CHAPTER 27

Political Union / Mathematics Educational Disunion: Building Bridges in European Didactic Traditions

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> *Thanks to the morbid estrangement the lunacy of nationality has produced and continues to produce between the people of Europe, likewise thanks also to the shortsighted and hasty handed politicians who are with its aid on top today and have not the slightest notion to what extent the politics of disintegration they pursue must necessarily be only an interlude...- Friedrich Nietzsche*[1]

As Nietzsche (1886) prophesized in the above quote, the "interlude" was nearly 100 or so years before the process of political and economic unification began in Europe. When one reads to the end of the above sentence in the book, Nietzsche declared "*Europe wants to be one*". The aim of this chapter is ambitious, that is we attempt to survey different traditions in mathematics education within Europe particularly in Germany, France and Italy. We chose these countries in particular because in spite of their geographic proximity, namely shared borders between Germany- France- Italy, the mathematics education research traditions reveal a rich heterogeneity, which make it particularly appealing for the purpose of examining similarities and differences and their historic origins.

Starting points, Motivation and Issues addressed in this Chapter

Using Robert Davis's[2] (1971) words, the intention of the chapter is also to be *highly suggestive* and result in the community examining the issue of theoretical traditions within mathematics education. It lies in the nature of any science dynamically and continually developed by human activity that its status as knowledge is always open to discussion and is critically put into question. This also applies to mathematics and to the various editions of the classic Courant–Robbins work, "*What is mathematics?*" which documented changes in mathematics over a period of more than 50 years in the core fields of interest. Thus, the rhetorical question "*What is*

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[1] Authors translation from Neitzsche's *Jenseits von Gut und Böse (1886)* (*Beyond Good and Evil)*
[2] Found in vol1,no1,p.71 of the *Journal of Children's Mathematical Behavior*

mathematics, really? (Hersh, 1997) does not really come as a surprise as it continues to question the status of mathematics. Hypothetically speaking, even more surprising would be a book titled "*What is mathematics in Europe, really"*? Is mathematics, unanimously structured according to the Mathematics Subject Classification[3] (MSC), really different in different countries[4]? One cannot overlook the fact that historically differing focal points of interest were put on the agenda at various times in history by mathematicians. For example the very productive mathematical field "*Foundations of Geometry*" was of central focus in Germany in the 1950s and the 1960s, whereas in France and the USA this was only of marginal interest. Another example can be given *vice versa*: Europe growing together, the activities of the European Mathematical Society (EMS) and bi-national projects also point to attempts at regional differences being integrated. However no concentrated attempts are being made to address issues concerning different research methodologies and research results. When one states the analogous question "*What is mathematics education in Europe, really?*, it is even more difficult to find an answer, as this field is much more heterogeneous – which motivates this chapter. The varying linguistic, cultural and educational traditions within Europe and the shifting borders due to the catastrophic events of the last century pose several difficulties. For instance, the first problem arises when we use the term Germany. The term Germany is imprecise due to particular historical phases. One of the results of World War II was the emergence of the *Federal Republic of Germany* (FRG) and the *German Democratic Republic* (GDR). Belonging to two rival blocs during the Cold War in the 50s and 60s, two different science traditions developed. Vol.35, no.4 of the *Zentralblatt für Didaktik der Mathematik* (ZDM, 2003) documented this development, in particular the articles by Borneleit,

[3] http://www.ams.org/msc/

[4] Thankfully, the times are finally over when mathematics publications in Germany in the 1930s and the 1940s of the last century would call themselves "*Deutsche Mathematik*" (German mathematics) on the basis of racist ideology (see Merthens, 1987).

Bruder, Burscheid, Griesel, Henning, Schupp, Toepell and Walsch. Another ambiguous linguistic term cannot be ignored here: the English term "mathematics education" only partially reflects the German term "*Mathematikdidaktik"* when addressing teaching and learning processes as a science related to mathematics. Maybe the term "didactics of mathematics" is a better match for the German term. In the English language the term "mathematics education" semantically focuses more strongly on real teaching and learning processes in mathematics lessons in schools, whereas the term "didactics of mathematics" stands more for an <u>independent discipline</u> hopefully having positive effects on teaching the subject in schools (when however not always understood as such). More important than the adequate translation of *Mathematikdidaktik* are differing intra-national views towards this term within Germany. Although one would naturally expect differing views about a new domain of inquiry at its inception, in our opinion, as of the present moment, not much has changed, which suggests we classify the field in Germany as being un-homogenous. There was a very broad discussion (Wittmann (1974, 1991; Freudenthal, 1974; Bigalke, 1974; Otte, 1974; Dress, 1974; Tietz ,1974; Zimmermann, 1981) in the 1970s on why the national view at that point in time was still pluralistic in Germany. This discussion lapsed and was then re-initiated by Vollrath in the article[5] from the German Sub-Commission of the International Commission on Mathematical Instruction in the pre-text of the International Congress of Mathematicians (1998) in Berlin, and devoted to aspects of mathematics lessons in schools. But there haven't been any serious discussions on why the view is pluralistic in Europe? These pluralistic aspects are more strongly emphasized in the research scene as opposed to the world of the textbook writers, from the point of view of practice, at least in Germany. To conclude this overview it would be a worthwhile enterprise to illuminate the link between both sectors, i.e., to follow up how effective the influence of didactic research is on

[5] http://www.didaktik.mathematik.uni-wuerzburg.de/history/meg/

concrete teaching and learning practice and also its impact on the various individual German *Bundesländer (states)*. This question seems to be a focal point of increasing importance for future PISA[6] surveys. The mutual dependencies between research and classroom practice are certainly an interesting research question, which brings into focus the question the broad and systematic effectiveness (or ineffectiveness) of educational research (Burkhardt & Schoenfeld, 2003). For the sake of comprehensiveness we should also point out that our theme viewed from different perspectives has always been a challenge to other colleagues (e.g., Bauersfeld, 1995; Burscheid, 1983; Steiner, 1991; Vollrath et al., 2004; ZDM, 1992). Our literature list is by no means exhaustive and also points out a dilemma yet to be resolved; that is most of the references are written in the German and French, which have presented a barrier to the interested international reader and community. Hopefully we remedy this unfortunate situation as well.

"Border" Conflicts

Broadly speaking, a careful perusal of mathematics education theories in Europe reveal "border conflicts" and disjointed schools of thought as to the nature and purpose of mathematics education research. That is not particularly amazing, because Europe has advanced primarily under a combination of economic and trade aspects, up-to-date monetary aspects and now slowly under politico-educational aspects. An explicit objective of PISA is to understand the determinants of educational processes – in the background of a Europe growing closer together. Formal standardization measures, for example for the comparability of university degrees (the co-called Bologna process) has got politically underway, however an expert discussion within the individual disciplines on the scope of scientific approaches in research on the European continent are still in its starting phases. Within these over-arching goals for the European Union are country-specific differences. For instance in Germany, uniformity in research traditions is

[6] Program for International Student Assessment

only coming very slowly. From the point of view of history, we point out that Germany became a unified country only around 1870/71, re-united again in 1919 during the Weimar Republic, then following World War II in 1949 it splintered yet again and re-united in 1990. Such catastrophic political changes are largely instrumental in institutional changes and the consequent diversity of present day research traditions within Germany. France on the other hand has been more or less a unified country since the establishment of the First republic in 1792, and has an even richer history of a centralized institutional set up. Italy became a unified country around 1871. Unlike the highly pluralistic view of didactics of mathematics in Germany, French didactic traditions reveal more uniformity. For instance, theories of mathematics education proposed by the *Brousseau-Chevellard* school of thought that tend to be more ecological (or global) in nature are routinely used as a framework by researchers working in France (Schoenfeld, 2002; Sriraman & English, 2005, 2006; Sriraman & Kaiser, 2006). In this school of thought, the approach to research is holistic and purports to address institutional (and broader social) dynamics that affect teacher-student-content interaction and the effect of the dynamics on the meaning and growth of mathematical knowledge. Part of the focus is also on the study of the growth of "mathematical" knowledge, i.e., on epistemology. As a Romance language, Italian is linguistically closer to French than German, however, the mathematics education traditions of Italy are distinctly different from both France and Germany as will be seen in an ensuing section.

On the other hand, approaches to mathematics education in the Anglo-Saxon[7] world: UK, Germany, and Scandinavia tend to be pragmatic in nature (Kaiser-Messmer, 1986), aimed at

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[7] The historical meaning of the term Anglo-Saxons is the conglomeration of different Germanic tribes from Angeln—a peninsula on the Baltic Sea, and what is now Lower Saxony, in the north-west coast of Germany. These tribes achieved dominance in southern Britain from the mid-5th century. In the next 4 centuries (around 900AD) they <u>unified</u> into a single people, the Anglo-Saxons, forming the basis for the modern day English country, people, language and culture.

informing practice, and heavily influenced by the ebb and flow of psychological-constructivist-social theories (e.g., Steffe, Cobb et al., 1996; Wittmann, 1995, 1998).

In the first edition of the Handbook, Schoenfeld (2002) provided broad brush strokes of trends within mathematics education in the 20th century, and argued that mathematics education is a relatively young discipline which partly explains the diversity in methods and perspectives that abound. Two of the issues examined in his chapter were the dangers of compartmentalization and superficiality. In his words:

> One must guard against the dangers of compartmentalization. It is all too easy to focus narrowly, ignoring or dismissing work or perspectives not obviously related to one's own. This can be costly, given the systemic and deeply connected nature of educational phenomena. Educators need a sense of the "big picture" and of how things fit together.

Bauersfeld (1995), however, saw the development in mathematics didactics somewhat differently and thus more positively. For many years, at least in Germany, working groups with special interests formed, such as empirical methods, learning difficulties, learning processes, computer science etc (see Bauersfeld, 1995 for specific details about these groups). Each of these groups held their own conferences. There were also smaller and more informal groups that worked on special topics and used specific methodologies. The large annual conventions of the *Gessellschaft für Didaktik der Mathematik* (GDM) tends to reflect a picture of the research scene in Germany. Many of the researchers in these groups apply methodologies from the social sciences to concrete research questions, openly criticize each others work, strive for relevant expert knowledge from other disciplines and more or less publish collaboratively. Occasionally these groups do step outside the German research scene and participate in larger meetings such as PME and ICME and place themselves in working or discussion groups which align with their views. From the viewpoint of complexity theory, one sees groups of researchers self-organizing into groups with particular emphases, such as the cognitive or social aspects of learning. As long as there is healthy interaction among the groups, it is conducive for the field of mathematics education in Germany. So a splintered approach is not necessarily a negative thing. However, the only problem of such independent groups is their reduced visibility at the European level, which could be circumvented if these groups formed coalitions and attempted to fit "things" together. This is another motivation for constructing a chapter of this nature in the European context. By presenting different mathematics education perspectives within Europe, we are hoping that the community will begin to examine how things fit together, commonalties and differences, uniformity or non-uniformity in traditions that can be bridged with the aim of better communication amongst researchers from different countries. This naturally leads to more general but fundamental questions, to which one would have to dedicate oneself, namely (1) what is the nature and purpose of mathematics didactics? Do multiple views on what is truly mathematics didactics contribute to "conflicts" and/or the different research traditions?

