WATTS THEOREM FOR SCHEMES: PRELIMINARY VERSION

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ABSTRACT. We describe obstructions to a direct-limit preserving right-exact functor between categories of quasi-coherent sheaves on schemes being isomorphic to tensoring with a bimodule. When the domain scheme is affine, all obstructions vanish and we recover Watts Theorem. We use our description of these obstructions to prove that if a direct-limit preserving right-exact functor F from a smooth curve is exact on vector bundles, then it is isomorphic to tensoring with a bimodule. This result is used in [2] to prove that the noncommutative Hirzebruch surfaces constructed in [4] are noncommutative $\mathbb{P}^1$-bundles in the sense of [7]. We conclude by giving necessary and sufficient conditions under which a direct-limit and coherence preserving right-exact functor from $\mathbb{P}^1$ to $\mathbb{P}^0$ is an extension of tensoring with a bimodule by a sum of cohomologies.

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1. Introduction

In this paper we describe a version of Watts Theorem for schemes. In order to motivate our results we first recall Watts Theorem:

Theorem 1.1. *Let T be a commutative ring, let R and S be T-algebras and suppose $F : \mathsf{Mod}R \to \mathsf{Mod}S$ is a T-linear right-exact functor commuting with direct-limits. Then there exists a T-central $R-S$-bimodule M such that $F \cong - \otimes_R M$.*

The bimodule M in the previous theorem is easy to describe. $M = F(R)$ as a right-module, and its left-module structure is defined as follows: for each $r \in R$, we let $\phi_r \in \operatorname{Hom}_R(R, R)$ denote right multiplication by r. For $m \in M$, we define $r \cdot m := F(\phi_r)m$.

It is natural to ask if such a result holds when the categories $\mathsf{Mod}R$ and $\mathsf{Mod}S$ are replaced by categories of quasi-coherent sheaves on schemes X and Y, $\mathsf{Qcoh}X$ and $\mathsf{Qcoh}Y$. In order to precisely pose the question in this context, we need to introduce some notation. To this end, if Z is a scheme, X and Y are Z-schemes, $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_{X \times_Z Y}$-module, and the projections $X \times_Z Y \to X, Y$ are denoted pr_1 and pr_2, we define

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E} := \operatorname{pr}_{2*}(\operatorname{pr}_1^* \mathcal{M} \otimes_{\mathcal{O}_{X \times_Z Y}} \mathcal{E}).$$

We make the further assumption that

$$- \otimes_{\mathcal{O}_X} \mathcal{E} : \mathsf{Qcoh}X \to \mathsf{Qcoh}Y,$$

which is automatic when, for example, $X \to Z$ is quasi-compact, separated and Z is affine.

Let $Z = \operatorname{Spec} T$. Given the statement of Theorem 1.1, it is natural to ask the following

Question 1.2. Let $F : \mathsf{Qcoh}X \to \mathsf{Qcoh}Y$ denote a T-linear, right-exact functor commuting with direct-limits. Is F isomorphic to tensoring with a bimodule, i.e. does there exist an object $\mathcal{E}$ of $\mathsf{Qcoh}\mathcal{O}_{X \times_Z Y}$ such that $F \cong - \otimes_{\mathcal{O}_X} \mathcal{E}$?

When X is affine, we show in Proposition 2.2 that the answer to this question is yes. In fact, the proof is a straightforward extension of the proof of Theorem 1.1, and is probably well known.

In general, the answer to this question is *no*, as the following example illustrates.

Example 1.3. [7, Example 3.1.3] Suppose $X = \mathbb{P}^1_k$ and $Y = Z = \operatorname{Spec} k$. If $F = \mathrm{H}^1(X, -)$, then F is Z-linear, right-exact, and commutes with direct-limits. However, as we will prove in Proposition 5.4, F is not isomorphic to tensoring with a bimodule.

The purpose of this paper is to describe the structure of T-linear right-exact functors $F : \mathsf{Qcoh}X \to \mathsf{Qcoh}Y$ which commute with direct-limits. In order to state our main result, we introduce notation and conventions which will be employed throughout this paper.

We let T denote a commutative ring, and $Z = \operatorname{Spec}T$. Throughout the paper, we assume all schemes and products of schemes are over Z. We assume X is a quasi-compact and quasi-separated scheme and Y is a quasi-separated scheme with structure morphisms $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$, respectively.

We note that the category

$$\mathsf{Funct}(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$$

of functors from $\mathsf{Qcoh}X$ to $\mathsf{Qcoh}Y$ is abelian, and we denote the full subcategory of T-linear functors (see Section 2 for a precise definition of *T-linear functor*) in $\mathsf{Funct}(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ by

$$\mathsf{Funct}_T(\mathsf{Qcoh}X, \mathsf{Qcoh}Y).$$

The category $\mathsf{Funct}_T(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ is abelian as well. We denote the full subcategory of $\mathsf{Funct}_T(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ consisting of right-exact functors commuting with direct-limits by

$$\mathsf{Bimod}_T(X - Y).$$

We denote the full subcategory of $\mathsf{Bimod}_T(X-Y)$ consisting of functors which take coherent objects to coherent objects by

$$\mathsf{bimod}_T(X - Y).$$

The following definition, studied in Section 4, plays a central role in our theory.

