why is Steiner's (1987) article of relevance today? It seems to us that Steiner was pointing to the (in-) differences of mathematics educators and psychologists to the epistemologies used by each side on what constitutes knowledge and learning. For instance, every paper dealing with the topic of epistemology in mathematics education cites the survey article of Sierpinska and Lerman (1996) addressing this issue. On the other side, every psychologically oriented publication in general reference papers e.g. of Schommer (1990), Schommer et al. (1992), Buehl and Alexander (2001), Hofer and Pintrich (1997), Clarebout et al. (2001) and especially that of Muis (2004). What are the reasons that these two research areas have never met and hardly communicated? The following paper regards itself as a small contribution toward an attempt to bridge these two strands. To find a rationale for this 'ignorance' we note that mathematics educators themselves have been intensively dealing with beliefs as a hidden variable for more than thirty years (Leder, Pehkonen & Törner, 2002). Although, their discussions are tightly intertwined with epistemological considerations, this is seldom recognized in most of their papers, and the term 'epistemology' is rarely mentioned explicitly. In this paper, we approach this issue in a pragmatic way. On the one hand, we focus on epistemology in general and show that its role has always been a prominent and integral topic in the history and philosophy of mathematics. On the other hand, we draw on the findings of various psychologists. Several new empirical results are also taken from the recent Ph.D. thesis (Rolka, submitted) of the student of the two authors. Last but not least, it should be noted that our research focus does not exclusively dwell on academic quality, which might be viewed as a purely philosophical issue, however, epistemological orientations and precepts play an important role in the everyday life of teaching and learning (see e.g. Sierpinski 1992b, Noss 2002, Noss et al. 1999, Köller et al. 2000).

Another example in the differences of epistemologies comes in the epistemology of what constitutes "mathematical creativity". For instance, in mathematics education, the epistemology of "mathematical creativity" is typically drawn from analyzing the writings and reflections of eminent mathematicians. Recently Burton (2004) proposed an epistemological model of "mathematician's coming to know" for consideration by the community, consisting of five interconnecting categories, namely the person and the social/cultural system, aesthetics, intuition/insight, multiple approaches, and connections. This model is grounded in the extensive literature base of mathematics, mathematics education, sociology of knowledge and feminist science, and purports to address four dominant challenges, namely "the challenges to objectivity, to homogeneity, to impersonality, and to incoherence." (p.17). Burton empirically tests this model by interviewing 70 mathematicians and qualitatively analyzing the data to generate the dominant themes. On the other hand in mainstream psychology, the approach to studying creativity uses quantitative methods and a systems approach to understanding the construct. Prominent among these are the "histriometric approach" (Simonton, 1984, 1994); the "systems approach" (Csikszentmihalyi, 1988, 2000); and the "investment theory approach" (Sternberg & Lubart, 1996, 2000) to studying creativity generated from historical and empirical data. These confluence theories both circumscribe and complement the categories of Burton's model. Burton's findings about the practices of mathematicians within the field of mathematics do cohere with the confluence theories of creativity but interestingly enough the references from the field of psychology are absent in the list of the book's references and vice versa. It seems to us that there is a further schism in the epistemologies of mathematics educators and psychologists conceptions of creativity.

In the domain of proofs in mathematics education, the DNR[1] theory created by Harel (2006) attempts to bridge the epistemologies of what constitutes proofs in the professional mathematics and mathematics education. Harel (2006) writes:

> *Pedagogically, the most critical question is how to achieve such a vital goal as helping students construct desirable ways of understanding and ways of thinking. DNR has been developed to achieve this very goal. As such, it is rooted in a perspective that positions the mathematical integrity of the content taught and the intellectual need of the student at the center of the instructional effort. The mathematical integrity of a curricular content is determined by the ways of understanding and ways of thinking that have evolved in many centuries of mathematical practice and continue to be the ground for scientific advances. To address the need of the student as a learner, a subjective approach to knowledge is necessary. For example, the definitions of the process of "proving" and "proof scheme" are deliberately student-centered. It is so because the construction of new knowledge does not take place in a vacuum but is shaped by one's current knowledge.*