The work of the ICMI study on mathematics education as a research domain (Sierpinska & Kilpatrick, 1998) juxtaposed numerous worldwide perspectives on mathematics education. This work could be further developed by (a) tracing and analyzing the history of country specific math-ed research perspectives, (b) outlining the similarities and differences between perspectives by examining their evolution within history, and (c) isolating common focal points to build bridges between traditions.

The transcripts of the audio-taped end reports of the fourteen working groups at the conclusion of the Fourth European Congress of Mathematics Education (CERME4) in St. Feliu de Guixols, Spain, 2005, indicated that many of the groups encountered difficulties in understanding the "domains of inquiry" of the others. This led to a deeper examination of the research papers in seven out of the fourteen working groups to examine whether research methods and conceptual frameworks contributed to this misunderstandings (Sriraman & Kaiser 2006). Based on this analysis, Sriraman & Kaiser (2006) argue that these miscommunications and misconceptions were in some cases a function of language[8] but in other instances could be clearly attributed to the unfamiliarity with motivating questions, the research designs and theoretical frameworks employed in mathematics education research by researchers in other countries. For instance while researchers trained in the French school of thought[9] speak with a specific grammar which uses terms such as *institutions, praxeologies, mileus, didactical contracts* and *anthropological theories*, researchers from some other countries in Europe speak in terms of a grammar that contains terms such as *operationalized variables, research design, instruments, reliability, validity, quantitative/qualitative design, instrumentation, data analysis* etc. The latter consists of terminology evolving from the shifts within theories in psychology (from associationism/behaviorism onto cognitive science) grounded in empirical methods, whereas the former is more aligned to terminology used in socio-cultural theories with words whose meanings are not satisfactorily transferable from French to English. This suggests that it is essential for the community to try and define terms understandable by the others. That is, develop the framework of a grammar that will allow for a wider dissemination of research findings from one school of thought to another. In ensuing sections of this chapter, we present brief overview of trends in theory usage within Europe (Sriraman & Kaiser, 2006) relevant for this discussion. The chapter fits within advances in frameworks and methodologies because of

[8] Some of the working groups at CERME4 jokingly remarked that communication was only possible in a new language called BBEFML, which stands for BBE (Basic Broken English) and FML (Francais mais lentement).
[9] This school of thought dominates theoretical frameworks used by researchers in France, Spain and South America as noted in the recently held Congress on the Anthropological Theory of Didactics in Baez, Spain.

the historical and contemporary reflections we engage in on the necessity to create interactions between researchers from different schools of thought. By painstakingly retracing the past, we hope to be able to isolate focal points which suggest ways in which the theoretical differences between the German, French and Italian schools of thought can be bridged (or networked) and made to interact in the present and future. In order to do so, we believe a necessary first step is to clearly present these different positions, how they came to be and how they are being used currently in terms of mathematics education research focus. Then we outline the differences and similarities between the various positions. The authors believe our long-term goal should be to encourage researchers entrenched in their "ideological" perspectives to open their minds to the perspectives of those "across the border".

Lerman (2000) explains that these ideological tendencies are a result of the field adopting theoretical frameworks via a process of recontextualization (Bernstein, 1996). In this process "different theories become adapted and applied, allowing space for the play of ideologies" (Lerman, 2000, p.19). Thus, our endeavor furthers the goal of the international community to better understand the rationale and the purpose of mathematics education perspectives adopted in certain geographic locations of the world and the aim of fostering communication between researchers (English, 2002; Sriraman & English, 2005).

What is *Mathematikdidaktik*? – German Viewpoint

As mentioned previously, the word "German" is ambiguous since educational research approaches in Germany splintered in the aftermath of World War II, with different ideologies and philosophical schools of thought developing in the former East (GDR) and the West (FRG) on research priorities for university educators - until reunification into one Germany in 1990. Currently the 16 *Bundesländer* in Germany reveal a rich heterogeneity in the landscape of mathematics teaching, teacher training and research methods, which manifests itself to insiders microscopically examining the TIMSS and PISA results. However, the reasons for this heterogeneity remain a mystery to outsiders. We macroscopically outline some historical reasons for this diversity. The *Bundestagung für Didaktik der Mathematik* created a platform for didacticians of mathematics at the *Pädagogische Hochschulen* (pedagogical colleges) and universities in the Federal Republic of Germany in 1967 to enable scientific exchange, for starting up common initiatives, which increasingly won the attention of others in the FRG and internationally with its developing scientific profile The protocols with summaries of the lectures offered an overview on the research results from didacticians of mathematics; these protocols became an established feature in the *Beiträge zum Mathematikunterricht* (see Vollrath et al., 2004). The *Zentralblatt für Didaktik der Mathematik* (ZDM, also known as International Reviews on Mathematical Education) was founded at the *Zentrum für Didaktik der Mathematik* at the University of Karlsruhe in 1969 for the documentation of publications in the field and as a journal providing analyses of current research. Its reviews of books and overviews on the state of didactics of mathematics research motivated critical discussions on fundamental questions in this discipline.

Since then these contributions have been integrated since into the *MathDi* database at the FIZ Karlsruhe. A research institute founded in 1973 (see Steiner 1975) at the *Institut für Didaktik der Mathematik* (IDM) at the University of Bielefeld, (in turn initiated by the Volkswagen Foundation) started with fundamental research in mathematics didactics, built up international contacts and expanded its activities to related sciences. Regretfully, for manifold reasons, the IDM did not manage to establish itself indefinitely and has been reintegrated into the university faculty. One can insofar conclude that research into mathematics didactics is at present only being conducted at German universities, partially financed by the *Deutsche Forschungsgemeinschaft* (DFG) and also within the scope of other programs. On the whole, mathematics education research presents a highly fragmentary picture. Individual persons or small groups shoulder most research projects. Larger-scaled research projects or institutions capable of organizing such research are missing (see also Burscheid, 2003; also Schupp, 2003). The consequences are obvious: whereas in the 70s the installation of professorships for mathematics didactics was generously initiated, this trend has been heading in the other direction since the 80s due to cutbacks in the employment of teachers in the education sector. This development is well mirrored by the history of the *Journal für Mathematikdidaktik* (JMD - first published in 1980), representing an important step towards attempts to further professionalize didactics of mathematics in the academic German-speaking world (see Hefendehl et al., 2004). As the title of this paper promised, we will present the views and the traditions of didactics of mathematics research in Germany and its historically developed paradigms and focal points of research interest. Such traditions define themselves as epochs starting quite spontaneously, slowly building up, reaching peaks, then partly declining without necessarily having to disappear. They are also intertwined[10], as demonstrated by Henning (2003), Toepell (2003) and Jahnke (1990). Didactics of mathematics established itself as an independent discipline at the end of the 60s – in Germany and internationally. However, reflections on processes of mathematics teaching and learning have a long-standing tradition in Germany (e.g., Griesel & Steiner, 1992). In fact they go back to the 19th century which produced a number of remarkable contributions to this discipline.

[10] It is only possible to a limited degree to chronologically organize these epochs on a linear basis. Due to the broad scope of research in didactics of mathematics it does not come as a surprise that some developments co-exist parallel to each other in different countries.

The Pedagogical Tradition of Mathematics Teaching - Mathematics as an Educational Value

Initially not only the didactics of mathematics but education research as a whole was dominated by a "value-oriented" paradigm historically focusing on educational aspects of teaching and learning. The early proponents of these theories of teaching and learning are recognizable names even to current researchers. Dominant among these early theorists was Adam Reise "the arithmetician" (1574) who stressed hand computation as a foundational learning process in mathematics. This emphasis is found in the pedagogical classics of the 19th century written by Johann Friedrich Herbart (1776-1841), Hugo Gaudig (1860-1923), Georg Kerschensteiner (1854-1932) (see Jahnke, 1990; Führer, 1997; Huster, 1981). The influence of this approach echoed itself until the 1960s in the so-called didactics of mathematics teaching in elementary schools to serve as a learning prerequisite for mathematics in secondary schools. Didactics of mathematics still generally orients itself on the standards und research questions of the individual discipline, whereby theoretical analyses dominate. If statements on the practice of teaching and learning were given at all, then they were mostly normative statements on the education system as a whole. Since 1920 didactics of mathematics has been organized into two main strands, the Elementary School and *Volksschule* at the newly founded *Paedagogische Hochschule*, and the *Gymnasium* (Grammar school) with didactics of mathematics seminars at mathematical institutes of some universities. This development continued after 1945 and has been increasingly converging since 1970.

Initiators of Mathematics Traditions in Didactics Research (20th Century)

In the early part of the previous century, mathematicians like Felix Klein (1849-1925) and in the 1960's Hans Freudenthal (1905-1990), became interested in the complexities of teaching and learning processes for mathematics in schools. The words "Erlangen Program" and "mathematization" in our vocabulary are the present day legacy of the contributions of Klein and Freudenthal to mathematics education. Klein characterized geometry (and its teaching) by focusing on related groups of symmetries to investigate mathematical objects left invariant under these groups, which was related to a portion of Klein's Habilitation in this area of geometry research. This was instrumental in more than 50% of the secondary schools in Germany replacing Euclidean geometry with a simplified version of transformation geometry of Felix Klein, called motion geometry until about the Royaumont Seminar in 1959 (see 3.1.3). The rise of structural mathematics enabled the progress of research mathematics in geometry. However the pedagogical recommendations based on what was happening in the research scene were more than often unreasonable for teachers to implement. The present day emphasis of using functions (or functional thinking) as the conceptual building block for teaching and learning algebra and geometry is reminiscent of a pre-existing (100 year old) Meraner Program from 1905 (see Krüger, 2001). During this epoch one also finds increasing interest in studying the psychological development of schoolchildren and its relationship to the principles of arithmetic (Behnke, 1961). This trend was instrumental in shaping German mathematics curricula in the 20th century with the objective of mediating calculus to students at the higher school levels (*Oberstufe* - 6th form). The most notable international development in this epoch was the foundation of the ICMI in 1908, presided by Felix Klein. One of the founding goals of ICMI was to publish mathematics education books accessible to both teachers and their students. First published in 1908 but still printed today are the recommendable books "*Elementarmathematik vom höheren Standpunkt'*" (Elementary mathematics from an elevated viewpoint) which clearly illuminate Klein's paradigm of school mathematics: *scientific foundation of school mathematics* and *accessibility through elementarization.* Kirsch (1978) tried to explain to the international community how this process of elementarization of higher mathematics works and wrote that this simplification (or elementarization) in school, used here in its form of providing an access to a subject, was a very tedious matter which often ended in the contrary: complications or even errors. In spite of that, some aspects of simplification specific to mathematics were: (1) to provide an access by concentrating on the hard core of a mathematical topic; (2) adding views from neighboring fields; (3) by recognizing and activating the students' previous knowledge or by changing the means of representation. Thus the term "multiple representations" in vogue today is in fact over 100 years old. We see Klein as one of the first initiators of the movement "elementarize" (or simplify) higher-level mathematics by founding it on a sound scientific (psychological) foundation. Mathematics educators like Lietzmann (1919) claimed that "didactic" principles were needed together with content to offer methodological support to teachers. This approach mutated over the course of the next 50 years well into the 1970s. The dominant metaphor employed by mathematics education researchers during this epoch was that of a gardener maintaining a small math garden analogous to ongoing research in a particular area of mathematics. The focus of research was on analyzing specific content and to use this as a basis to elaborate on instructional design[11] (Reichel, 1995; Steiner, 1982). This approach is no longer in vogue but was instrumental in creating a schism between mathematicians and "mathematics-didaticians", partly analogous to the math wars in the United States. In this context the image of mathematics-didaticians was prototypically viewed a mediator who - parallel to his teaching practice - should cultivate a little mathematical "garden". Regardless of a critical basic stance one cannot ignore that a rich amount of literature on didactics of mathematics (almost all of it in the German language) has been produced over the last hundred years, focusing didactically on content analysis, working out mathematical background theories for particular lesson planning

[11] This tradition is also known as *Stoffdidaktik* in the German speaking world.

schemes, concepts and methodological developments (see Steiner 1982). However the quantity of such literature is on the decline. In these beginnings of the field in Germany, the "mathematics" tends to be more dominant with little or no emphasis on psychological or social frameworks, less so today.

