ON CERTAIN CONTROL PROBLEMS FOR A CLASS OF SINGULARLY PERTURBED PARABOLIC EQUATIONS

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Abstract. We consider singularly perturbed parabolic equations that have alternating boundary layer type solutions: two boundary layer type solutions (upper and lower) may exist, and at certain instants of time the switching between these two solutions is observed. The problem formulations for these parabolic equations contain parameters that can be chosen to regulate the duration of periods corresponding to upper solution and to lower solution stages. We present a biological model for which it is possible to choose the values of control parameters in such a way that only one of the mentioned above boundary layer type solutions persists.

Key words. singular perturbations, parabolic equations, boundary function method, Dirichlet boundary conditions, control

AMS subject classifications. 34E10, 35B05, 35B25

1. Introduction. In Vasil'eva [4] and Vasil'eva and Kalachev [6] the following singularly perturbed (s.p.) parabolic equation was studied:

$$(1.1)\quad \varepsilon^2(u_{xx} - u_t) = (u^2 - 1)(u - \phi(t)) = F(u, x, t), \quad 0 < x < 1, \quad -\infty < t < \infty.$$

Here $\varepsilon > 0$ is a small parameter, and $\phi(t)$ is a 2π-periodic function of t. For equation (1.1) a 2π-periodic solution satisfying various boundary conditions was constructed.

The degenerate equation, obtained from (1.1) by setting $\varepsilon = 0$, has three roots: $u = -1$, $u = \phi(t)$, and $u = 1$. For these roots to not intersect in the interval $0 \le x \le 1$ for any t it is sufficient that the inequality $-1 < \phi(t) < 1$ is satisfied. Evidently, $F_u(u, x, t)$ satisfies the following inequalities:

$$\left.\frac{\partial F}{\partial u}\right|_{u=1,\, u=-1} > 0, \qquad \left.\frac{\partial F}{\partial u}\right|_{u=\phi} < 0.$$

In [6], using numerical and analytical techniques, the following result was obtained: a boundary value problem for (1.1) may have a solution that is a contrast structure of alternating type (CSAT) where the, so-called, upper boundary layer type solution close to $u = 1$ in the interval $x \in (0, 1)$ and the, so-called, lower boundary layer type solution close to $u = -1$ in the interval $x \in (0, 1)$ may at certain instants of time switch from one to the other with the rate of transition process of order $O(1/\varepsilon)$ (see, e.g., Vasil'eva et al. [7]. It is common to use the name *halt* for the stages when the solution is close to $u = 1$ or $u = -1$, and the stages when the transitions take place are referred to as the *run*.

In this paper we show how the control can be introduced to regulate the properties of CSATs. In particular, we show how to affect the duration of the halt stages, and how to force the solution to stay close to only one of the boundary layer type solutions (upper or lower one) for changing t.

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2. Boundary layer type solutions for the case of Dirichlet boundary conditions. Let us specify the following boundary conditions for equation (1.1):

$$u(0,t,\varepsilon) = u_0, \quad u(1,t,\varepsilon) = u_1, \tag{2.1}$$

and, without loss of generality, let us assume that $-1 < u_1 < u_0 < 1$.

For the upper boundary layer functions important near the left and the right boundary points of the domain $x \in [0,1]$ we introduce notations v^+ and w^+, respectively. Similarly, for the lower boundary functions important near the left and the right end points we introduce notations v^- and w^-.

The equations for corresponding boundary functions may be written as follows (see, e.g., Vasil'eva et al. [5]:

$$\begin{gathered} \frac{d^2v^+}{d\eta_0^2} = ((v^+)^2 - 1)(v^+ - \phi(t)), \\ \eta_0 = x/\varepsilon, \qquad v^+(0) = u_0; \end{gathered} \tag{2.2}$$

$$\begin{gathered} \frac{d^2w^+}{d\eta_1^2} = ((w^+)^2 - 1)(w^+ - \phi(t)), \\ \eta_1 = (x-1)/\varepsilon, \qquad w^+(0) = u_1. \end{gathered} \tag{2.3}$$

Similar equations can be written for v^-, w^-.