Definition. An object F of $\mathsf{Funct}_T(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ is *totally global* if for every affine open immersion $u : U \to X$, $Fu_{i*} = 0$.

In order to generalize Watts Theorem, we first study an assignment, which we call the *Watts functor*,

$$W : \mathsf{Bimod}_T(X - Y) \to \mathsf{Qcoh}X \times Y,$$

whose construction was sketched in [7, Example 3.1.3]. We prove that it is functorial (Subsection 5.2), left-exact (Proposition 5.1), compatible with affine localization (Proposition 5.2), and has the property that if $F = - \otimes_{\mathcal{O}_X} \mathcal{F}$ then $W(F) \cong \mathcal{F}$ (Proposition 5.4). It follows from Propostion 2.2 that if X is affine, then $F \cong - \otimes_{\mathcal{O}_X} W(F)$.

We then work towards our main result, proven in Section 6:

Theorem 1.4. *Suppose each object of* $\mathsf{Qcoh}X$ *admits an epimorphism from a flat object. If* $F \in \mathsf{Bimod}_T(X - Y)$*, then there exists a natural transformation*

$$\Gamma_F : F \longrightarrow - \otimes_{\mathcal{O}_X} W(F)$$

such that

1. *if F is exact, then* $\ker \Gamma_F = 0$ *(Proposition 6.1),*
2. *if $- \otimes_{\mathcal{O}_X} W(F)$ is right-exact then* $\operatorname{cok} \Gamma_F = 0$ *(Proposition 6.2),*
3. *if $F = - \otimes_{\mathcal{O}_X} \mathcal{F}$ for some object $\mathcal{F}$ in* $\mathsf{Qcoh}X \times Y$ *then Γ_F is an isomorphism (Proposition 6.4), and*
4. *Γ is compatible with affine localization (Proposition 6.7).*

It follows readily from (4) that $\ker \Gamma_F$ *and* $\operatorname{cok} \Gamma_F$ *are totally global (Corollary 6.8). Hence, if X is affine, Γ_F is an isomorphism.*

It follows that if $F \in \mathsf{Bimod}_T(X - Y)$, then $- \otimes_{\mathcal{O}_X} W(F)$ serves as the "best" approximation of F in the following sense (Corollary 6.5):

Corollary. *Let $\mathcal{F}'$ be an object of* $\mathsf{Qcoh}X \times Y$ *and suppose $F' := - \otimes_{\mathcal{O}_X} \mathcal{F}'$ is an object of* $\mathsf{Bimod}_T(X - Y)$*. If $\Phi : F \to F'$ is a morphism in* $\mathsf{Bimod}_T(X - Y)$*, then Φ factors through Γ_F. If $- \otimes_{\mathcal{O}_X} W(F)$ is right-exact, then Φ factors uniquely.*

In Section 7, we apply Theorem 1.4 to the case that the domain scheme X is a smooth curve. We prove the following

Theorem 1.5. *If X is a smooth curve and F is an object of $\mathsf{Bimod}_T(X-Y)$, then* $\operatorname{cok}\Gamma_F = 0$ *(Proposition 7.5), so that there exists an exact sequence*

$$0 \longrightarrow \ker\Gamma_F \longrightarrow F \xrightarrow{\Gamma_F} - \otimes_{\mathcal{O}_X} W(F) \longrightarrow 0.$$

If F is exact on vector-bundles, then Γ_F is an isomorphism (Corollary 7.8).

In the final section, Section 8, we use Theorem 1.5 to attack the simplest classification problem not addressed by Proposition 2.2: that of describing the category $\mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$ when k is algebraically closed. To this end, we prove the following (Corollary 8.13):

Theorem 1.6. *Suppose k is algebraically closed and let $F \in \mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$. If* $\ker\Gamma_F$ *is right-exact, then for some $m, n_i \geq 0$, there is an exact sequence*

$$0 \longrightarrow \oplus_{i=-m}^{\infty} H^1(\mathbb{P}^1, (-)(i))^{\oplus n_i} \longrightarrow F \xrightarrow{\Gamma_F} - \otimes_{\mathcal{O}_X} W(F) \longrightarrow 0$$

in $\mathsf{Funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$.

We also give necessary and sufficient conditions on $F \in \mathsf{bimod}_T(\mathbb{P}^1 - \mathbb{P}^0)$ for $\ker\Gamma_F$ to be right-exact (Corollary 8.15).