"Genetic" Mathematics Instruction: Ineffective Visionary Bridges (1960 – 1990)

The basic orientation in math lessons at all class levels and in all school forms towards mathematics as an academic discipline in Germany became compulsory in the didactics of mathematics at the end of the 60s and has been dominant since then. This orientation has reasons based on political decisions in the educational sector, which elevated the status of the former *Volksschullehrer* teaching at the Grundschule and Hauptschule (Elementary and Secondary Modern School) and legitimized their academic status of being educated at universities, which in turn led to higher salary levels. The former *Volksschullehrer* viewed this as a step in approximation towards the scientific standards of the Grammar School teacher (see Griesel, 2003). Whereas orientation towards academic standards was wholly accepted in Germany, there was dispute on the concrete forms it should take on. Already in the 60s there were two academic orientations labeled *rigorous* and *genetic* (or *rigorous* and *moderate* - Lenne, 1969, p.84). Whereas the "rigor" school was orientated towards the actual state of academic mathematics, the "genetic" school paid equal attention to the cognitive development of the pupil. This school demanded orientation towards natural learning and the epistemological development of the pupil parallel to an orientation towards the substance of mathematics, whereby this substance is not wholly extracted from university lectures on mathematics but also possibly from "deeper" levels (see Griesel, 2003). The word "genetic" was used to exemplify an approach to mathematics instruction to prevent the danger of mathematics taught completely via procedures (Lenné, 1969). Several theorists stressed that mathematics instruction should be focused on "genetic" i.e., natural construction of mathematical objects. This can be viewed as an earlier form of constructivism. This approach on mathematics education did however not gather momentum. The word "genetic" occurred frequently in the didactics research literature - until the 1990's (Törner & Sriraman, 2005, 2006a).

The New Math (1960 – 1975)

Parallel to the New Math movement occurring in post-Sputnik United States, an analogous reform movement took place in Germany (mostly in the West, but partly adopted by the East). Superficial inspection seems to point to a realization of Klein's dream of teaching and learning mathematics by exposing students to its structure. This reform took on the effect of motivating mathematicians and scientists to work within and together with teacher training, the resulting outcome being a lasting influence on mathematics instruction during this epoch. Unlike the United States, teachers were able to implement to an extent, a structural approach to mathematics in the classroom. This can be attributed to the fact that during this epoch there was no social upheaval in Germany, unlike in the USA where pressure for social reform in the classroom (equality and individualized instruction) interfered with this approach to mathematics education. The fact that the German "New Math" did not survive the tide of time indicates that there was difficulty in implementing it effectively.

The Birth of Mathematics Didactics as a Research Discipline (1975)

Defining a particular year for the start of a research tradition in Germany is of course problematic. Setting this date with the foundation of an academic society, namely the *Gesellschaft für Didaktik der Mathematik* (German Mathematics Didactics Society) can be viewed as arbitrary. Next to the first meeting of the annual symposium of this society in

Saarbrücken in 1975 there are other factors also pointing to this year. The internal preparation for the 3rd International Congress of Mathematics Education (ICME) in 1976 in Karlsruhe (Germany) played a noteworthy role and incited manifold research efforts with intensively and broadly conducted discussions on the position of mathematics didactics in issues of the *International Reviews on Mathematics Education* (ZDM) in 1975-76.

The Second International Mathematics Study/SIMS, played a crucial role in shaping the mathematics education scenario in Europe today, more so than TIMSS and PISA The various meetings and encounters organized by SIMS were very enriching in identifying critical differences among several traditions. Also, the 3rd International Congress of Mathematics Education/ICME 3, mentioned above deserves more attention. ICME 3 was very important in defining lines of research. Indeed, the current areas of priority of ICMI Studies are a result of ICME 3. It also brought, to the open, socio-political issues. ICME 3 differs radically from ICME 1 and ICME 2, in its structure and organization, in priorities and in the selection of the main speakers. The role of the IPC took a different character. The important financing of the preparatory phase, with significant contributions of the Volkswagen Foundation and of UNESCO, made ICME 3 unique. We see both, ICME 3 and SIMS, as very influential in shaping, or reshaping, the scenario of Mathematics Education in Europe (D'Ambrosio, personal communication).

While the new mathematics movement was subject to a host of criticisms, one positive outcome was the foundation of the *Gesellschaft für Didaktik der Mathematik* (German Mathematics Didactics Society) - which stresses that didactics of mathematics is a science concerned with placing mathematical thinking and learning on a sound theoretical (and empirically verifiable) foundation. This was a radical step for mathematics education research in

Germany, one that consciously attempted to move away from the view of a math educator as a part-time mathematician (recall Klein's garden). Needless to say, we could easily write an entire book if we wanted to spell out the ensuing controversy over the definition of this new research discipline in Germany (see Bigalke, 1974; Dress, 1974; Freudenthal, 1974; Griesel, 1974; Laugwitz, 1974; Leuders, 2003; Otte, 1974; Tietz, 1974; Wittmann, 1974, 1992). However the point to be taken from the birth of this society and a new science discipline is that the very debate we have undertaken here, i.e. to globally outline theoretical traditions of mathematics education within Europe, has in fact many localized manifestations such as in Germany.

In this epoch math lessons were viewed comprehensively. Initially didactics of mathematics (or more appropriate: methodology of mathematics) as a discipline was only to be found at the *Pädagogische Hochschule* and similar institutions, thus only reaching student teachers for the *Grundschule* and *Hauptschule.* This led to the consciously or unconsciously perceived stigma that didactical-methodological knowledge was needed to teach children with deficiencies in mathematics. Besides this – so the attitude - the immanent values of mathematics work have positive effects simply due to their existence. Thus 1981 can be viewed as a further milestone when for the first time the Annual Federal Symposium for Mathematics Didactics was held for the first time at a university – the traditional bastion for educating and teaching prospective Grammar school teachers (see Schupp, 2000).

Mathematics Teaching and Learning - A Sociocultural and an Individualistic Process (1980 – Today)

Defining this epoch also appears to have an arbitrary character. We suggest an agreement here to relate this date to the publication date of the highly acknowledged Bauersfeld paper (1980) titled *Hidden Variables*. Around 1980 various studies drew attention to what was going on below the surface in mathematics classrooms. (e.g., Clements, 1996 for further references; Erlwanger, 1975). Bölts (1977) reviewed and gave reasons for the touchy and clouded approach of mathematics didactics in West Germany in these years. The author urges sociological studies to theoretically and empirically define the task demanded from mathematics education in view of qualifications and social behavior of the pupils. Several fields are laid out in which contemporary didactics of mathematics is quite insufficient or lacking self-consciousness. One of the consequences of founding a new academic discipline was the creation of new theories to explain better the phenomenon of mathematics acquisition. The progress in cognitive science together with interdisciplinary work with social scientists led to the creation of "partial" paradigms about how learning occurs. Bauersfeld's (1988, 1995) views of mathematics and learning mathematics as a socio-cultural process within which the individual operates can be viewed as one of the major contributions to theories on mathematics education (Törner & Sriraman, 2005, 2006a).

An Orientation Crisis - The Conundrums Posed by New Technology (1975 – Today)

Weigand's (1995) work poses the rhetorical question as to whether mathematics learning and teaching is undergoing yet another crisis. This message steps into line with the international chorus of voices e.g., NCTM. The advent of new technologies opened up a new realm of hitherto unimagined possibilities for the learner, as well as researchable topics for mathematics educators. The field of mathematics education in Germany oriented itself towards the issues of teaching and learning mathematics with the influx of technology. However the implications of redefining mathematics education, particularly the "*hows*" of mathematics teaching and learning in the face of new technology, poses the conundrum of the necessity to continually reorient the field, as technology continually evolves (see Noss / Hoyles (1995); Hoyles et al. (1999) for an ongoing global discussion).

TIMSS and PISA - The Anti-Climax (*1997 – Today*)

The results of TIMSS and PISA brought these seven aforementioned epochs to a collision - with mathematics educators and teachers feeling lesser appreciated in the wake of the poor national results in Germany. These assessments also brought mathematicians and politicians back to the table for defining major policies affecting the future of mathematics education in Germany. Mathematics education now finds itself in the midst of a new crisis because the results of these assessments paint German educational standards in a dim global light (Neubrand et al., 1999). A statistically detailed inspection of the results indicate that poor scores can also be related to factors other than flaws in the mathematics curriculum, and/or its teaching and learning, namely to socioeconomic and cultural variables in a changing modern German society. Thus mathematics education in Germany would now have to adapt to those forces and trends creating similar problems in other regions of the globe (see Burton, 2003; Steen, 2001).

Didactique des Mathématiques -The French Tradition

Analogous to our claim that there is no English equivalent to the German term *Mathematikdidaktik*, the same linguistic difficulty occurs (see Chevellard, 1999a) with the term "*Didactique des Mathematiques*"(henceforth DdM). The best English explanation for DdM is the study of the process of the dissemination of mathematical knowledge, with more emphasis on the study of teaching. The French term also encompasses the study of the transformations produced on mathematical knowledge by those learning it in an institutional setting. DdM as a field of science lies at the intersection of mathematics, epistemology, history of mathematics, linguistic psychology and philosophy. As is the case in Germany, research in DdM occurs within specific departments in the institutionalized setting of universities, with international networks of collaborators and regular conferences.