Harel's views echo the recommendations of William Thurston, the 1982 fields medal winner, who contributed a chapter entitled On Proof and Progress in Hersh (2006). Thurston outlines for the lay person (1) what mathematicians do, (2) how (different) people understand mathematics, (3) how this understanding is communicated, (4) what is a proof, (5) what motivates mathematicians, and finally (6) some personal experiences. Thurston stresses the human dimension of what it means to do and communicate mathematics. He also gives numerous insights into the psychology of mathematical creativity, particularly in the section on what motivates mathematicians. Mathematics educators can draw great satisfaction from Thurston's writings, particularly on the need for a community and communication to successfully advance ideas and the very social and variant nature of proof, which depends on the sophistication of a particular audience.

It seems to us that although there are dissonances in the terminology used by psychologists, mathematics educators and mathematicians when speaking about the same construct, there are some similar elements which can lay the foundation of a common epistemology. But several hurdles exist in creating common epistemologies for the diverse audience of researchers working in mathematics education. Harel (2006b) in his commentary to Lester's (2005) recommendations to the mathematics education research community (for developing a philosophical and theoretical foundation), warns us of the dangers of oversimplifying constructs that on the surface seem to be the same. Harel (2006b) writes that a major effort has been underway for the last two decades to promote argumentation, debate and discourse in the mathematics classroom. He points out that scholars from multiple domains of research have been involved in this initiative, i.e., mathematicians, sociologists, psychologists, classroom

[1] DNR= *duality, necessity*, and *repeated-reasoning*

teachers, mathematics educators and such. Harel (2006b) writes:

> *However, there is a major gap between "argumentation" and "mathematical reasoning" that, if not understood, could lead us to advance mostly argumentation skills and little or no mathematical reasoning.* ***.*** *Any research framework for a study involving mathematical discourse ...[w]ould have to explore the fundamental differences between argumentation and mathematical reasoning, and any such exploration will reveal the critical need for deep mathematical knowledge. In mathematical deduction one must distinguish between status and content of a proposition (see Duval, 2002). Status (e.g., hypothesis, conclusion, etc.), in contrast to content, is dependent only on the organization of deduction and organization of knowledge. Hence, the validity of a proposition in mathematics—unlike in any other field—can be determined only by its place in logical value, not by epistemic value (degree of trust). Students mostly focus on content, and experience major difficulties detaching status from content. As a consequence, many propositions in mathematics seem trivial to students because they judge them in terms of epistemic values rather than logical values. Another.. [s]ource of difficulty for students, is that in the process of constructing a proof, the status of propositions changes: the conclusion of one deductive step may become a hypothesis of another. These are crucial characteristics that must hold in any form of mathematical discourse, informal as well as formal (!). In argumentation and persuasion outside mathematics, on the other hand, they are not the main concern: the strength of the arguments that are put forward for or against a claim matters much more.*

Concluding Points

We take a relativitistic stance akin to Perry (1970), namely a scientific-sociological view and decide that in the long term the field has to develop a locally appropriate philosophy and an associated epistemology depending on the area of investigation (such as beliefs or creativity or proofs etc).

Perry's epistemological positions are as follows:

1. Acknowledges absolute knowledge handed down by authority.
2. Acknowledges differences of opinion that are the result of poorly qualified authority.
3. Acknowledges uncertainty as temporary.
4. Acknowledges relativistic knowledge as the exception to the rule
5. Acknowledges absolute knowledge as the exception to the rule
6. Apprehends the need for personal commitment in a relativistic world.
7. Initial commitment is made.
8. Exploring commitment.
9. Acknowledges commitment as an ongoing, complex, and evolving process.