We conclude the introduction by mentioning one further application of Theorem 1.5. In [4], Ingalls and Patrick show that the blow-up of a noncommutative weighted projective space is a noncommutative Hirzebruch surface in an appropriate sense. More precisely, they show that the blow-up is a projectivization of an exact functor $F : \mathsf{Qcoh}\mathbb{P}^1 \to \mathsf{Qcoh}\mathbb{P}^1$ which commutes with direct-limits. It follows from Theorem 1.5 that $F \cong - \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{F}$ where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$-module. In [2], we provide a proof that $\mathcal{F}$ is a locally free rank two $\mathcal{O}_{\mathbb{P}^1}$-bimodule and that the noncommutative Hirzeburch surface Ingalls and Patrick construct is a noncommutative ruled surface in the sense of [7].

Our discovery of Theorem 1.4 was motivated by a question posed by M. Van den Bergh in [7]. To describe his question, we recall a definition.

Definition 1.7. Let X and Y be noetherian. A quasi-coherent $\mathcal{O}_{X\times Y}$-module $\mathcal{E}$ is a sheaf-bimodule if for every coherent subsheaf $\mathcal{F} \subset \mathcal{E}$, the restriction of the projections $\mathrm{pr}_{1,2} : X \times Y \to X, Y$ to $\operatorname{Supp}\mathcal{F}$ are finite.

In [7], M. Van den Bergh asks how to characterize the image of the category of sheaf-bimodules in $\mathsf{Bimod}_T(X - Y)$. He notices that if $\mathcal{F}$ is a sheaf-bimodule, then $- \otimes_{\mathcal{O}_X} \mathcal{F}$ is exact on vector-bundles. Conversely, one can ask when $F \in \mathsf{Bimod}_T(X - Y)$ is isomorphic to $- \otimes_{\mathcal{O}_X} \mathcal{F}$ for some sheaf-bimodule $\mathcal{F}$. Theorem 1.5 shows that if X is a smooth curve, exactness of F on vector-bundles is sufficient to conclude that $F \cong - \otimes_{\mathcal{O}_X} \mathcal{E}$ for some object $\mathcal{E} \in \mathsf{Qcoh}(X \times Y)$. In [2], sufficient conditions are given for when $\mathcal{E}$ is a sheaf-bimodule.

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2. Watts Theorem

The purpose of this section is to prove that the naive generalization of Watts Theorem holds when the domain scheme is affine (see the statement of Proposition 2.2 for a precise formulation of this statement). The result is used implicitly in [7, Example 3.1.3]. We include the proof here since we could not find a proof in the literature and since we use the result often in the sequel. We first recall the following definition, which is invoked in the statement of Proposition 2.2.

Definition 2.1. An element $F \in \mathsf{Funct}(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ is *T-linear* if the diagram

$$\begin{array}{ccc} T \times \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) & \rightarrow & T \times \operatorname{Hom}_{\mathcal{O}_Y}(F\mathcal{M},F\mathcal{N}) \\ \downarrow & & \downarrow \\ \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) & \rightarrow & \operatorname{Hom}_{\mathcal{O}_Y}(F\mathcal{M},F\mathcal{N}) \end{array}$$

whose horizontal arrows are induced by F, and whose vertical arrows are induced by global sections of the structure maps $\mathcal{O}_Z \to f_*\mathcal{O}_X$ and $\mathcal{O}_Z \to g_*\mathcal{O}_Y$ respectively, commutes.

Proposition 2.2. *If X is affine, then the inclusion functor*

$$\mathsf{Qcoh}(X \times Y) \to \mathsf{Bimod}_T(X - Y)$$

induced by the assignment $\mathcal{F} \mapsto - \otimes_{\mathcal{O}_X} \mathcal{F}$ is an equivalence of categories.

Proof. Let R be a commutative T-algebra such that $X = \operatorname{Spec} R$. We need to prove that if $F \in \mathsf{Bimod}_T(X-Y)$, then there exists a quasi-coherent $\mathcal{O}_{X\times Y}$-module $\mathcal{F}$ such that $F \cong - \otimes_{\mathcal{O}_X} \mathcal{F}$, and that the inclusion functor is fully faithful. The idea behind the proof is to copy the proof of Watts Theorem and then glue. We proceed in several steps.

Step 1: We show that $F(\mathcal{O}_X)$ canonically corresponds to a quasi-coherent $\mathcal{O}_{X\times Y}$-module, $\mathcal{F}$. If $U \subset Y$ is affine open with $U = \operatorname{Spec} A$, we define a sheaf on $X \times U$, $\mathcal{N}_U$, by defining an $R \otimes_T A$-module N. As a right A-module we let $N = F(\mathcal{O}_X)(U)$. If we let

$$\mu_r \in \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$$

correspond to multiplication by $r \in R \cong \Gamma(X, \mathcal{O}_X)$, then we give N an R-module structure by defining $r \cdot n := F(\mu_r)(U)n$ for $n \in N$. It remains to show that N is T-central, but this follows directly from the fact that F is T-linear. We conclude that N is an $R \otimes_T A$-module, hence corresponds to a quasi-coherent $\mathcal{O}_{X\times U}$-module, $\mathcal{N}_U$. We next note that the sheaves $\mathcal{N}_U$ glue to give a quasi-coherent $\mathcal{O}_{X\times Y}$-module, which we call $\mathcal{F}$. For, if $q : X \times U \to U$ is projection, $q_*\mathcal{N}_U = F(\mathcal{O}_X)|_U$, so that we may define gluing isomorphisms $\alpha_{U,V} : \mathcal{N}_U|_{U\cap V} \to \mathcal{N}_V|_{V \cap U}$ as the identity map.