Historical Roots (1600-1900)

As a starting point for the French discussion and to illustrate its current place in the field of science in France we briefly outline its historical origins. Numerous writings on the history of didactic traditions (Holmes and Mclean, 1989; Kaiser, 2002; Pepin, 1999) suggest that humanism played a major role as the general philosophy of education in both England and Germany. On the other hand the French educational philosophy mutated from humanism to an "encyclopedic" tradition (or Encylopaedism[12]) as seen in the massive works of Denis Diderot (1713-1784), Charles Monstequieu (1689-1755), Francois Voltaire (1694-1778), Jean-Jacques Rousseau (1712-1778) and many others who were instrumental in paving the way for the French revolution. It is particularly interesting that many of these philosophers took a deep interest in the fundamental questions of learning which are still unresolved today. For example Voltaire was as much an educational theorist as he was a linguist and social activist; Rousseau outlined a comprehensive philosophy of education in the *Emile*. Rousseau theorized that there was one developmental process common to all humans, its earliest manifestation was seen in children's curiosity that motivated them to learn and adapt to the surroundings. A detailed discussion of these works is beyond the scope of this chapter but it helps establish the encyclopaedic roots of the French traditions. Just as politics and philosophy have been deeply intertwined in French society, so has philosophy and education. The French educational system was grounded on the principles of égalité (equality) and laicité (secularism) with mathematics as one of the many subjects important to develop a person's rational faculties (Pepin, 1999). A documented concern for improving mathematics education has been present for over a hundred years as seen in the founding of the journal *L'Enseignment Mathématique* in 1899 by Henri Fehr and Charles-Ange

[12] The definition of the word Encyclopaedism in the online dictionary (wordreference.com) suggests that the word means eruditeness, learnedness, scholarship and falls within the same categorical tree as psychology, cognition (knowledge, noesis), content, education and letters.

Laisant. Furinghetti (2003) in her introduction to the monograph celebrating 100 years of this journal writes:

> The idea of internationalism in mathematics education was crucial to the journal right from its very beginning…the two editors had proposed in 1905 to organize an international survey on reforms needed in mathematics education, asking in particular opinions on the conditions to be satisfied by a complete-theoretical and practical-teaching of mathematics in higher institutions (p. 12)

This call was instrumental in the formation of the International Commission on Mathematics Instruction (ICMI), which we alluded to in the German context and illustrates the overlap of histories. The journal also initiated the study of mathematical creativity. This is a very important event as it brought into relevance the field of psychology and the attention of Jean Piaget and mathematicians within the fold (see Furinghetti, 2003, pp.36-37).

The Influence of Mathematicians (1890- 1960)

The historical influence of prominent French mathematicians on mathematics education is seen particularly in textbooks used, the structure and focus of the content and the unique characteristics of teacher training. For instance, entry into teacher education programs is extremely competitive and includes substantial course work in university level mathematics, much more in comparison to universities in the U.S and Germany. The system in France is highly centralized with only a small proportion of students gaining entry into engineering programs and researcher or teacher training programs typically at the secondary level. The inference here is that these students are exposed to higher level mathematics content for a prolonged time period irrespective of whether they want to be teachers or researchers. From the point of view of mathematics education research, the influence of prominent mathematicians and philosophers on subsequent epistemologies of mathematics education is best evident in the fact that the works of Henri Poincaré (1908) and Léon Brunschwicg (1912) influenced subsequent works of Bachelard (1938), Jean Piaget (1972) and Dieudonné (1992). The emphasis of the French mathematics curriculum at all levels on logical reasoning, encouraging elements of proof, developing mathematical thinking and facilitating discovery contains elements from the writings of Piaget, Poincaré and Dieudonné. For instance, based on numerous longitudinal and cross-cultural analyses of textbooks (Haggarty & Pepin, 2002; Pepin & Haggarty, 2001), teaching and learning practices (Pepin, 1999, 2002) in England, Germany and France, Pepin (1999) writes:

> The reasoning and training-of-the-mind aspect of mathematics is repeatedly emphasized by French teachers and the researcher could see this conviction in practice in the classroom. Pupils had to reason (sometimes with rigorous proof) their results and they were given cognitive activities (problem-solving) to discover mathematics for themselves…[F]rench teachers' perception of different facets of mathematics (inter-connectedness of concepts, process-orientated, entitlement) resulted in a picture of mathematics as a whole. (p.141)

From the point of view of teacher education, historically speaking, the first established institutes for continuing teacher education within established university structures started in 1969 with the founding of Institutes for Research into Mathematics Education (IREM) (Barbin, 1995, 1997). In order to understand the emphasis on rigor in mathematics education programs, one of the major influences were the Bourbaki.

Bourbaki → New Math/Modern Mathematics[13] *←Dieudonné and the Royaumont Seminar (1930-1970)*

[13] We wish to point out that a lot of literature incorrectly calls the French movement for a reform of Mathematics by New Math. The French movement was called Modern Mathematics, while New Math was identified with the

> For example, it is well known that Euclidean geometry is a special case of the theory of Hermitian operators in Hilbert spaces – Dieudonné

It has become fashionable to criticize formal treatments of mathematics in the current post-constructivist phase of mathematics education research as well as to point to the shortcomings and failings of New Math. However the New math period was crucial from the point of view of sowing the seeds of reform in school curricula at all levels in numerous countries aligned with the United States in the cold war period as well as initiated systemic attempts at reforming teacher education. In fact many of the senior scholars in the field today owe part of their formative experiences as future mathematicians and mathematics educators to the New Math period. However the fundamental ideas of New Math were based on the massive work of the Bourbaki. The Bourbaki were a group of mostly French mathematicians, who began meeting in the 1930s and aimed to write a thorough (formalized) and unified account of all mathematics, which could be used by mathematicians in the future (see Bourbaki, 1970). The highly formal nature of mathematics textbooks following the Boubarki tradition is evident in examples such as the "bourbakized" definition of $2^{\sqrt{2}}$ as the supremum of a suitable set of rational powers of 2 (Sriraman & Strzelecki, 2004). It is commonly agreed that New Math was one of outcomes of the Bourbakists, who systematized common threads from diverse mathematical domains into a coherent whole and influenced policy makers in the 1950's and early 1960's to attempt an analogous logical math program for schools (Pittman, 1989). The mathematical community became interested in mathematics education stimulated by both their war time experiences as well the new importance that mathematics, science and technology had achieved in the public

movement which sprung out of the *School Mathematics Study Group/SMSG*. They had conceptually different approaches. Indeed, as we have remarked there has been little, or no mention, of these conceptual differences of the French and the American movements.

eye. This resulted in mathematicians and experts from other fields designing curriculums for schools (e.g., *School Mathematics Study Group or SMSG*). One must understand that the intentions of mathematicians like Max Beberman and Edward Begle was to change the mindless rigidity of traditional mathematics. They did so by emphasizing the *whys* and the *deeper structures* of mathematics rather than the *hows* but it in hindsight with all the new findings on the difficulties of changing teacher beliefs it seems futile to impose a top-down approach to the implementation of the New Math approach with teacher "upgrades" via summer courses on university campuses. The global impact of New Math as a result of the *Royaumont Seminar* is *not one* that is well documented in the literature[14], particularly the huge influence it had on changes in mathematics content taught in schools (Dieudonné, 1961; Moon, 1986). Given no mention of this seminar in extant mathematics education histories constructed (Bishop, 1992; Kilpatrick, 1992) we deem it important to fill this gap in the literature. The prominent French mathematician and Bourbakist, Jean Dieudonné played a significant role in initiating these changes. The Royaumont Seminar was held in 1959 in France (OEEC, 1961), organized chiefly by the Organization for European Economic Co-operation, attended by 18 nations (including Germany, France and Italy) catalyzed New Math into a more global "Western" phenomenon. Dieudonné, who chaired one of the three sections of this seminar, made his famous declaration that "Euclid must go" (see Dieudonné, 1961). The subsequent report released in 1961 led to the systematic disappearance of Euclidean geometry from the curricula of most participating countries. In fact the original SMSG materials included Euclidean geometry. Thus, the influence of prominent Bourbakists on New Math in Europe was instrumental in changing the face of mathematics education completely. We remind readers that the emergence of the discipline

[14] D'Ambrosio (2003) briefly alluded to the Royaumont Seminar and wrote that this seminar "established the major guidelines for the movement known as *Modern Mathematics*" (p. 309).

"Mathematics Education" in the beginning of the 20th century had a clear political motivation. This political motivation became amplified within the Modern Mathematics and New Math movements. Economic and strategic developments were the main supporters of the movements. Both the patronage of the OEEC (Organization for European Economic Co-operation) for the European movement and the public manipulation of the public opinion in the United States, are clear indications of the political motivation of the movements.

The Birth of Didactique (1970-)

In spite of the history presented in the previous section, not every prominent French mathematician was enamored by New Math's promise of modernizing mathematics. In his address to the 2nd International Congress of Mathematics Education, René Thom (1923-2002) was unsparing in his criticism:

> Mathematics having progressed, so we are told, considerably since Cauchy, it is strange that in many countries the syllabuses have not done like wise. In particular, it is argued that the introduction into teaching of the great mathematical 'structures' will in a natural way simplify this teaching, for by doing so, one offers the universal schemata which govern mathematical thought. One will observe that neither of these two objectives is, to be precise 'modern' nor even recent. The anxiety about teaching mathematics in a heuristic or creative way does not date from yesterday (as Professor Polya's contribution to congress thought shows). It is directly descended from the pedagogy of Rousseau and one could say without exaggeration that modern educators could still be inspired by the heuristic pedagogy displayed in the lesson that Socrates gave to the small slave of Menon's.[15] As for the advancement of mathematics which would necessitate a re-organisation of syllabuses, one needs only point to the embarrassment and uncertainty of

[15] Thom is referring to the Fire Dialogues of Plato.

> modern theorists in dating the alleged revolution which they so glibly invoke: Evariste Galois, founder of group theory; Weierstrass, father of rigour in analysis; Cantor, creator of set theory; Hilbert, provider of an axiomatic foundation for geometry; Bourbaki, systematic presenter of contemporary mathematics, so many names are called forth at random, and with no great theoretical accuracy, to justify curricular reform. (Thom, 1973, pp.194-195)

One direct inference to be made from Thom's criticism was that mathematics reform initiated by New Math was not anchored in any mathematics education/didactics research per se, and was simply being done on a whim by invoking individuals in history who had made seminal contributions to mathematics that resulted in what is now called modern mathematics. Parallel to the birth of *Mathematikdidaktik* as a separate academic discipline in Germany in the 1970's, in France the society of researchers engaged in DdM was founded in the 1973. Guy Brousseau and Gérard Vergnaud are widely regarded as the founders of this society. Among the systemic research initiatives engaged in by this group is the adaptation of the specific grammar (definitions, theoretical constructs etc) from Brousseau's (1997) theory of didactical situations (TDS) as a theoretical framework in mathematics education research, as well as the significant extension of Brousseau's theory by Yves Chevellard into the anthropological theory of didactics (ATD). These theoretical developments are further described in the next sections of the chapter. The role of serendipity in the evolution of ideas is seen in the fact that Brousseau adapted Bachelard's (1938) theory of epistemological obstacles into the setting of education, particularly the researching of teaching. Vergnaud, a student of Jean Piaget, on the other hand, was extending Piaget's work on cognitive psychology into a theory of learning, and his work is widely known in the literature (e.g., see Vergnaud, 1996).