It seems to us that the Steiner's theses are an attempt to bridge the world of the working mathematician with that of the student and teacher of mathematics. Steiner's stance is quite relativistic (particularly #3 and #4). How would one resolve the writings of Goldin (2003) in light of Perry's positions? Would Goldin reject such a categorization as inappropriate for mathematics? Again, Hersh (2006), continuing in the tradition of Lakatos (1976) presents mathematics as a cultural activity, which can be fallible at times. In particular he explains more about how mathematicians work, and acknowledges a kind of relativistic conception in the world of the working mathematician [where people comfortably go back and forth between Platonism and Formalism]. We think that Platonism is more of a psychological construct for today's mathematician as opposed to a philosophical construct. The relevance of Steiner's theses clearly is the fact that mathematics education has not succeeded in arriving at any philosophy that fits well with teaching and learning processes somewhat akin to futile attempts of squaring the circle! The great debate between social and radical constructivists did not really solve any problems. What we suggest is the creation of local philosophies based on the particular problems such as the success of researchers working in the domain of mathematical beliefs and proof schemes. We must grant also a limited validity period to such local philosophies given the volatile nature of mathematics education. We are reminded of the quote by Euclid, which says that there is really no high road to mathematics. Similarly there can never be any final curriculum or school book which will alleviate the basic problems confronting the field.

References

Aguirre, J. (2006). "There's all kinds of math": Teachers' domain specific beliefs in response to mathematics education reform. Preprint.

Burton. L. (2004). *Mathematicians as Enquirers*. Dordrecht (The Netherlands): Kluwer Academic Publishers.

Buehl, M. M., Alexander, P. A., & Murphy, P. K. (2001). Beliefs about schooled knowledge: Domain specific or domain general? . *Contemporary*

Educational Psychology, 27, 415-449.

Clarebout, G.; Elen, J.; Luyten, L.; Bamps, H.. (2001). Assessing Epistemological Beliefs: Schommer's Questionnaire Revisited. *Educational Research and Evaluation: An International Journal on Theory and Practice*, 7 (1), 53-77.

Csikszentmihalyi, M. (1988). Society, culture, and person: A systems view of creativity. In R. J. Sternberg (Ed.), *The nature of creativity: Contemporary psychological perspectives* (pp. 325-339). Cambridge University Press.

Csikszentmihalyi, M. (2000). Implications of a systems perspective for the study of creativity. In R. J. Sternberg (Ed.), *Handbook of creativity* (pp. 313-338). Cambridge: Cambridge University Press.

Davis, P.J. (1972). Fidelity in mathematical discourse: is one and one really two? *American Mathematical Monthly,* 79, 252-263.

Davis, P.J.; Hersh, R. (1980). *The mathematical experience*. Boston: Birkhäuser.

Davis, P.J.; Hersh, R. (1986). The real mathematician. In: T. Tymoczko (Ed.), *New directions in the philosophy of mathematics* (pp. 177 - 184). Boston: Birkhäuser..

Duval, R. (2002). Proof understanding in mathematics. Proceedings of 2002 International Conference on Mathematics: Understanding Proving and Proving to Understand (pp. 23-44). Taiwan: National Taiwan Normal University, Department of Mathematics.

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In C. Keitel, P. Damerow, A. Bishop & P. Gerdes (Eds.), *Mathematics, Education and Society* (pp. 99 - 101). Paris: UNESCO Science and Technology Education Document Series No 35.

Ernest, P. (1991). *The philosophy of mathematics education*. London: Falmer Press.

Fenstermacher, G.D. (1978). A philosophical consideration of recent research on teacher effectiveness. In L.S. Shulman (Ed.), *Review of research in education* 6 (pp. 157 - 185). Ithasca (IL): Peacock.

Goldin, G.A. (2003). Developing complex understandings: on the relation of mathematics education research to mathematics. *Educational Studies in Mathematics*, 54, 171--202.

Grigutsch, S. (1996). *"Mathematische Weltbilder" bei Schülern: Struktur, Entwicklung, Einflussfaktoren*. Dissertation. Duisburg: Gerhard-Mercator-Universität Duisburg, Fachbereich Mathematik.

Harel, G. (2006a). What is mathematics? A pedagogical answer to a philosophical question. To appear in R.B. Gold & R. Simons (Eds.), *Current Issues in the Philosophy of Mathematics From the Perspective of Mathematicians*. Mathematical American.