In the next four steps, we construct a natural transformation $\Theta : - \otimes_{\mathcal{O}_X} \mathcal{F} \to F$ by defining Θ locally then gluing. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module and let $U \subset Y$ be an affine open subset.

Step 2: We define $\Theta_{\mathcal{M}}(U) : \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{F}(U) \to F\mathcal{M}(U)$. We note that

$$\begin{aligned} \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{F}(U) &= \operatorname{pr}_{2*}(\operatorname{pr}_1^* \mathcal{M} \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F})(U) \\ &= (\operatorname{pr}_1^* \mathcal{M} \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F})(X \times U) \\ &\cong \mathcal{M}(X) \otimes_R F(\mathcal{O}_X)(U). \end{aligned}$$

Hence, in order to define $\Theta_{\mathcal{M}}(U)$, it suffices to construct an $\mathcal{O}_Y(U)$-module map $w : \mathcal{M}(X) \otimes_R F(\mathcal{O}_X)(U) \to F\mathcal{M}(U)$. This is constructed as in the proof of Watts Theorem, as follows. Suppose $m \in \mathcal{M}(X)$, $n \in F\mathcal{O}_X(U)$, $r \in R$, and

$$\mu_m \in \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M})$$

corresponds to multiplication by m in $\operatorname{Hom}_R(R, \mathcal{M}(X))$. Thus,

$$F\mu_m \in \operatorname{Hom}_{\mathcal{O}_Y}(F\mathcal{O}_X, F\mathcal{M})$$

and

$$\begin{aligned} F(\mu_{mr})(U)(n) &= F(\mu_m\mu_r)(U)(n) \\ &= F(\mu_m)(U)F(\mu_r)(U)(n) \\ &= F(\mu_m)(U)(rn). \end{aligned}$$

Hence, the function $w(m \otimes n) := F(\mu_m)(U)(n)$ extends to a well defined homomorphism of $\mathcal{O}_Y(U)$-modules $w : \mathcal{M}(X) \otimes_R F(\mathcal{O}_X)(U) \to F\mathcal{M}(U)$, which in turn corresponds to a map of $\mathcal{O}_Y(U)$-modules $\Theta_{\mathcal{M}}(U) : \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{F}(U) \to F\mathcal{M}(U)$.
Step 3: We show that the maps $\Theta_{\mathcal{M}}(U)$ glue to give a map of $\mathcal{O}_Y$-modules

$$\Theta_{\mathcal{M}} : \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{F} \to F\mathcal{M}.$$

Suppose $U, V \subset Y$ are affine open. Since Y is separated, $U \cap V$ is affine as well. Then the commutivity of the diagram

$$\begin{array}{ccc} (\mathcal{M} \otimes_{\mathcal{O}_X} F\mathcal{O}_X)(U) & \overset{\Theta_{\mathcal{M}}(U)}{\to} & F\mathcal{M}(U) \\ \downarrow & & \downarrow \\ (\mathcal{M} \otimes_{\mathcal{O}_X} F\mathcal{O}_X)(U \cap V) & \overset{\Theta_{\mathcal{M}}(U\cap V)}{\to} & F\mathcal{M}(U \cap V) \end{array}$$

whose verticals are restriction, follows from the fact that, for all $m \in \mathcal{M}(X)$, $F(\mu_m)$ is an $\mathcal{O}_Y$-module morphism, hence is compatible with restriction. Hence, if $\{U_i\}$ is an open affine cover of Y consisting of a basis for the topology on Y, the morphisms $\Theta_{\mathcal{M}}(U_i)$ glue to give a morphism

$$\Theta_{\mathcal{M}} : \mathcal{M} \otimes_{\mathcal{O}_X} F\mathcal{O}_X \to F\mathcal{M}.$$

Step 4: We show that $\Theta_{\mathcal{O}_X}$ is an isomorphism. Since we need only check that $\Theta_{\mathcal{O}_X}$ is an isomorphism at each point, it suffices to check the result affine locally, which is routine.
Step 5: We show that the maps $\Theta_{\mathcal{M}}$ define a natural transformation $\Theta : -\otimes_{\mathcal{O}_X} \mathcal{F} \to F$. Let $\phi : \mathcal{M} \to \mathcal{N}$ denote a morphism of $\mathcal{O}_X$-modules. We must show that

$$F(\phi)\Theta_{\mathcal{M}} = \Theta_{\mathcal{N}}(\phi \otimes_{\mathcal{O}_X} \mathcal{F}).$$