The Brousseau School of thought: Theory of Didactical Situations (1970-)

Guy Brousseau's (1981, 1986, 1997, 1999a, 1999b) theory of didactical situations (TDS) is a holistic theory. Simply put TDS studies the complexity inherent in any situation involving the interaction of teacher-student-content (a three-way schema). Broadly speaking TDS attempts to single out relationships that emerge in the interaction between learners-mathematics –the milieu. The milieu typically includes other learners, the concepts learned by students as well as prior conceptual machinery present in the student's repertoire and available for use. The interesting thing about TDS is the fact that its conceiver began his career as an elementary school teacher in Southwestern France and attributed the foundational ideas of his theory to his formative experiences as a practicing teacher in the 1950's. Much later, when reflecting on the origins of his theory Brousseau (1999b) said:

> This three-way schema is habitually associated with a conception of teaching in which the teacher organizes the knowledge to be taught into a sequence of messages from which the student extracts what he needs. It facilitates the determination of the objects to be studied, the role of the actors, and the division of the study of teaching among sundry disciplines. For example, mathematics is responsible for the content, the science of communication for the translation into appropriate messages, pedagogy and cognitive psychology for understanding and organizing the acquisitions and learnings of the student.

At this juncture, we will also point out the fact that Brousseau developed TDS with some practical ends in mind. That is, to ultimately be able to help teachers re-design/engineer mathematical situations and classroom practice so as to facilitate understanding. Again, in Brousseau's (1999b) own words:

> The systematic description of didactical situations is a more direct means of discussing with teachers what they are doing or what they could be doing and of considering a practical means for them to take into account the results of research in other domains. A theory of situations thus appeared as a privileged means not only of understanding what teachers and students are doing, but also of producing problems or exercises adapted to knowledge and to students, and finally a means of communication between researchers and with teachers.

TDS is very much a constructivist approach to the study of teaching situations (Artigue,1994) and "founded on the constructivist thesis from Piaget's genetic epistemology (Balacheff, 1999, p.23). It could be thought of as a special science complete with theoretical considerations and methodological examples for a detailed study of mathematics teaching within an institutional setting. TDS includes a specific grammar with specific meanings for terms such as *didactical situation, adidactical situation, milieu, didactical contract* etc. Taken in its entirety TDS comprises of all the elements of what is today called situated cognition. The only difference is that TDS is particularly aimed at the analysis of teaching and learning occurring within an institutional setting. The most significant contribution of TDS to mathematics education research is that it allows researchers from different theoretical traditions to utilize a uniform grammar to research, analyze and describe teaching situations. One example of this possibility is seen in the recent special volume of *Educational Studies in Mathematics* (2005, vol. 59, no.s 1-3) in which 9 empirical studies conducted in Europe used the "classroom situation" (in its entirety) as the unit of analysis. Such a uniform approach was made possible largely because of the utilization of Brousseau's TDS and Chevellard's ATD (next section) as the common theoretical framework.

However the research sites at which these studies were conducted were predominantly in France, and Spain, which have historically used these frameworks.

The Chevellard School of Thought: Anthropological Theory of Didactics (ATD)

The Anthropological theory of didactics (ATD) is the extension of Brousseau's ideas from within the institutional setting to the wider "Institutional" setting. Artigue (2002) clarifies this subtlety by saying that:

> The anthropological approach shares with "socio-cultural" approaches in the educational field (Sierpinska and Lerman, 1996) the vision that mathematics is seen as the product of a human activity. Mathematical productions and thinking modes are thus seen as dependent on the social and cultural contexts where they develop. As a consequence, mathematical objects are not absolute objects, but are entities which arise from the practices of given institutions. The word "institution" has to be understood in this theory in a very broad sense…[a]ny social or cultural practice takes place within an institution. Didactic institutions are those devoted to the intentional apprenticeship of specific contents of knowledge. As regards the objects of knowledge it takes in charge, any didactic institution develops specific practices, and this results in specific norms and visions as regards the meaning of knowing or understanding such or such object (p. 245).

The motivation for proposing a theory much larger in scope than TDS was to move beyond the cognitive program of mathematics education research, namely classical concerns (Gascon, 2003) such as the cognitive activity of an individual explained independently of the larger institutional mechanisms at work which affect the individuals learning. Chevellard's (1985, 1992, 1999a) writings essentially contend that a paradigm shift is necessary within mathematics education, one that begins within the assumptions of Brousseau's work, but shifts its focus on the very origins of mathematical activity occurring in schools, namely the institutions which produce the knowledge (K) in the first place. The notion of didactical transposition (Chevellard, 1985) is developed to study the changes that K goes through in its passage from scholars/mathematicians→curriculum/policymakers→teachers→students. In other words, Chevellard's ATD is an "epistemological program" which attempts to move away from the reductionism inherent in the cognitive program (Gascon, 2003). Bosch, Chevellard & Gascon (2005) clarify the desired outcomes of such a program of research:

> ATD takes mathematical activity institutionally conceived as its primary object of research. It thus must explicitly specify what kind of general model is being used to describe mathematical knowledge and mathematical activities, including the production and diffusion of mathematical knowledge. The general epistemological model provided by the ATD proposes a description of mathematical knowledge in terms of mathematical praxeologies whose main components are types of tasks (or problems), techniques, technologies, and theories. (pp. 4-5).

It is noteworthy that the use of ATD as a theoretical framework by a large body of researchers in Spain, France and South America resulted in the inception of an International Congress on the Anthropological Theory of Didactics, held in 2005 in Baeza, Spain. The aim of this particular Congress and future congresses is to propose a cross-national research agenda and identify research questions which can be systematically investigated with the use of ATD as a framework.

The Italian Tradition[16]- A Broad Overview

[16] The Italian section is a bit shorter than the German and French sections because excellent English publications are available which chronicle its development (e.g., Arzarello & Bussi, 1998; Barra, Ferrari, Furinghetti et al., 1992; Bagni, Furinghetti, & Spagnolo, 2004; Bazzini, 1988; Furinghetti, 2006). We give a sufficiently broad overview with elements necessary for the purpose of comparing it with the German and French traditions.

The Italian tradition is uniquely different from the German or French traditions and can be best characterized as a "historic" tradition (Fasanelli et al.,2000; Furinghetti, 1997, 2002, 2006) where the field of mathematics didactics evolved out of mathematics and still bears a "relationship" with the latter (Bussi,1998; Furinghetti, 2002, 2006; Grugnetti & Spinoza, 1999). This "relationship", described in more detail in the next section, has been tenuous at times because of the academic power of mathematicians (in the past) to overturn reform initiatives started by teachers. Like France, the educational system in Italy is highly centralized with schools run mainly by the State (Furinghetti & Barnett, 1998). The term historic is used in the sense that there is a pedagogical tradition in the use of historical development of mathematical concepts (concept-didactics) in the teaching of mathematics at the school (e.g., Bagni, 1997) and university level. Fasanelli et al (2000) point to legislation in place since 1923 for the schooling programs of 14-16 year olds which states that upper secondary students after two years are expected to be able to "put in historical perspective some significant moments of the evolution of mathematical thinking" (p. 9). While this tradition was also present in the university setting where mathematicians were in charge of the instruction, more recently there have been legislative attempts aimed at the training of teachers. For instance Grugnetti & Speranza (1994) wrote.

> …a law passed in 1990 established a two-year postgraduate course for "abilitazione", a compulsory qualification for those aspiring to teach. There is <u>almost universal agreement</u> about the presence of specific didactics, history and epistemology in the curriculum… [t]he central issue is: What history (and what epistemology) are suitable for didactics?

The emphasis on the important role of the history of mathematics in mathematics education research is one that has been sporadically addressed. Furinghetti & Radford (2002) traced the evolution of *Haeckel's (1874) law of recapitulation* from the point of view that parallelism is inherent in how mathematical ideas evolve and the cognitive growth of an individual (Piaget & Garcia, 1989). In other words the difficulties or reactions of those who encounter a mathematical problem can invariably be traced to the historical difficulties during the development of the underlying mathematical concepts. The final theoretical product (namely the mathematical theorem or object), the result of the historical interplay between phylogenetic and ontogenetic developments of mathematics, where phylogeny is recapitulated by ontogeny, has an important role in pedagogical considerations (Bagni, 2005; Bagni & Furinghetti, 2004; Furinghetti & Radford, 2002). The humanistic mathematics network (The HPM group) and the subsequent humanistic movement recently re-initiated by the Mathematical Association of America to improve and contextualize the mathematics required of university students can be viewed as outstanding offshoots of the Italian tradition.

The Strong Influence of Mathematicians

Like Germany, the nation of Italy was originally a collection of splintered states with different systems of instruction. The unification of these states occurred in 1861, after which one of the main concerns of the newly formed government was the systematization of the school curriculum. As seen in our re-construction of the mathematics education histories of Germany and France, the influence of mathematicians in positions of academic power played a significant role in the shaping of mathematics education in Italy as we know it today. In her reflections of the evolution of mathematics education in Italy, Furinghetti (2006) remarks:

> In Italy things evolved in a different way. The academic power of mathematicians choked the timid attempts of rebellion to the use of the Elements. A sentence in the mathematics programs issued after Italian unification epitomizes the official attitude towards mathematics in school: "mathematics is a gymnastic of the mind." This view was not unanimously accepted (especially by school teachers) and ironic references to this expression are present in papers appeared in the following years. (pp. 100-101)

This quote indicates the overlap of histories from the point of view that methods of teaching geometry, the schism between teachers and mathematicians, and particular textbooks to be used were at the center of controversy in Italy. Analogous to the contributions of Felix Klein in Germany and the influence of Poincaré and Dieudonné in France, the Italian mathematicians Guiseppe Peano (1858-1932) and Luigi Cremona (1830- 1903), among others played a major role in shaping mathematics education in Italy. Peano's seminal contributions to the foundations of mathematics, in particular his axiomatization of the natural numbers in terms of sets was published in a pamphlet *Arithmetices principia, nova methodo exposita* around 1889 (Gillies, 1982). Around this same time period, Peano undertook the ambitious task of re-organizing the presentation of mathematics in the textbooks of that time period. The *Formulario Mathematico* project (Kennedy, 1973, 1974, 1980) aimed at revising all the known theorems at that point in time into the language of mathematical logic in order to facilitate their delivery to students and teachers. Furinghetti (2006) reflected on this phase of Italian mathematics education history and mused that although Peano's goals in the *Formulario Mathematico* project were utopian, "his enthusiasm and good willingness attracted secondary teachers who collaborated with him and…[c]onstitutes an early example of a mixed group of university professors and school teachers working on didactic problems." (p.102). In spite of the shortcomings of Peano's approach, we view his sympathy for the pedagogical problems of teachers as remarkable and as exemplary as those of Klein and Freudenthal, and unfortunately not recognized as much outside of Italian circles. At the turn of the last century there was also a major re-organization of the treatment of geometry in Italian textbooks based on Hilbert and Peano's axiomatic treatments that resulted in changing the spirit of the Elements, much to the chagrin of the teachers. Another major influence of mathematicians in Italy is in the role that entrance exams play for students wishing to enter various university departments. Furinghetti & Barnett (1998) state that "in the tests for entering university faculties with limited places there are always items concerning mathematics. This fact reflects an underlying belief (perhaps unconsciously) that mathematics is a means for sorting students" (p.341). The numerous historical analyses conducted by Furinghetti reveals that the preparation of future mathematics teachers has a very high degree of overlap with those of professional mathematicians. Although some university courses offer pedagogical topics, these courses are usually optional. Another important element of the Italian tradition is that university mathematics educators are situated within departments of mathematics and mostly teach content courses. Furinghetti and Barnett (1998) remark that "a kind of schizophrenia exists in their professional lives between the research carried out and the topics taught in the university" (p. 343).