Asymptotic solution, u^+, close to $u = 1$ has the form (cf, e.g., vasil'eva et al. [5]):

$$u^+ = -1 + v^+ + w^+ + O(\varepsilon). \tag{2.4}$$

Analogously, for asymptotic solution, u^-, close to $u = -1$, we write:

$$u^- = 1 + v^- + w^- + O(\varepsilon). \tag{2.5}$$

The above formulas were justified under certain conditions on u_0 and u_1, e.g., in Nefedov [?]. For the existence of the upper boundary layer type solution (2.4) we require that the straight vertical line $v^+ = u_0$ intersect in the phase plane $(v^+, dv^+/d\eta_0)$ the separatrix that enters the saddle point (1,0) for $\eta_0 \to \infty$, and the vertical line $w^+ = u_1$ intersect in the phase plane $(w^+, dw^+/d\eta_1)$ the separatrix that enters the saddle point (1,0) for $\eta_1 \to -\infty$.

Conditions involving u_0 and u_1 that must be satisfied for the boundary layer type solution to exist are expressed as follows (see Vasil'eva [3]): for the upper solution we must have

$$\phi(t) < \min(\phi_0^+, \phi_1^+), \quad \text{where} \quad \phi_0^+ = \frac{3}{4}\frac{(u_0+1)^2}{(u_0+2)}, \qquad \phi_1^+ = \frac{3}{4}\frac{(u_1+1)^2}{(u_1+2)}; \tag{2.6}$$

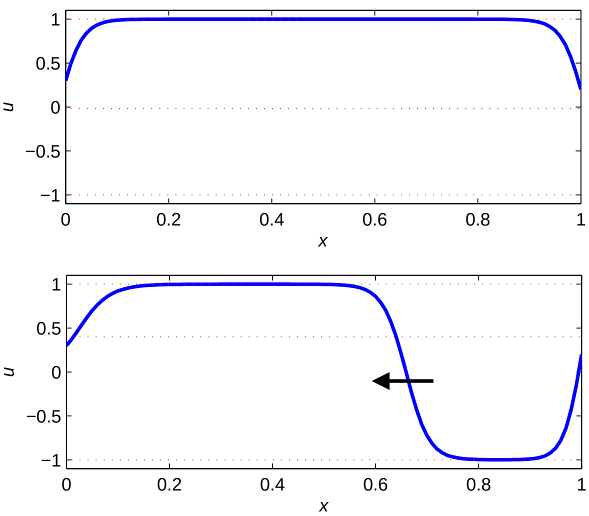
for the lower solution we must have

$$\phi(t) > \min(\phi_0^-, \phi_1^-), \quad \text{where} \quad \phi_0^- = \frac{3}{4}\frac{(u_0-1)^2}{(u_0-2)}, \qquad \phi_1^- = \frac{3}{4}\frac{(u_1-1)^2}{(u_1-2)}. \tag{2.7}$$

Assume that at a given instant of time the upper solution is observed and $\phi(t)$ is increasing. Then, as t grows, at a certain instant of time one of the conditions, e.g., for the existence of the upper boundary layer type solution may cease to be satisfied. This leads to initiation of a switch from upper to lower boundary layer type solution via the *run* phase with duration of order $O(\varepsilon)$. Similarly, as t continues to increase, $\phi(t)$ starts to decrease and, at another certain instant of time, the lower boundary layer type solution will switch to the upper one via another *run* phase, etc. That is, we observe the CSAT.

There exits a number of numerical results related to the problem described above (see [3], [6]), however, the strict mathematical description of all the connected stages, *runs* and *halts* combined sequentially, of the asymptotic solution of this problem is not yet available. We note that the upper and lower solutions that appear after respective *runs* are not exactly the ones given by leading order terms in (2.4), (2.5). They differ from the solutions defined by these formulas by a quantity of order $O(\varepsilon)$. Numerically, the original problem is solved as the initial-boundary value problem while the CSAT appears as a limiting solution as $t \to \infty$. For sufficiently small ε the solution of the initial-boundary value problem rapidly approaches the CSAT. While the exact CSAT representation for all times is not available, the "partial" description introduced in [3], [4], [6] is sufficient for analyzing corresponding statements of control problems.