It suffices to check the result point-wise, hence affine locally on $U \subset Y$. To this end, if $m \in \mathcal{M}(X)$ then we must show

$$(F\phi)(U)(F\mu_m)(U) = F(\mu_{\phi(X)(m)})(U).$$

But

$$\begin{aligned} (F\phi)(U)(F\mu_m)(U) &= F(\phi\mu_m)(U) \\ &= F(\mu_{\phi(X)(m)})(U) \end{aligned}$$

where the last line follows from the equality $\phi\mu_m = \mu_{\phi(X)(m)}$.
Step 6: We show that Θ is a natural equivalence. The proof of this step follows the classical proof of Watt's Theorem (see [6, Lemma 11.2], for example).
Step 7: We prove the inclusion functor is full. Let $\Phi : - \otimes_{\mathcal{O}_X} \mathcal{E} \to - \otimes_{\mathcal{O}_X} \mathcal{F}$ be a natural transformation. Our strategy is as follows: we first prove that there exists a morphism $\phi : \mathcal{E} \to \mathcal{F}$ such that $\Phi_{\mathcal{O}_X} = \mathrm{id}_{\mathcal{O}_X} \otimes \phi$. We will then use the naturality of Φ to show that $\Phi_{\oplus \mathcal{O}_X} = \mathrm{id}_{\oplus \mathcal{O}_X} \otimes \phi$ for a direct sum of $\mathcal{O}_X$'s and $\Phi_{\mathcal{M}} = \mathrm{id}_{\mathcal{M}} \otimes \phi$ for an arbitrary $\mathcal{M} \in \mathsf{Qcoh} X$.

In order to construct $\phi : \mathcal{E} \to \mathcal{F}$ with the desired property, we first construct a map

$$\delta : \mathrm{pr}_{2*}\,\mathcal{E} \to \mathrm{pr}_{2*}\,\mathcal{F}$$

as follows: On $U \subset Y$ open affine, we define $\delta(U)$ to be the map making the diagram

$$\begin{array}{ccc} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{E}(U) & \xrightarrow{\Phi_{\mathcal{O}_X}(U)} & \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F}(U) \\ \cong\downarrow & & \downarrow\cong \\ \mathcal{O}_X(X) \otimes_{\mathcal{O}_X(X)} \mathcal{E}(X \times U) & & \mathcal{O}_X(X) \otimes_{\mathcal{O}_X(X)} \mathcal{F}(X \times U) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{pr}_{2*}\,\mathcal{E}(U) & \xrightarrow[\delta(U)]{} & \mathrm{pr}_{2*}\,\mathcal{F}(U) \end{array} \tag{1}$$

whose verticals are canonical isomorphisms, commute. The fact that the $\delta(U)$'s glue to give a morphism $\delta : \mathrm{pr}_{2*}\,\mathcal{E} \to \mathrm{pr}_{2*}\,\mathcal{F}$ is straightforward.

We now claim that δ is a $\mathrm{pr}_{2*}\,\mathcal{O}_{X\times Y}$-module morphism. It suffices to prove this affine locally on Y, so let $U \subset Y$ be affine open. The claim reduces to showing that if $r \in \mathcal{O}_X(X)$ and $\mu_r \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$ corresponds to multiplication by r, then the diagram

$$\begin{array}{ccc} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{E}(U) & \xrightarrow{\Phi_{\mathcal{O}_X}(U)} & \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F}(U) \\ {\scriptstyle (\mu_r\otimes\mathcal{E})(U)}\downarrow & & \downarrow{\scriptstyle (\mu_r\otimes\mathcal{F})(U)} \\ \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{E}(U) & \xrightarrow[\Phi_{\mathcal{O}_X}(U)]{} & \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F}(U) \end{array}$$

commutes. This follows from the naturality of Φ.

By the preceding claim and [3, II, ex. 5.17], there exists

$$\phi \in \mathrm{Hom}_{\mathcal{O}_{X\times Y}}(\mathcal{E}, \mathcal{F})$$

such that $\mathrm{pr}_{2*}\,\phi = \delta$. Since the diagram formed by replacing $\Phi_{\mathcal{O}_X}$ by $\mathrm{id}_{\mathcal{O}_X} \otimes \phi$ in (1) commutes, it follows that $\Phi_{\mathcal{O}_X} = \mathrm{id}_{\mathcal{O}_X} \otimes \phi$.