Harel, G. (2006b). Mathematics education research, its nature, and its purpose: a discussion of Lester's paper. *Zentralblatt für Didaktik der Mathematik (ZDM)*, 38(1), 58-62.

Hersh, R. (1979). Some proposals for reviving the philosophy of mathematics. *Advances in mathematics*, 31, 31 – 50.

Hersh, R. (1991). Mathematics has a front and a back. New directions in the philosophy of mathematics. *Synthese*, 88 (2), 127 - 133.

Hersh, R. (1997). *What is Mathematics, Really?* New York: Oxford University Press.

Hersh, R. (2006). *18 Unconventional Essays on the nature of mathematics*. Springer: New York.

Hofer, B.K.; Pintrich, P.R. (1997). The development of epistemological theories: Beliefs about knowledge and knowing and their relation to learning. *Review of Educational Research* 67 (1), 88 - 140.

Köller, O.; Baumert, J. & Neubrand, J. (2000). Epistemologische Überzeugungen und Fachverständnis im Mathematik- und Physikunterricht. In Baumert, J. et al. 2000. *TIMSS / III - Dritte Internationale Mathematik - und Naturwissenschaftsstudie - Mathematische und naturwissenschaftliche Bildung am Ende der Schullaufbahn.* Band 2: Mathematische und physikalische Kompetenzen am Ende der gymnasialen Oberstufe (pp. 229 – 270). Opladen: Leske + Budrich.

Lakatos, I. (1976). *Proofs and Refutations*. Cambridge: Cambridge University Press.

Leder, G.C.; Pehkonen, E. & Törner, G. (2002). *Beliefs: A Hidden Variable in Mathematics Education?* Dordrecht: Kluwer Academic Publishers.

Lester, F. (2005). On the theoretical, conceptual, and philosophical foundations for research in mathematics education. *Zentralblatt für Didaktik der Mathematik (ZDM)*, 37(6), 457-467.

Muis,K.R. (2004). Personal epistemology and mathematics: a critical review and synthesis of research. *Review of Educational Research*, 74 (3), 317 - 377.

Noss, R. (2002). Mathematical Epistemologies at work. *For the Learning of Mathematics. An International Journal of Mathematics Education*, 22(2), 2-13.

Noss, R.; Pozzi, S.; Hoyles, C. (1999). Touching epistemologies: meanings of average and variation in nursing practice. *Educational Studies in Mathematics*, 40 (1), 25-51.

Perry, W. G. (1970). *Forms of intellectual and ethical development in the college years*. New York: Holt, Rinehart & Winston.

Pajares, M.F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research*, 62 (3), 307 –

332.

Paulsen, M.B. & Wells, C.T. (1998). Domain differences in the epistemological beliefs of college students. *Research in Higher Education,* 39 (4), 365 - 384.

Rolka, K. (2006). *Eine empirische Studie über Beliefs von Lehrenden an der Schnittstelle Mathematikdidaktik und Kognitionspsychologie.* Dissertation. Duisburg: Universität Duisburg-Essen. Fachbereich Mathematik.

Schoenfeld, A.H. (1998). Toward a theory of teaching-in-context. *Issues in Education*, 4 (1), 1 - 94.

Schommer, M. (1990). Effects of beliefs about the nature of knowledge on comprehension. *Journal of Educational Psychology,* 82 (3), 498 - 504.

Schommer, M., Crouse, A. & Rhodes, N. (1992). Epistemological beliefs and mathematical text comprehension: Believing it is simple does not make it so. *Journal of Educational Psychology,* 84 (4), 435 - 443.

Sierpinska, A. (1992). On understanding the notion of function - Aspects of epistemology and pedagogy. In G. Harel & E. Dubinsky (Eds.), *The concept of functions: Aspect of epistemology and pedagogy.* MAA Notes, Vol. 25 (pp. 25—38). Washington, DC: Mathematical Association of America.

Sierpinska, A.; Lerman, S. (1996). Chapter 22: Epistemologies of mathematics and of mathematics education. In Bishop, A.J. et al . (Eds.) (1996) *International Handbook of Mathematics Education* (Vol. 4). (pp. 827 - 876). Dordrecht: Kluwer.