Example 1. Consider a particular choice of $\phi(t) = a \sin t = 0.5 \sin t$ (similar to that considered in [3], [6], [7]). Let $u_0 = 0.2 > u_1 = 0.1$. Without loss of generality we assume that at $t = 0$ the upper boundary layer type solution is observed, $\phi(0) = 0$, and $\phi(t)$ increases. For the above mentioned choice of u_0, u_1 it turns out that $a = 0.5 > \phi_0^+ \approx 0.4909 > \phi_1^+ \approx 0.4321$ (see (2.6)), and the break up of u^+ occurs via the break up of w^+: the moving threshold starts at the right end of the spatial interval $[0, 1]$ and travels through the whole interval over the time period of order $O(\varepsilon)$ (*run* phase). This leads to the appearance of the lower boundary layer type solution (see Figure 1).



(a)

(b)

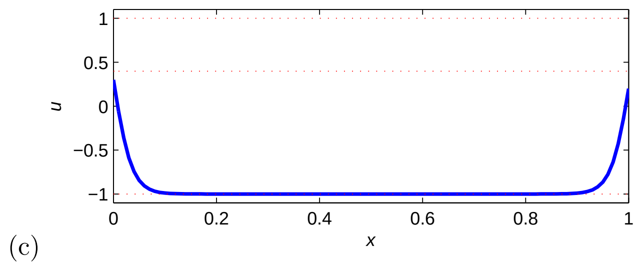


Figure 1. Instantaneous solution profiles for (a) *halt*: the upper solution, (b) *run*, (c) *halt*: the lower solution. The switch from the upper to the lower solution takes place via a moving threshold that is initiated at the right end of the interval $x \in [0, 1]$.

Let us use notation $t = t_1$ for the instant of time when the function w^+ breaks up. This instant of time is defined as the first root of the equation

$$\phi_1^+ \approx 0.4321 = \phi(t_1) = 0.5\sin(t_1)$$

that follows after $t = 0$. The lower solution appears almost instantly (the rate at which the threshold moves is $O(1/\varepsilon)$), so in the leading order approximation we may assume that the lower solution appears at the same instant of time $t = t_1$.

The lower solution exists until the instant of time $t = t_2$ defined as the the first root of the equation

$$\phi_0^- \approx -0.2667 = \phi(t_2) = 0.5\sin(t_2)$$

that follows after $t = t_1$. The lower solution breaks up via break up of v^- for decreasing $\phi(t)$. As t increases the process periodically repeats itself. Evidently, $\phi(t_1) > 0$ and $\phi(t_2) < 0$ (see Figure 2). It is clear from Figure 2 that the duration of *halt* stage corresponding to upper solution approximately equals $(t_2 - t_1)$, and the duration of *halt* stage corresponding to lower solution equals $2\pi - (t_2 - t_1)$. This result indicates that the duration of various *halt* stages is defined by t_1 and t_2 which, in turn, depend on u_0 and u_1. Thus, by changing u_0 and u_1 one can regulate the relative durations of various stages of the periodic solution.

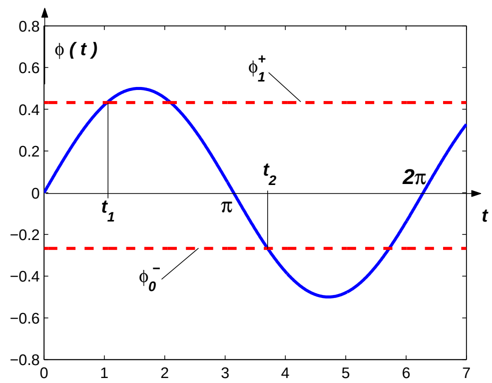


Figure 2. The transitions between different types of solutions are initiated at instants of time t_1 and t_2.

In Example 1 the lower and upper solutions were alternating between each other. Next, we address the following question: How can one specify the boundary conditions so that the transition takes place at most once and then the solution stays either in the lower or in the upper state.