Next, we show that if $\oplus\mathcal{O}_X$ is a direct sum of structure sheaves, then $\Phi_{\oplus\mathcal{O}_X} = \mathrm{id}_{\oplus\mathcal{O}_X} \otimes \phi$. To this end, if $\alpha_i : \mathcal{O}_X \to \oplus\mathcal{O}_X$ denotes inclusion of $\mathcal{O}_X$ in the ith summand of $\oplus\mathcal{O}_X$, naturality of Φ implies that the diagram

$$\begin{array}{ccc} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow{\Phi_{\mathcal{O}_X} = \mathrm{id}_{\mathcal{O}_X} \otimes\phi} & \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F} \\ {\scriptstyle \alpha_i\otimes_{\mathcal{O}_X}\mathcal{E}}\downarrow & & \downarrow{\scriptstyle \alpha_i\otimes_{\mathcal{O}_X}\mathcal{F}} \\ (\oplus\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow[\Phi_{\oplus\mathcal{O}_X}]{} & (\oplus\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F} \end{array}$$

commutes. Since i is arbitrary, it follows that $\Phi_{\oplus\mathcal{O}_X} = \mathrm{id}_{\oplus\mathcal{O}_X} \otimes \phi$.

Now let $\mathcal{M} \in \mathsf{Qcoh}X$ and let $\pi : \oplus\mathcal{O}_X \to \mathcal{M}$ be an epimorphism. Then the diagram

$$\begin{array}{ccc} (\oplus\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow{\mathrm{id}_{\oplus\mathcal{O}_X} \otimes\phi} & (\oplus\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F} \\ {\scriptstyle \pi\otimes\mathcal{O}_X\mathcal{E}}\downarrow & & \downarrow{\scriptstyle \pi\otimes\mathcal{O}_X\mathcal{F}} \\ \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow[\Phi_{\mathcal{M}}]{} & \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{F} \end{array}$$

commutes by the naturality of Φ and the verticals are epimorphisms. It follows that $\Phi_{\mathcal{M}} = \mathrm{id}_{\mathcal{M}} \otimes \phi$ so that inclusion of $\mathsf{Qcoh}X \times Y$ in $\mathsf{Bimod}_T(X - Y)$ is indeed full.

Step 8. We show the inclusion functor is faithful. If $\phi : \mathcal{E} \to \mathcal{F}$ is such that $- \otimes_{\mathcal{O}_X} \phi = 0$, then $\mathrm{pr}_{2*}\, \phi = 0$ so that, for all affine open subsets $U \subset Y$,

$$\phi(X \times U) = 0.$$

Since $X \times U$ is affine open in $X \times Y$, there exists an open affine cover $\{V_i\}$ of $X \times Y$ such that $\phi(V_i) = 0$ for all i. Thus, $\phi = 0$. □

3. Basechange, the Projection Formula, and Compatibilities

Our construction of the Watts functor and proof of Theorem 1.4 depends, in a fundamental way, on the existence and properties of two canonical isomorphisms which are constructed using basechange and projection formulas. The purpose of this section is to describe these isomorphisms as well as several fundamental compatibilities involving them.

Throughout this section, we let $u : U \to X$ denote an open affine immersion, we let $v = u \times \mathrm{id}_Y$, and let we $p, q : U \times Y \to U, Y$ denote projections.

We begin with some preliminary observations. We note that the diagram

$$\begin{tikzcd} U \times Y \arrow[r, "p"] \arrow[d, "v"'] & U \arrow[d, "u"] \\ X \times Y \arrow[r, "\mathrm{pr}_1"'] & X \end{tikzcd}$$

is a fiber square. We claim the basechange and projection formula morphisms

$$\mathrm{pr}_1^*\, u_* \longrightarrow v_* v^*\, \mathrm{pr}_1^*\, u_* \xrightarrow{\cong} v_* p^* u^* u_* \longrightarrow v_* p^*$$

and

$$v_* - \otimes_{\mathcal{O}_{X\times Y}} - \to v_* v^*(v_* - \otimes_{\mathcal{O}_{X\times Y}} -) \xrightarrow{\cong} v_*(v^* v_* - \otimes_{\mathcal{O}_{U\times Y}} v^* -) \to v_*(- \otimes_{\mathcal{O}_{U\times Y}} v^* -)$$

induced by unit and counit morphisms of (u^*, u_*) and (v^*, v_*), and by the distributivity of tensor products and pullbacks, are isomorphisms. To this end, we note that it suffices to prove that they are isomorphisms over subsets of the form $V \times W$ where $V \subset X$ and $W \subset Y$ are open affine subsets. This reduces the claim to a straightforward affine computation, which we omit.