General Trends in Focus of Mathematics Education Research in Italy

Arzarello & Bussi (1998) analyzed trends that occurred with mathematics education research in Italy with the purpose of creating a framework via which findings within the Italian context could be communicated in a meaningful way to the international community. It is remarkable that the historic origins of mathematics education research in Italy consistently clustered around concept-based didactics and innovation in the classroom. Concept-based didactics can be best characterized as historic tradition tracing its lineage to prominent Italian mathematicians like Guiseppe Peano, Luigi Cremona and others and can be best described as mathematicians consciously trying to present mathematical ideas dependent on culture, in a form understandable to students without sacrificing form, language and logical rigor. One could say that this research was primarily focused on the mathematics and less on the learning processes. On the other hand "innovation in the classroom" was research engaged in by classroom teachers and focused on improving practice, namely the instruction of specific mathematical content. That is the aim of this research was on the teaching and learning processes within the social context of the classroom. These two trends gradually evolved into research focused on cognition with experimental designs bearing resemblance to empiricism in psychology. This has subsequently branched into research concerned with the developing of models for teaching and learning. The historical trends in the Italian context bear more resemblance to those that occurred in Germany over the similar time period. However it remains distinctive in its own right. In general one can say that Italian researchers have striven to maintain a relationship with the community of mathematicians by exploring the points of interaction between mathematics history and didactics (Bagni,1997; Furinghetti, 1997), as well mathematics itself and didactics (Bussi & Bazzini, 2003) and framing carefully analyzed recommendations in which the two communities (mathematicians and didacticians) can benefit from one another. The Italian tradition has been a focal point for recent initiatives between researchers in France and Germany as seen in the initiatives started by the INTER-IREM Commission[17] in France and conferences organized by Jahnke and colleagues in Germany in 1992. These aspects will be explored in further detail in a subsequent section of the paper.

[17] Épistemologie et Histoire des Mathématiques

Recent Research Trends and Theory Usage in Europe Based on CERME4 Proceedings

Given the major contributions of these three traditions, what type of impact have they had on current research in Europe, particularly those in Germany, France and Italy? What types of theoretical frameworks are used by researchers in these countries? Are they entrenched in perspectives which evolved out of their countries or are they using frameworks from else where? The summaries and analysis of research reports from seven of the fourteen working groups at CERME4 indicate that in domains such as embodied cognition; affect and beliefs, argumentation and proof, there was some uniformity in the use of theoretical frameworks and the types of research questions addressed (Sriraman & Kaiser, 2006). The theoretical frameworks of these groups were international in character. However in the domains of mathematical structures, algebraic thinking, geometric thinking and modeling there was a much wider variation in the types of theoretical frameworks used, the focus of the research and motivating questions, as well as tendencies of researchers from various European countries to use "home grown" frameworks. Sriraman & Kaiser (2006) wrote that for instance, in the working group on proof and argumentation, authors that were most frequently cited in theoretical frameworks (in alphabetical order) were Balacheff, Coe& Ruthven, Fischbein, Healy & Hoyles, Polya, and Toulmin. The papers from France were grounded in theoretical frameworks developed by French researchers. However, the papers from Germany and the UK also cited many French sources in their reports that suggested signs of a healthy cross-fertilization of ideas. Some of the papers in this working group examined the misalignment of textbook proof content with curricular goals. Another paper examined the differences in validation schemes employed by textbooks in Germany and France and proposed a framework via which the two could be compared. One overwhelming theme in all the papers was the social/institutional nature of proof. In terms of available theoretical frameworks that can inform research in this domain, particularly classroom studies investigating students' understanding of proof, one possibility is Chevellard's Anthropological Theory of Didactics (ATD). This framework would be particularly useful to compare studies conducted across classrooms in different countries on the dynamics of proof, particularly the transposition of meaning in its passage through from scholars/mathematicians→curriculum/policymakers→ teachers→students. The papers in the working group on algebraic thinking also showed tendencies to use "home grown" frameworks. The papers from France predominantly used the frameworks of Brousseau, Chevellard, and Bachelard. The papers from Italy were classical historical studies in the Italian tradition of integrating the history of mathematics in didactics. In this particular case, the authors were tracing the phylogenesis of the development of notions within equations and equalities, and how analogies from one domain were not always transferable to the other with historical evidence to support the difficulties encountered in the classroom by students. The paper from Germany utilized frameworks from the international literature on early algebraic thinking. In the working group on modeling similar tendencies were present which are further discussed. Sriraman & Kaiser (2006) came to the following conclusions: More often than not, the theoretical frameworks used by the papers from France tended to be steeped in their particular didactic traditions (Brousseau, Chevellard, Duval), the Italian papers tended towards historical studies of mathematical notions with didactic recommendations based on the relationship between phylogeny and ontogeny. Although the German papers were the most heterogeneous and at first did not reveal tendencies to favor home-grown frameworks, a closer inspection revealed that a subset of these papers did include substantial citations of research conducted by focus groups within Germany, as previously described in the paper when we presented Bauersfeld's viewpoint on mathematics education research. We also find it interesting that this recent analysis of CERME4 papers reveal localized versions of patterns seen in our historical analysis of the development of mathematics education research traditions in these countries. The question confronting us at this stage is what ought to be done to increase the interaction of researchers from these countries? What are the historical and contemporary links and ties between these traditions that can enable future generations of researchers to find common ground?

Mediating Resolutions and a Common Agenda

Does a common intellectual past foster teacher/researcher mobility in Germany-France-Italy?

The historic evolution of didactic traditions in Germany, France and Italy as traced in the previous sections reveals several commonalties that can serve as a fertile source of discussion and for mediating a common agenda. In all three countries mathematics didactics has its origins in the humanistic movement that emerged in post Renaissance Europe. Humanism is typically viewed as a literary and cultural movement that began its spread through Western Europe in the 14th and 15th centuries. However histories of the Renaissance(e.g., Garin,1972; Kristeller, 1979; Trinkaus, 1983) indicate that humanism was also an educational curriculum with its roots in 14th century Italy with Guarino Veronese (1374-1406) in Ferrara and Vittorino da Feltre (1373-1446) at Mantua who began using Quintilian texts as a model for their educational program. The Italian poet and lawyer, Francesco Petrarca, or *Petrarch* (1304-74), is called the “father of humanism”. In this “educational” program, students learned Latin, Greek and studied the works of the great philosophers like Plato, Aristotle, and Cicero. This pretty much laid the foundations of education in Western Europe. The Platonic Academy was founded in Florence in 1450 by Cosimo de'Medici and created a shift in the humanistic focus from social and political concerns to the study of human nature and the cosmos. The widespread transmission of classical texts with the serendipitous advent of the printing press led the humanistic movement into Germany, France and England where it mutated differently. In France, as we previously described, one of the mutations of humanism was the encyclopaedic tradition, which is characterized by the principles of universality, rationality and utility (Kaiser, 2002; Mclean, 1990; Pepin, 1999a, 1999b). In Germany, humanism in its new forms, mutated into a child-centered epistemology (or an emphasis on the individual), which have influenced the educational traditions of Anglo-Saxon countries. In spite of this shared intellectual past, the present day cultural traditions of mathematics teaching in Germany, France and Italy are very varied. There are similarities though. For instance:

> A German teacher going to France would find the transition relatively easy in terms of mathematics. In both countries the notion of mathematics is relatively formal and structured, with theorems and proof being part of it. The conversational style of teaching would also be similar in France, possibly slightly more focused, but German teachers might be encouraged… to attend in-service courses in order to learn about discovery and problem-solving approaches (cognitive activities) within the subject teaching (Pepin, 1998, p. 12).

Informed by studies of mathematics teachers in Germany, France (and England) and analysis of pedagogical practices, Pepin argued that mathematics teachers (and we claim researchers) interested in teaching (and researching) in mathematics classrooms in other European countries need to be acutely aware of differences in culture, expectations and support structures, implicitly assuming language is not a barrier. Although comparative surveys of mathematics education on an international scale (e.g., Atweh & Clarkson, 2001; Atweh, Clarkson, & Nebres, 2003; Clarke, 2001) indicate that "globalization" and "internationalization" has been instrumental in curricular trends becoming somewhat similar, this is not true in our analysis of these three major didactic traditions within Europe and much more complicated because of their very different linguistic, cultural and mathematical heritages. On the one hand, it is easy for engineers, scientists and mathematicians to transpose themselves into institutions across Europe and be familiar with the culture and professional problems the same is unfortunately not true in mathematics education. Therefore, mathematics education research on the issue of factors facilitating mathematics teacher and researcher mobility between different countries and across traditions is a problem that will eventually confront the European Union in the future. The line of inquiry has already been initiated (Mclean, 1990; Kaiser, 1999, 2002; Pepin, 1999a, 1999b, Haggarty & Pepin,2002) but is in need of further development and presents a multitude of possibilities as a long term common agenda for mathematics education researchers in Germany-France-Italy. We next examine commonalties in other focal points of German, French, and Italian traditions, beginning with similarities in the mathematical heritages of these countries.

Common Mathematical Focal Points

The history of mathematics didactics in Germany, France, and Italy reveals the major influence of numerous prominent mathematicians Poincaré, Dieudonné (France); Cremona, Peano (Italy); Klein, Behnke[18] (Germany). In fact the field of mathematics education in general has benefited from the contributions of mathematicians like Klein, Freudenthal (see Bass, 2005), and pedagogically inclined mathematicians like Polya, Dienes (see Bishop, 1992). Bass (2005) reflected on the relationship between mathematicians and mathematics education and particularly those of the mathematicians Klein and Freudenthal.

[18] Seminal work in functional analysis, complex analysis, differential equations, and influenced many next generation mathematicians who became prominent.