Example 2. Let $u_0 = u_1 = 0.8$ and, once again, $\phi(t) = a \sin t = 0.5 \sin t$, so that $\phi(t)$ values belong to the interval $[-0.5, 0.5]$. According to (2.6), (2.7),

$$\phi_0^+ = \phi_1^+ = 0.8679, \qquad \phi_0^- = \phi_1^- = 0.0250.$$

Since $0.8679 > a = 0.5$, if at $t = 0$ the upper solution is observed, it will not break up for any $t > 0$. If, on the other hand, at $t = 0$ the lower solution takes place, it will break down and switch to the upper solution for $t = t^*$ defined as the first root of

$$0.5 \sin(t^*) = \phi_0^- = \phi_1^- = 0.0250$$

that follows after $t = 0$. For our choice of $\phi(t)$ and of boundary conditions, $t^* \approx \pi + 0.0063$. The upper solution will persist for all $t > t^*$.

Example 3. Analogous situation holds when, e.g., $u_0 = u_1 = 1.5$, i.e., the boundary conditions are both greater than the maximal of the roots $u = 1$ of the equation $F(u, x, t) = 0$. In this case the lower and the upper solutions may also exist, and the transition from lower to upper boundary layer type solution may be observed observed (see Figure 3).

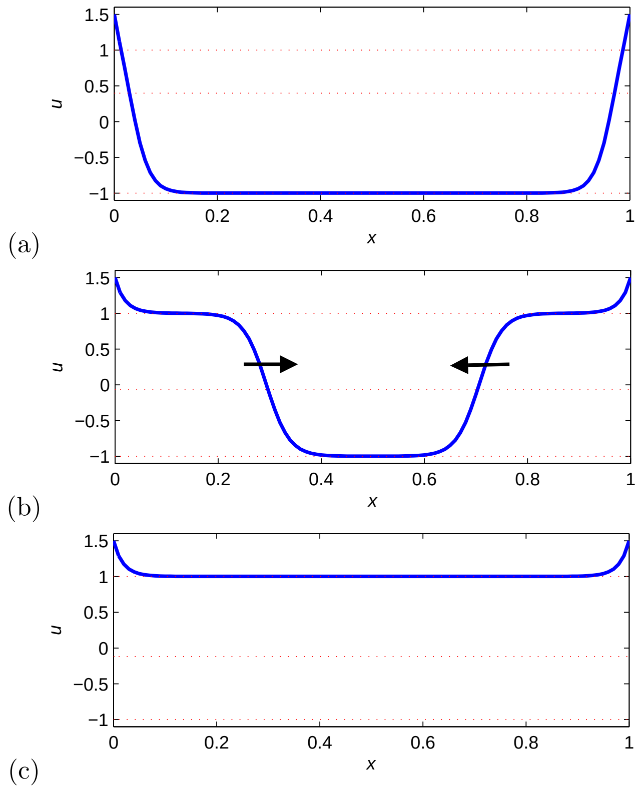
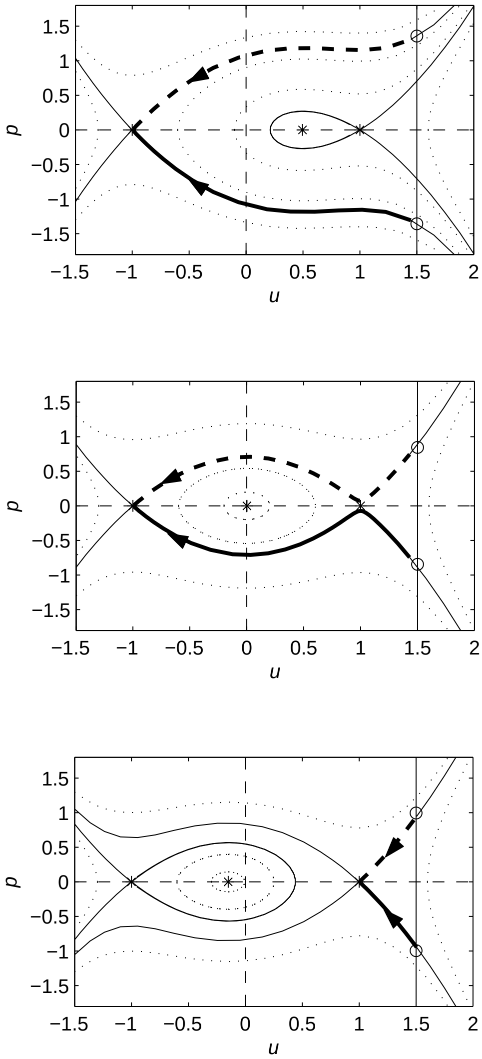


Figure 3. Instantaneous solution profiles for different instants of time for the choice of boundary conditions $u(0, t, \varepsilon) = u(1, t, \varepsilon) = 1.5$.