Simonton, D.K. (1984). *Genius, creativity and leadership*: *Histriometric inquiries*. Cambridge, MA: Harvard University Press.

Simonton, D.K. (1994). *Greatness: Who makes history and why?* New York: Guilford.

Steiner, H.-G. (1966a). Mathematisierung und Axiomatisierung einer politischen Struktur. *Der Mathematikunterricht,* 12 (3), 66-86.

Steiner, H.-G. (1966b). Äquivalente Fassungen des Vollständigkeitsaxioms für die Theorie der reellen Zahlen. *Math.-Phys. Semesterberichte,* 13, 180-201.

Steiner, Hans-Georg. (1974). Bemerkungen zur Wittmann'schen Axiomatisierung des Gruppierungsbegriffs von J. Piaget und Hinweise auf weitere didaktisch relevante Modelle. In: *Die Schulwarte* 8 /9, S. 95 -102; ferner: *Beiträge zum Mathematikunterricht* 1973 (pp. 239 - 246). Hannover: Schroedel.

Steiner, H.G. et al. (1984). Theory of Mathematics Education (TME). ICME 5 - Topics and Miniconference. *JDM-Occasional Paper No. 54*. Bielefeld, 1984

Steiner, H.G. (1985). Theory of Mathematics Education (TME): An Introduction. *For the Learning of Mathematics*, 5, 2, 11-17

Steiner, H.G. (1987). Philosophical and epistemological aspects of mathematics and their interaction with theory and practice in mathematics education. *For the Learning of Mathematics,* 7 (1), 7 - 13.

Steiner, H.G. (1992). Recent and coming activities of the International study group on 'Theory of Mathematics Education (TME)'. *Zentralblatt Didaktik Mathematik (ZDM),* 24 (2), 66-67.

Sternberg. R. J., & Lubart, T.I. (1996). Investing in creativity. *American Psychologist, 51*, 677-688.

Sternberg. R. J., & Lubart, T.I. (2000). The concept of creativity: prospects and paradigms. In R. J. Sternberg (Ed.)., *Handbook of creativity* (pp. 93-115). Cambridge: Cambridge University Press.

Stewart, I. (1995). Bye-bye Bourbaki - paradigm shifts in mathematics. *Mathematical Gazette,* 79 (486), 496 - 498.

Thom, R. (1973). Modern mathematics: does it exist? In A.G. Howson (Ed.), Developments in Mathematical Education. *Proceedings of the Second International Congress on Mathematical Education.* (pp. 194 - 209). Cambridge: Cambridge University Press.

Thompson, A.G. (1982). *Teachers' conceptions of mathematics and mathematics teaching: Three case studies.* Unpublished doctoral thesis dissertation. University of Georgia.

Thompson, A.G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D.A. Grouws (Ed.), *Handbook of Research on Mathematics Learning and Teaching* (pp.127 - 146). New York: Macmillan Publishing.

Törner, G. (2002). Mathematical beliefs. In Leder, G.C.; Pehkonen, E. & Törner, G. (2002). *Beliefs: A Hidden Variable in Mathematics Education?* (p. 73--94). Dordrecht: Kluwer Academic Publishers.

von Glasersfeld, E. (1983). Learning as a Constructive Activity, *Proceedings of PME-NA*, 1, 41-69.

Wittenberg, A.I. (1965). Priorities and responsibilities in the reform of mathematical education: An essay in educational metatheory. *Enseign. Math.* II. Ser., 11, 287 - 308.

Authors
Prof. Dr. Günter Törner,
Institute of Mathematics
University of Duisburg-Essen
Duisburg Campus
Lotharstr. 63 / 65
D-47048Duisburg
Germany

E-mail: guenter.toerner@uni-due.de

Bharath Sriraman, PhD
Associate Professor of Mathematics
Editor, *The Montana Mathematics Enthusiast*
http://www.montanamath.org/TMME
Dept. Of Mathematical Sciences
The University of Montana
Missoula, MT 59812
USA
E-mail: sriramanb@mso.umt.edu