Let $\mathcal{E} \in \mathsf{Qcoh}U \times Y$ and $\mathcal{F} \in \mathsf{Qcoh}X \times Y$. We define canonical isomorphisms

$$u^* - \otimes_{\mathcal{O}_U} \mathcal{E} \longrightarrow - \otimes_{\mathcal{O}_X} v_* \mathcal{E} \tag{2}$$

and

$$u_* - \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow - \otimes_{\mathcal{O}_U} v^* \mathcal{F}, \tag{3}$$

natural in $\mathcal{E}$ and $\mathcal{F}$, as follows: The map (2) is defined to be the composition

$$\begin{aligned} u^* - \otimes_{\mathcal{O}_U} \mathcal{E} &\xrightarrow{=} q_*(p^* u^* - \otimes_{\mathcal{O}_{U\times Y}} \mathcal{E}) \\ &\xrightarrow{\cong} q_*(v^*\, \mathrm{pr}_1^* - \otimes_{\mathcal{O}_{U\times Y}} \mathcal{E}) \\ &\xrightarrow{=} \mathrm{pr}_{2*}\, v_*(v^*\, \mathrm{pr}_1^* - \otimes_{\mathcal{O}_{U\times Y}} \mathcal{E}) \\ &\xrightarrow{\cong} \mathrm{pr}_{2*}(\mathrm{pr}_1^* - \otimes_{\mathcal{O}_{X\times Y}} v_* \mathcal{E}) \\ &\xrightarrow{=} - \otimes_{\mathcal{O}_X} v_* \mathcal{E} \end{aligned}$$

where the second morphism comes from the equality $\mathrm{pr}_1 v = up$ and the fourth morphism is the projection formula.

We define the map (3) as the composition

$$\begin{aligned} u_* - \otimes_{\mathcal{O}_X} \mathcal{F} &\xrightarrow{=} \mathrm{pr}_{2*}(\mathrm{pr}_1^* u_* - \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F}) \\ &\xrightarrow{\cong} \mathrm{pr}_{2*}(v_* p^* - \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F}) \\ &\xrightarrow{\cong} \mathrm{pr}_{2*} v_*(p^* - \otimes_{\mathcal{O}_{U\times Y}} v^*\mathcal{F}) \\ &\xrightarrow{=} q_*(p^* - \otimes_{\mathcal{O}_{U\times Y}} v^*\mathcal{F}) \end{aligned}$$

where the second morphism is basechange and the third morphism is the projection formula.

Naturality of (2) and (3) follows from naturality of basechange and the projection formula.

The remainder of this section is devoted to the proof that (2) and (3) satisfy three compatibilities. The first says that (2) and (3) are compatible with the units and counits of the adjoint pairs (u^*, u_*) and (v^*, v_*) (Lemma 3.1). The second says that if $\widetilde{U} \xrightarrow{\tilde{u}} U$ is an open affine immersion and $\tilde{v} = \tilde{u} \times \mathrm{id}_Y$, then (2) and (3) are compatible with the canonical isomorphisms $(u\tilde{u})^* \cong \tilde{u}^* u^*$ and $(v\tilde{v})^* \cong \tilde{v}^* v^*$ (Lemma 3.2). The third says that (2) and (3) are compatible with affine basechange (Lemma 3.3).

Lemma 3.1. *The diagram*

$$\begin{tikzcd} - \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r] \arrow[d] & - \otimes_{\mathcal{O}_X} v_* v^* \mathcal{F} \arrow[d] \\ u_* u^*(-) \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r] & u^*(-) \otimes_{\mathcal{O}_U} v^* \mathcal{F} \end{tikzcd} \tag{4}$$

whose top horizontal and left vertical are induced by unit morphisms, whose right vertical is the inverse of (2) and whose bottom horizontal is induced by (3), and the diagram

$$\begin{tikzcd} - \otimes_{\mathcal{O}_U} \mathcal{E} & - \otimes_{\mathcal{O}_U} v^* v_* \mathcal{E} \arrow[l] \\ u^* u_*(-) \otimes_{\mathcal{O}_U} \mathcal{E} \arrow[u] & u_*(-) \otimes_{\mathcal{O}_X} v_* \mathcal{E} \arrow[l] \arrow[u] \end{tikzcd} \tag{5}$$

whose top horizontal and left vertical are induced by counit morphisms, whose bottom horizontal is induced by (2), and whose right vertical is (3), commutes.

Proof. To prove that the basechange and projection formulas are isomorphisms,

We next show that (4) commutes. It suffices to show that the diagram

$$\begin{tikzcd} \mathrm{pr}_1^*(-) \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F} \arrow[r] \arrow[d] & \mathrm{pr}_1^*(-) \otimes_{\mathcal{O}_{X\times Y}} v_* v^* \mathcal{F} \arrow[d] \\ \mathrm{pr}_1^* u_* u^*(-) \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F} \arrow[d] & v_*(v^* \mathrm{pr}_1^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^* \mathcal{F} \arrow[d] \\ v_* p^* u^*(-) \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r] & v_*(p^* u^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^* \mathcal{F}) \end{tikzcd} \tag{6}$$

whose top horizontal and upper-left vertical are induced by unit morphisms, whose upper-right vertical and bottom horizontal are projection formulas, whose bottom-left vertical is basechange and whose bottom-right vertical is canonical, commutes.