The formulas (2.6), (2.7) cannot be used any more to estimate the time of switching. However, the analysis on the phase plane (see Figure 4) can help to understand the process (same as in [6]).



(a) (b) (c)

Figure 4. Boundary functions on a phase plane: trajectories shown by the solid lines correspond to the left boundary layer functions, and trajectories shown by the dashed lines correspond to the right boundary layer functions. Three different cases correspond to the following solution stages: (a) lower boundary layer type solution (the original solution profile is shown in Figure 3(a); figure notations: $u = v^-$, $p = d(v^-)/d\eta_0$ for the solid line, $u = w^-$, $p = d(w^-)/d\eta_1$ for the dashed line); (b) situation observed before the start of transition from the lower to upper boundary layer type solution via moving thresholds (figure notations: $u = v^-$, $p = d(v^-)/d\eta_0$ for the solid line, $u = w^-$, $p = d(w^-)/d\eta_0$ for the dashed line); (c) upper boundary layer type solution (the original solution profile is shown in Figure 3(c); figure

notations: $u = v^+$, $p = d(v^+)/d\eta_0$ for the solid line, $u = w^+$, $p = d(w^+)/d\eta_1$ for the dashed line).

In particular, assume that the solution is currently in the lower (*halt*) stage, and that $\phi(t)$ is decreasing. This situation is shown in Figure 3(a), and corresponding boundary functions are presented in Figure 4(a), where solid line describes the left boundary layer function (important near $x = 0$) with arrow indicating its behavior for $\eta_0 \to \infty$; dashed line describes the right boundary layer function (important near $x = 1$) with arrow indicating its behavior as $\eta_1 \to -\infty$.

As the value of $\phi(t)$ decreases, at the instant of time t^* when the, so-called, cell is formed (i.e., the situation is observed where the 2 saddle points in the phase plane are connected by 2 symmetric heteroclinic orbits; see Figure 4(b)) the lower boundary layer type solutions ceases to exist, and the solution switches to the upper boundary layer one (see Vasil'eva et al. [5] for more discussion). The switch occurs via traveling thresholds (*run* phase) that appear simultaneously at both endpoints of the interval $x \in [0, 1]$ (see Figure 3(b)). In Figure 4(b) the boundary functions are shown for the instant of time just before the start of the *run* phase.

After the *run* phase is over, the system will stay in the upper solution (*halt*) stage for all $t > t^*$. Corresponding instantaneous solution profile and the boundary functions are shown in Figure 3(c) and Figure 4(c), respectively.

3. Application: mathematical biology problem. Now, let us discuss a more complex case related to a particular biological application. We consider the following non-dimensionalized problem for parabolic equation modeling spruce budworm population in a forest (Murray [1], page 417, equation (14.85)):

$$\varepsilon^2(Du_{xx} - \alpha u_t) = -ru(1 - u/q) + up(u) = F(u), \tag{3.1}$$

$$u|_{x=0} = u_0 > 0, \quad u|_{x=1} = u_1 > 0,$$

where D, r and q are parameters (D is the diffusion coefficient, r is the growth rate coefficient, q is the forest specific carrying capacity of the insect population), $up(u)$ is the term that represents predation with

$$p = \frac{u}{1 + u^2},$$

$0 < \varepsilon \ll 1$ is a small parameter. The only difference between (3.1) and equation (14.85) in Murray [1] is the presence of ε^2 term multiplying the derivatives. Such s.p. model corresponds to biologically meaningful situation where the adjustment of spruce budworm population to carrying capacity and its predation (e.g., by birds) occur on a faster time scale compared to changes of the carrying capacity (due to forest growth or decay, etc.) and insect spread via diffusion. Without loss of generality we consider a more general case where the boundary conditions are positive. The case of zero boundary conditions can be studied in a similar way.