> Although their engagements in the domain were quite different, Felix Klein and Hans Freudenthal exemplify the long history of mathematicians' interest in pre-college mathematics education. Each brought his aesthetic dispositions about mathematics to his view of the desirable nature of young learners' encounters with mathematics. Each considered in fine grain the special issues important to the mathematical integrity of the school curriculum … [K]lein examined and re-wrote a vision of the school level curriculum, offering a view of that curriculum that situated it in the larger mathematical landscape[19]. He also sought to model both that mathematicians had important contributions to make and that humility in those contributions was essential… [a]nd each helped to establish the legitimacy and possible nature of mathematicians' involvement in mathematics education. (p. 421)

Such reflections are by no means new. Behnke (1954) and Drenckhahn (1958) took stock of the international scene in mathematics education and pointed out general trends that are relevant for our paper and the current Zeitgeist. As noted previously the ICMI was founded nearly 100 years ago, with the first meeting in 1908, and consisted of a significant number of mathematicians interested in the issues of teaching and learning. The period 1911-1954 was marred by wars on the European continent where nothing significant happened[20]. Subsequently a resolution of the subcommittee CIEM (Commission Internationale de l'Enseignement Mathématique[21]), with representatives from different nations were invited to report on the practices of mathematics instruction for 16-21 year olds in their respective countries at the International Congress of Mathematics (ICM) in Amsterdam in September 1954 (see Behnke, 1958). In spite of the

[19] We have explained Klein's viewpoint in detail in previous sections within the German context

[20] Although we are suggesting that nothing significant in mathematics education happened in terms of systemic initiatives to better mathematics education, we must point that the success of many nations during the World Wars hinged largely on the efforts of mathematicians on projects that determined the end outcome of the wars.

[21] Commission for Mathematics Teaching

upheavals caused by war, at this point in time, there was an excellent group of established mathematicians who were interested in didactics, and who were instrumental in launching our field internationally. The only point of strain was that the mathematical community was more focused on the 16-21 age-group because they were exclusively responsible for the mathematical teacher education of this group, whereas the didacticians expressed interest in the teaching and learning of younger age groups. This basic orientation explains much, at least in Germany, a positive and a negative role ascribed to the mathematicians. That is, mathematicians were interested in focusing on specific content matter and in clarifying the fundamental ideas necessary for students to understand the content often from a more general standpoint, analogous to Klein's specific interest in his "garden of geometry". However the concerns of mathematicians like (1) Freudenthal: creating meaningful learning experiences for younger students' of mathematics; (2) Dienes: extensive work on conveying the structural nature of mathematics through manipulatives and mathematical games (Dienes, personal communication; Sriraman & Lesh, 2007); (3) Polya: pedagogy which facilitates mathematical discovery and most recently; (4) Hyman Bass' involvement with mathematics education research groups in Michigan on the development of a new theory called MKT (mathematical knowledge for teaching), reveal symbiotic ties to the community of mathematicians. Such ties are an important and common linking point for mathematics education research in Germany, France and Italy, which already have rich historical ties to mathematicians. The question then is how can mathematics education researchers utilize this connection and/or benefit from this commonalty? There are several avenues for such an interaction, some of which are already in existence:

"Content specific" didactics.

A large dilemma for the mathematicians is that they always refer to the formal training aims of mathematics instruction justifying, without being able to point out actually what the real meaning of "formal training" is. Does this encompass the wide range of mathematical instruction in school? If so, then this is a common focal point of both mathematicians and mathematics education researchers. Mathematics education research at the elementary levels attempts to inform teaching and learning with continual modifications dependent on the changing relevance of mathematics for today's world. However at the university level, this is not the case, although it makes sense to keep mathematics in close contact with the neighboring sciences of physics and astronomy which makes heavy use of it. As the Italian tradition has characterized it, concept based didactics can be best described as mathematicians consciously trying to present mathematical ideas dependent on culture, in a form understandable to students without sacrificing form, language and logical rigor. On the other hand in Germany, mathematicians do not view this as their primary responsibility, which they think should be delegated to the didacticians. As mentioned earlier this tradition is known as Stoffdidaktik in Germany and was present until the late 70's. Our historical analysis reveals that this could be an area of fruitful exchange between mathematicians and mathematics educators in and between these three countries (examples, Bagni, 2005; Blum & Törner, 1983; Dorier, 2000; Tietze, Klika & Wolpers, 1997).

History of mathematics.

In Germany, the place of history of mathematics is not regarded highly in the mathematics community. There are perhaps two or three departments that focused on this area of research which in the past gave it some measure of respectability. On the other hand in the GDR the history of mathematics was more highly regarded (which can be attributed to socialist consciousness in GDR). In the annual meetings of the Deutsche Mathematik Vereinigung (DMV), history, philosophy and didactics were put together in the "none of the above" section, meaning they could not be treated as one of the traditional areas of inquiry within mathematics. At the international level the International Congress of Mathematics also designates "mathematics education" and "history" into sections typically called "Popularization of mathematics". Within Germany, this has changed in the recent past due to the interventions of the second author, whereby didactics and history were split into independent sections. In Italy apparently the historic tradition is substantially deeper and has attracted many more researchers than in Germany (Furinghetti, personal communication; Bussi & Bazzini, 2003) and there is a substantial effort to change the association of "didactics" with popularization or entertainment among the community of mathematicians. One of the benefits of mathematics education researchers worldwide (Sfard, 1991; Vinner, 1988) becoming interested in the Italian tradition of historical analysis of phylogeny-ontogeny of mathematical notions, have focused the community to work on the didactical transposition of difficult notions, which in turn can be used by mathematicians in their courses in the university setting. Once again this is not a new area of interaction as evidenced in the use of *Haeckel's law of recapitulation* by Blum & Törner (1983), Klein (1968), Toeplitz (1927), and Freudenthal (1973), which are all outstanding examples of the benefits of such research to build bridges between traditions.

The Potential to Inform Philosophical Questions

Thom, Steiner, Lakatos, Bachelard, Piaget, Freudenthal, and many others have engaged in deep philosophical musings about the basis of epistemology, methodology and philosophy of mathematics education. Mathematics education researchers have sporadically investigated fundamental philosophical questions important for mathematics education. Lerman (2000) suggests that this interest in the philosophy of mathematics stems from the impact of Lakatos' (1976) classical book *Proofs and Refutations*, particularly its emphasis of mathematics as a humanistic, quasi-empirical activity subject to fallibility. There exists a body of work frequently cited in the mathematics education literature (Davis and Hersh, 1981; Dawson, 1969; Ernest, 1985, 1991; Kitcher, 1983; Sriraman, 2006a; Tomoczko, 1986), which build on the Lakatosian approach of viewing mathematics teaching and learning as a humanistic and quasi-empirical activity. The considerable body of studies in mathematics education on the nature of proof further builds on some of these previously cited studies. This fruitful area of exchange between mathematicians and mathematics education researchers is in need of being developed (e.g., Hersh, 2006; Sriraman, 2006b).

Lerman (2006) based on his empirical analysis of PME reports from 1985-2005, contended that "the multiplicity and divergence" of theoretical orientations are neither surprising nor necessarily damaging to the field. More importantly Lerman (2006) observed that the plurality of theories can be traced to the intellectual communities and the creative products produced by the communities within which researchers are situated in. Within PME, these intellectual communities in the last 20 years have increasingly been sociology, philosophy, semiotics, and anthropology. Based on his extensive experience and overview of reported research in numerous journals, Lester (2005) posited that researchers typically situated their research within a combination of theoretical and practical frameworks, which he termed a conceptual framework. However Lester pointed out to the problem of misalignment between researchers' philosophical orientation and the research conducted as a source of conflict when collecting and analyzing data. Both Lerman (2006) and Lester (2005) suggest we pay careful attention to the underlying inquiry systems. This presents numerous avenues of investigation for mathematics education researchers within the German-French-Italian traditions interested in bridge building and better communication. We pose several questions that are wide open for further investigation. (1) What are the underlying philosophical orientations of mathematics education researchers in the German, French and Italian traditions? (2) What is the underlying purpose of research in various countries (German, French and Italian)? (3) Do these three traditions have different epistemological orientations on the nature of learning? (4) How do these traditions answer the most fundamental question, namely, what is mathematics? Steiner (1987) posed analogous question to the community on the philosophical and epistemological aspects of mathematics and their interaction with theories of mathematics education which 20 years later, have by and large been left unanswered (Törner & Sriraman, 2007).

Modeling Perspectives

Modeling perspectives have been increasingly adopted in Germany, Italy, and France (e.g., ICMI 14 study, CERME proceedings). Could this be the typology of a common language? The recent discussions of the working group on modeling and applications at CERME 4 as reported by Sriraman & Kaiser (2006) indicated that different approaches permeated the current European work on modeling and it came as no surprise that in general there did not exist a homogeneous understanding of modeling similar to the situation assessed twenty years ago (Kaiser-Messmer, 1986). In her previous analysis, Kaiser-Messmer (1986) distinguished the following perspectives internationally: (a) a pragmatic perspective focussing on utilitarian goals, such as the ability of learners to apply mathematics to solve practical problems, and (b) a scientific-humanistic perspective which was oriented towards mathematics as a science and humanistic ideals of education with focus on the ability of learners to create relations between mathematics and reality. The situation today suggests that although research on applications and modeling has developed further, it has become more fragmented with differentiations arising from the two traditions described above. The perspectives were: (a) Epistemological modeling; (b) Realistic or applied modeling; (c) Educational modeling; (d) Cognitive modeling, and (e) Contextual modeling. At CERME4, after a substantial period of absence, attempts from Romance language speaking countries (France and Spain) were brought into the discussion on applications and modeling. These approaches were based on a theory-related background and relied on "home-grown" theories like Chevellard's ATD and Brousseau's TDS. In these approaches less importance was ascribed to reality in the examples that purported to address modeling and many of the examples dealt with "intra-mathematical modeling" (namely the transformation of the mathematics itself). By using Chevellard's notion of praxeologies, these researchers were essentially claiming that every mathematical activity is identified as a modeling activity. As a consequence these approaches show a strong connection to the scientific-humanistic perspective mainly shaped by the early Freudenthal. In his earlier work, Freudenthal (1973) viewed mathematization as local structuring of mathematical and non-mathematical fields by means of mathematical tools and distinguished between local and global mathematizations. In global mathematization the process of mathematizing is viewed as the development of mathematical theory. Using Treffers (1987) notions of horizontal mathematising (reality to mathematics and vertical mathematizing, (working within mathematics itself), inside mathematics, it seems that the French approach to modeling fits within the vertical characterization and can be viewed as epistemological modeling. On the other hand, German approaches came from the pragmatic perspective and approached realistic modeling, in which authentic examples from industry and science played an important role. Besides these quasi polarising approaches, namely epistemological modeling and realistic modeling, other European papers consisted of integrative approaches within educational modeling and put the structuring of learning processes and fostering the understanding of concepts as its main focus. However, this educational modeling may also be interpreted as continuation of the scientific-humanistic approaches in its version formulated by Freudenthal in his late years and the continuation of his work (Delange, 1987; Treffers,1987) for whom real-world examples and their interrelations with mathematics become a central element for the structuring of teaching and learning mathematics. Sriraman & Kaiser (2006) concluded that while new developments were occurring in the modeling research in Europe, these new approaches were by and large compatible with the pre-existing didactic traditions in various countries. One major confronting the European community, particularly researchers in Germany-France-Italy is the need to clearly identify the underlying assumptions and positions of the various modeling approaches, which would foster discussion and better understanding between the epistemological school of thought and the realistic school of thought.