First, let us concentrate on the stationary case of (3.1),

$$\varepsilon^2 Du_{xx} = -ru(1 - u/q) + up(u). \tag{3.2}$$

The degenerate equation ($\varepsilon = 0$) written as

$$0 = -ru(1 - u/q) + up(u) \tag{3.3}$$

has one of the roots $u = 0$ and the other possible roots satisfying the equation ($s = -r/q$):

$$0 = -r - su + p(u), \quad \text{or} \quad r + su = p(u). \tag{3.4}$$

Geometrically, possible non-trivial roots of this equation correspond to points of intersection of the straight line $f_1 = r + su$ with the curve $f_2 = p(u)$. In Figure 5 in the (s, r)-parameter plane we show the regions of parameter values that produce zero, one, two, and three roots (these regions are marked with numbers 0, 1, 2, and 3, respectively). The boundaries of the region where three solutions are possible were found by simultaneously solving (3.4) and equation $s = p'(u)$ with constraint $u > 0$ (solid lines); for the three roots to exist it is also necessary to have $s < 0$, which produces the vertical boundary of the region (dashed line). Once again, we note that the three *biologically meaningful* ($u > 0$) roots exist when $r > 0$ and $s < 0$.

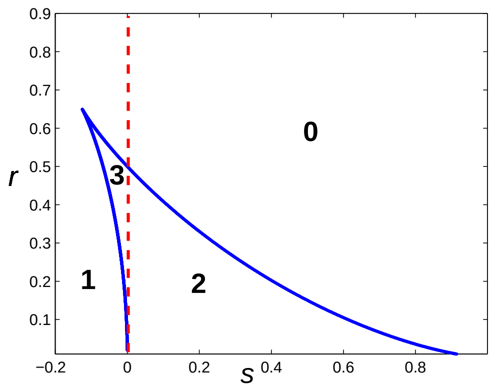


Figure 5. In the (s, r) plane the regions of parameter values that correspond to zero, one, two, and three roots of (3.4) are marked with numbers 0, 1, 2, and 3, respectively.

In Figure 6 we show the situation where three intersections of f_1 and f_2 are observed for a particular choice of parameter values from the region 3 presented in Figure 5 ($r = 0.45$, $s = -0.05$). We use U_i ($i = 1, 2, 3$) to denote the three non-trivial biologically meaningful roots satisfying $0 < U_1 < U_2 < U_3 < \infty$.

Let us check the stability properties of the roots (according to Tikhonov). We write

$$\frac{\partial F}{\partial u} = -r - 2su + p(u) + up'(u),$$

and thus,

$$\left.\frac{\partial F}{\partial u}\right|_{u=0} = -r < 0,$$

which means that $u = 0$ is always unstable.

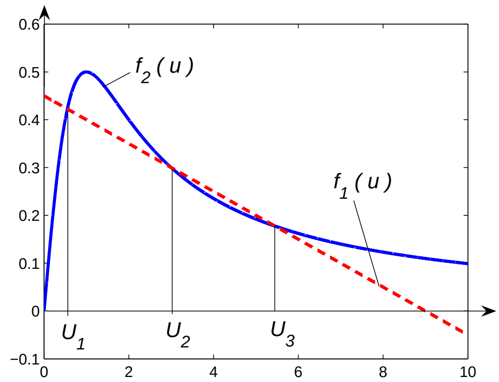


Figure 6. Three roots of (3.4) obtained for a particular choice of parameter values ($r = 0.45$, $s = -0.05$).

According to (3.4) on every non-trivial root we have ($i = 1, 2, 3$):

$$\left.\frac{\partial F}{\partial u}\right|_{u=U_i} = U_i(-s + p'(U_i)). \tag{3.5}$$

Since $U_i > 0$ the sign of (3.5), and thus, the stability of corresponding root are determined by the sign of expression $(-s + p'(U_i))$. In Figure 7 we show the graph of the function $f = -r - su + p(u)$. The points of intersection of this function with $f = 0$ axis correspond to the non-zero roots $0 < U_1 < U_2 < U_3$ of the equation $F(u) = 0$ (see also (3.4)). We note that we have $f'(u) = -s + p'(u)$, so by checking the slopes $f'(U_i)$ of the tangent lines to the graph of $f(u)$ evaluated at various roots we can make conclusions about their stability.

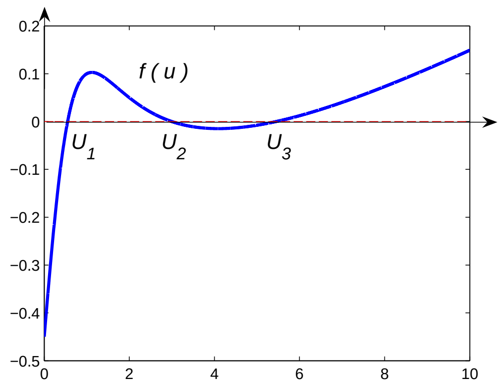


Figure 7. For a particular choice of parameter values ($r = 0.45$, $s = -0.05$) the slopes of the tangent lines to the graph of $f(u)$ are positive at $u = U_1$, $u = U_3$, and negative at $u = U_2$.

It is clearly seen from Figure 7 that $f'(U_1) > 0$, $f'(U_2) < 0$, $f'(U_3) > 0$, and thus

$$\left.\frac{\partial F}{\partial u}\right|_{u=U_1,U_3} > 0, \tag{3.6}$$

and

$$\left.\frac{\partial F}{\partial u}\right|_{u=U_2} < 0. \tag{3.7}$$

So, the roots $u = U_1$ and $u = U_3$ are stable, while the root $u = U_2$ is unstable.

To initiate the transition from one type of the boundary layer type solution to another the *cell* must exist on the phase plane (that is used to describe the boundary functions; see Figure 4(b) from Example 3) for certain value s^* of parameter s. The condition for existence of the *cell* is given by the relation (cf., e.g., [5]):

$$I = \int_{U_1}^{U_3} F(u,s)du = 0. \tag{3.8}$$

The above integral (3.8) can be easily computed:

$$I = \int_{U_1}^{U_3} \left(-ru - su^2 + \frac{u^2}{1+u^2}\right) du = \left.\left(-r\frac{u^2}{2} - s\frac{u^3}{3} + u - \arctan u\right)\right|_{U_1}^{U_3}, \tag{3.9}$$

and the value of $s = s^*$, for which $I = 0$, can be estimated numerically. We obtain that

$$s^* \approx -0.0485.$$

For this parameter value, we have $U_1 \approx 0.5524$, $U_2 \approx 2.8636$, and $U_3 \approx 5.8716$.

Next, we consider the original time dependent equation (3.1) with specified boundary conditions, and the constant parameter values for which the stationary upper boundary layer type solution (corresponding to high insect population) is locally stable. Assume that the initial conditions belong to the domain of attraction of this solution. Then, as time increases, the solution of the original problem will tend to this upper boundary layer stationary solution.

Now, if we allow some of the parameters, e.g., s, to change, the condition needed for initiation of a transition from the upper to the lower boundary layer type solution (existence of the *cell* on the phase plane mentioned above) may be satisfied for some particular parameter values, e.g., for $s = s^* \approx -0.0485$ in the special example case mentioned above.

Finally, to make sure that the solution stays in the low insect populations state after the switch from high insect population state, we must check that the boundary conditions satisfy the inequalities: $u_0 < U_1$ and $u_1 < U_1$.

The biological implications for insect control immediately follow from the analysis presented above. The assumptions on boundary conditions mean that a forest domain under consideration must be surrounded by the regions that are "almost" bug free. This may be accomplished, e.g., if the forest is surrounded by patches of land that are not covered with trees, or if the surrounding patches are sprayed with insecticide. Parameter r is species related and it is usually cannot be easily changed without, e.g., extensive use of chemical pesticides (that one tries to avoid). So, if the system is in the high infestation state (corresponding to upper boundary layer type solution), one has to decrease parameter s beyond s^* value to initiate the transition from upper to lower boundary layer type solution that corresponds to low spruce budworm population state. The reduction of s is equivalent to decrease in the carrying capacity q for the species in a given forest. This, in turn, can be accomplished by partial forest clearing, i.e., removal of a fraction of trees. The important observation that follows from model analysis is that the clearing must be performed only once. Then even though the number of trees in the forest will increase, and as a result the carrying capacity for the spruce budworm will increase as well, the equilibrium insect population will continue to stay at the low state as long as the conditions on the boundary values at the boundaries of the forest domain are satisfied (i.e., the forest is surrounded by the domains with small insect population, e.g., due to spraying or due to small number of trees).