Curriculum

There are some commonalties in curricular changes in these countries, such as the mutation of geometry from Euclid's version, the controversies over modern treatments (Chevellard, 1992b), the use of various content strands such as algebra and history in the curriculum as documented by numerous ICMI studies, elements of Bourbaki style mathematics. Tests like TIMSS and PISA have further documented deficiencies in the mathematical repertoire of students in these countries, which was instrumental in emphasizing a modeling based curriculum in Germany by Blum and colleagues. Adda (1991) wrote that reforms, often considered as too radical, are more and more frequent in France. Each time these reforms have tried to fight against the student's failure in mathematics. But in spite of these reforms in Germany and France, invariant groups of students in "social-cultural terms" repeatedly do not experience success. The reason behind this is that there exists a “hidden curriculum”, which defines educational tasks, and the school as an institution established norms and expectations regardless of the next innovative curriculum in place. Given this background, is a common European curriculum possible?

Concluding Points and Visions

Globalization is not a new phenomenon. In the story of mankind, there have always been dynamical cultural interchanges. Cultural Dynamics plays an important role in defining new forms of thought. A big, intensive, but localized process, which we may, improperly, also call “globalization”, occurred in Europe, in the expansion of Christianity in the Middle Ages, in the shadow of the Roman Empire. In the late Middle Ages, States began to take shape as components of a new form of Empire. The scenario resulting from this process of European “globalization”, prevails until now. In the sort of jig-saw puzzle which characterize the political dynamics present in this process, the idea of a Nation became strong. States and Nations are different concepts, as well as Political Dynamics and Cultural Dynamics (D’Ambrosio, personal communication). The political dimension of this process prevailed and something vaguely called State/Nation began to take shape as the primary unit of the European scenario. The Empire which emerged in the Late Middle Ages and the Renaissance as the assemblage of such State/Nations, although fragile, mainly due to power struggle, favored the development of the ideological, intellectual and material bases for building up the magnificent structure of Science and Technology, anchored in Mathematics, supporting a capitalistic socio-economic structure. The expanding capitalism, supported by religious ideology and a strong Science and Technology, had, as a consequence, a new form of globalization, now effectively engaging the entire globe. The great navigations and the consequent conquest and colonization, completely disclosed the fragility of a possible European Empire. The internal contradictions of State, as a political arrangement, and of Nation, as a cultural arrangement, emerged, in many forms. Religious and linguistic conflicts, even genocide, within a State/Nation became not rare facts. Indeed, they are not over. As a result of all this process, Education was, probably, the most affected institution. Similar educational proposals, even curricula, are noticed. The influence of national characteristics interfered with objectives derived from the new World scenario. The development of Science and Technology, obviously related to the educational systems, was unequal. Interchanges intensified. The Industrial Revolution made Science and Technology a determinant of progress. Hence, the enormous competition among European States, which intensified during the 19th century and early 20th century, raised Science and Technology, which became increasingly dependent on Mathematics, to top priority. Mathematics became key for development. We see this as responsible for the great advances in Mathematics Education, since the early years of the 20th century. A good Mathematics Education was regarded as favoring industrial and economical, hence political, competitiveness.

The European conflicts of the 20th century, mainly WWI and WWII, revived the ideal of a united Europe (the word Empire became proscribed), a Europe of countries without frontiers. This ideal was designed and implemented. An economical integration was partially achieved and a constitution, although very controversial, will eventually be reached. It is necessary to overcome the obstacles in the name of a common interest of the States. At the same time, the preservation of national character cannot be overlooked, and to favor cultural manifestation crossing the traditional frontiers is a fact. Cultural and educational programs, bringing together different traditions, are encouraged and substantially financed. Testing, applied to the different countries, has, as a consequence, the homogenization of curricula. But a new component, affecting all countries, both their State strand and their Nation strand, interferes with this

European project. The decolonization, in the post-WW II era, brought made frontiers unsustainable. The proposal of opening the frontiers among Europeans has, as a consequence, their permeability. Cheap labor force favored this permeability. Now, we see a new generation, uneasy with a number of problems, such as being strangers in cultural environments that are, in many cases, hostile, and without perspective of employment. Social tensions are mounting. Daily news is perturbing to say the very least. Although we rely on historical analyses, it has everything to do with what is going on today and with the ideal of educational union. But this aim is now complicated by the multinational presence of non-European national traditions within the European States looking for union. The rapid and intense demographic movements of both European national groups and organized non-European nationals (both legal citizens and illegal immigrants) grow as important factors, which are rapidly becoming dominant factor in reflections about European Education, particularly Mathematics Education. While we have opted to steer out of a deeper political discussion underlying mathematics education trends, a discussion of the political scenario in which we, as Mathematics Educators, act is warranted if we wish to build bridges across national traditions. We cannot continue teaching good mathematics in the margin of the main issues affecting the present and the future. Mathematics and its teaching are not done in the margin of the core issues affecting society. The current Zeitgeist, calls for a broad attention to political issues surrounding Mathematics Education.

The contents of the entire chapter so far can be viewed in terms of a metaphor. The metaphor is that of a Banyan tree (*Ficus benghalensis*), which grows in the tropics. This tree is most unusual because its starts out with a main trunk like any other tree but on reaching a certain level of maturity starts to produce hanging roots from the branches. When these hanging roots get long enough and make contact with the soil, they in turn become other trunks and the tree grows enormously large[22]. In the long term, i.e., over the period of centuries, it is nearly impossible to count the order in which these new trunks formed and to forget that they are all connected. This metaphor characterizes European didactic traditions. The birth of the main trunk is analogous to Humanism, with the myriad branches being the theological, philosophical, artistic, political, social, literary, and scientific traditions that flowered in the post-Renaissance phase. The hanging roots of these branches in turn produced trunks which are today distinct domains of knowledge such as mathematics, physics, music, astronomy, history, theology, the liberal arts, other forms of humanism, etc. The subordinate trunk of mathematics in turn produced other branches and hanging roots among which are the numerous didactic traditions and the next generation of subordinate trunks. Stepping away from the tree and viewing its growth from a 600 year perspective as we have done (in accelerated time) reveals the myriad interconnected links of the three didactic traditions. The common links we have unraveled have revealed trends/tendencies, common links in the historic evolution of these three didactic traditions but they are by no means complete and hopefully will prompt others to investigate more connections. The histories of mathematics education in Germany, France, and Italy reveal major tendencies/trends. These histories are best viewed as epochs with an underlying metaphor for the accepted theories of each epoch. Felix Klein's metaphor of a mathematics educator was a mathematics gardener tending to all aspects of a specialized domain within this discipline, including its teaching and learning aspects. This epoch was also seen in Italy in Peano's valiant attempts to rest school textbooks entirely on a logical foundation (his area of research) and to an extent in France in mathematicians and philosophers' interest in education. This then shifted on to a focus on the structure of modern mathematics itself and partly to the teacher as a

[22] The largest banyan tree is said to be on the island of Sri Lanka, with over 350 large trunks and more than 3,000 small ones. (see http://www.itsnp.net/common/btree.html)

"transmitter" of structural mathematics in the 1960's during the "Modern Mathematics" epoch. This was followed by an epoch where the science of mathematics education and learned societies were established in Germany, France, and Italy, separate from the mathematics society, and both the teacher and student came into focus bringing forth attempts of delineating theories for this new science such as Guy Brousseau's *Didactique* and Bauersfeld's socio-cultural theories. New technologies have now shifted the focus of theories to accommodate how learning occurs at the human-machine interface. Finally TIMMS and PISA focused on linking assessment with mathematical literacy, an issue with societal and political ramifications for all three countries.

Can communication be elevated from individuals/researchers within colleague networks in different countries to communication at the institutional levels? For instance the American Mathematical Society (AMS) and Deutsche Mathematik Vereinigung (DMV) communicate regularly and hold joint meetings, UMI (Unione Mathematica Italiana- Italian Mathematical Society) and DMV have also recently initiated a joint meeting. ERME (European Researchers in Mathematics Education) is now mutating into a professional society that has opened up a window of possibility for better communication between European countries. Our varied linguistic background does not bias us towards any specific language. But can journals from these countries open a "window" of communication in English. For instance NOMAD (Nordic Studies in Mathematics Education) in Scandinavia offers one issue a year in English. ZDM has now switched completely to English and started to devote one special issue per year to the extended reports of current research within Germany. Italian researchers have also been consistently trying to communicate with the international community about research developments in their country (Arzarello & Bussi, 1998; Bagni, Furinghetti & Spagnolo, 2004). The research efforts from France typically come into the international view through individual initiatives (e.g., the writings of Artigue, Balacheff, Brousseau[23], Chevellard, Lagrange), occasional chapters within the ICMI study series and Kluwer/Springer mathematics education library and CERME proceedings. It would be beneficial if an annual publication of one of the French journals could be devoted to reports in English on current research within their country.

In all three countries, one important historic commonalty is journals aimed at teachers. In Italy the first journal focused on mathematics teaching was established in 1874, which did not take roots, but a few years later *Mathesis* was another journal founded by mathematics teachers (Furinghetti, 2006). Similarly *the l'Enseignment Mathematique* (founded in 1899 in Geneva), *The Mathematical Gazette* (founded in 1894 in the UK), and the *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht* (founded in 1870 in Germany) all of which contain elements of the German tradition of StoffDidaktik and the Italian tradition of Concept-based Didactics. In the spirit of these seminal journals, another possibility is to begin a new European journal series under the umbrella of ERME with volumes dedicated to particular traditions and eventually collaborations between researchers across traditions in the spirit of ESM's recent attempt to present papers grounded in one framework.

Although there have been sporadic initiatives due to individuals like Hans-Georg Steiner for bi-national meetings between these countries (e.g., Steiner & Bazzini, 1989), for the most part mathematics education researchers in these traditions do not communicate regularly at the institutional levels. Under the umbrella of the European Mathematical Society, mathematics researchers from Germany-France-Italy regularly initiate bi-national meetings (e.g., 2005-Mainz, 2006-Torino, Perugia-2007). Why is this not a reality for mathematics education researchers as well? One can give the "old" argument that mathematics education societies are

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[23] Guy Brousseau published a very interesting article in German on the state of conceptual frameworks and research done in France. Brousseau,G. (1986). Forschungstendenzen der Mathematik in Frankreich. *Journal für Mathematikdidaktik*, 2, 2/3, 95-118.

about a 100 years *younger* than their mathematical counterparts, and are now in their "adolescence". But we think it is time to finally grow up!

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