We end this section with an illustrative numerical example; see Figure 8.

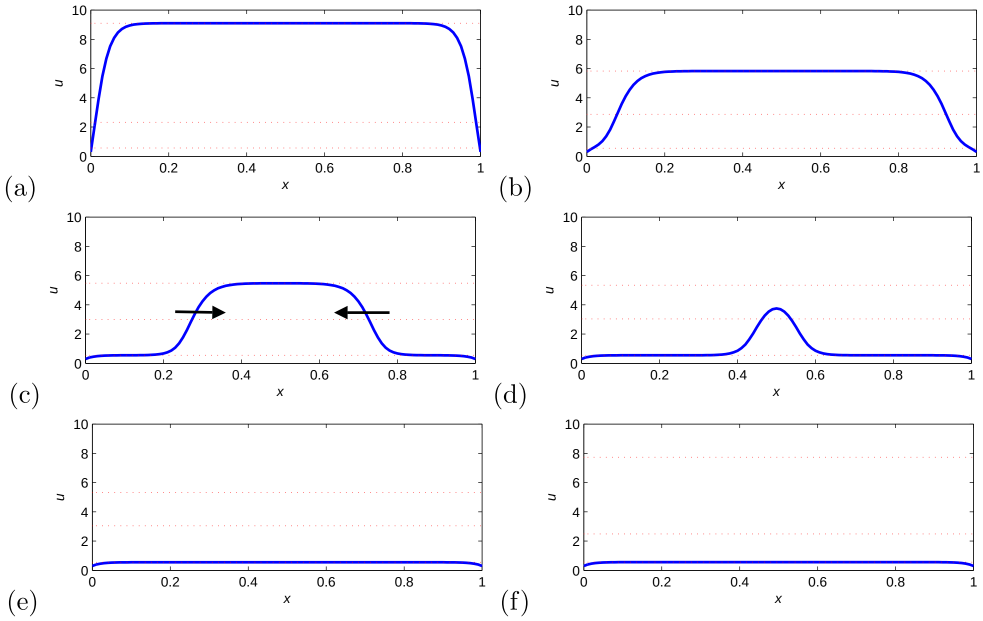


Figure 8. Instantaneous solution profiles for problem (3.1) with slowly changing parameter $s = s(t)$. The switch from the upper to the lower boundary layer type solution takes place via moving thresholds that are initiated simultaneously at both ends of the interval $x \in [0, 1]$.

Here $r = 0.45$, and parameter s is a slowly varying function (of t): $s(t) = -0.045 + 0.0075 \cdot \sin(3t)$. The boundary conditions are $u(x, 0) = u(x, 1) = 0.3 > 0$. At the initial instant of time the upper boundary layer type solution phase is observed (this phase corresponds to high insect infestation); see Figure 8(a). Dotted lines indicate corresponding solutions $0 < U_1 < U_2 < U_3$ of (3.4). As $s(t)$ changes for increasing t, the value $s = s^* \approx -0.0485$ is reached at certain $t = t^*$. This initiates the transition from upper to lower boundary layer type phase of the solution (see Figure 8(b),(c),(d)). As t continues to increase, the solution stays in the lower boundary layer type phase (corresponding to low insect infestation); see Figure 8(e),(f). For the instant of time corresponding to situation presented in Figure 8(f), the parameter values are such that both the upper and the lower boundary layer type solutions may exist. However, the transition from lower to upper solution does not take place, which exactly corresponds to the results discussed in this section.

4. Conclusion. In this paper we introduced the methodology for analyzing certain types of control problems for s.p. parabolic partial differential equations. We explained how to choose the boundary conditions to specify the duration of time time periods within which the periodic solution of the original problem stays in the upper or in the lower boundary layer type phase. We also studied the situation where the switch from one boundary layer type solution to the other occurs only once, and after that the resulting type of solution persists. We applied our approach to the analysis of spatially dependent spruce budworm model. Our study of the model produced a number of practical recommendations that ensure that the long term spruce budworm population stays in the lower population state and switch to the higher population state does not occur.

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