Next, we claim the diagram

$$
\begin{tikzcd}
\mathrm{pr}_1^* \arrow[r] \arrow[d] & v_* v^* \mathrm{pr}_1^* \arrow[d] \\
\mathrm{pr}_1^* u_* u^* \arrow[d] & v_* p^* u^* \arrow[d] \\
v_* v^* \mathrm{pr}_1^* u_* u^* \arrow[r] & v_* p^* u^* u_* u^*
\end{tikzcd} \tag{7}
$$

whose top horizontal and left verticals and bottom-right vertical are induced by unit morphisms, and whose bottom horizontal and upper-right vertical are canonical, commutes. The claim follows by splitting (7) into two subdiagrams via the morphism

$$v_* v^* \mathrm{pr}_1^* \longrightarrow v_* v^* \mathrm{pr}_1^* u_* u^*$$

induced by the unit of (u^*, u_*), and noticing that each commutes by the naturality of the unit.

The claim implies that, in order to show (6) commutes, it suffices to show that both squares of the diagram

$$
\begin{tikzcd}
\mathrm{pr}_1^*(-) \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F} \arrow[r] \arrow[d] & \mathrm{pr}_1^*(-) \otimes_{\mathcal{O}_{X\times Y}} v_* v^* \mathcal{F} \arrow[d] \\
v_* v^* \mathrm{pr}_1^*(-) \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F} \arrow[r] \arrow[d] & v_*(v^* \mathrm{pr}_1^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^* \mathcal{F}) \arrow[d] \\
v_* p^* u^*(-) \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F} \arrow[r] & v_*(p^* u^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^* \mathcal{F})
\end{tikzcd} \tag{8}
$$

whose top horizontal and top-left vertical are induced by unit morphisms, whose top-left vertical and middle and bottom horizontals are projection formulas, and whose bottom verticals are canonical, commutes. The bottom square commutes by the naturality of the projection formula, while to prove the top square of (8) commutes, it suffices to show that the diagram

$$
\begin{tikzcd}
- \otimes_{\mathcal{O}_{X\times Y}} - \arrow[r] \arrow[d] & - \otimes_{\mathcal{O}_{X\times Y}} v_* v^*(-) \arrow[d] \\
v_* v^*(-) \otimes_{\mathcal{O}_{X\times Y}} - \arrow[r] & v_*(v^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^*(-))
\end{tikzcd}
$$

whose right vertical and bottom horizontal are projection formulas, and whose left vertical and top horizontal are induced by units, commutes. By the symmetry of the inputs in the bifunctor $- \otimes -$, it suffices to prove that both squares in the diagram

$$
\begin{tikzcd}
- \otimes_{\mathcal{O}_{X\times Y}} - \arrow[r] \arrow[d] & v_* v^*(- \otimes_{\mathcal{O}_{X\times Y}} -) \arrow[r] \arrow[d] & v_*(v^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^*(-)) \\
v_* v^* - \otimes_{\mathcal{O}_{X\times Y}} - \arrow[r] & v_* v^*(v_* v^* - \otimes_{\mathcal{O}_{X\times Y}} -) \arrow[r] & v_*(v^* v_* v^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^*(-)) \arrow[u]
\end{tikzcd}
$$

whose left arrows are unit morphisms, whose right horizontals are induced by the canonical isomorphism

$$v^*(- \otimes_{\mathcal{O}_{X\times Y}} -) \xrightarrow{\cong} v^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^*(-) \tag{9}$$

and whose right horizontal is the counit morphism, commute. The left square of this diagram commutes by naturality of the unit morphism, while the commutivity of the right square follows from the commutivity of square

$$\begin{array}{ccc} v_*v^*(- \otimes_{\mathcal{O}_{X\times Y}} -) & \longrightarrow & v_*(v^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^*(-)) \\ \downarrow & & \downarrow \\ v_*v^*(v_*v^* - \otimes_{\mathcal{O}_{X\times Y}} -) & \longrightarrow & v_*(v^*v_*v^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^*(-)) \end{array}$$

whose verticals are induced by units and whose horizontals are (9). This square commutes by naturality of (9).

The proof that (5) commutes is similar to the proof that (4) commutes, and we omit it. □

Lemma 3.2. *Suppose $\widetilde{U} \subset U$ are open affine subschemes of X, with inclusion morphisms $\tilde{u} : \widetilde{U} \to U$ and $u : U \to X$. Let $\tilde{v} = \tilde{u} \times \mathrm{id}_Y$, let $v = u \times \mathrm{id}_Y$, and let $\mathcal{F}$ be an object of $\mathsf{Qcoh} X \times Y$. Then the diagrams*

$$\begin{array}{ccc} - \otimes_{\mathcal{O}_{\widetilde{U}}} \tilde{v}^*v^*\mathcal{F} & \longrightarrow & \tilde{u}_*(-) \otimes_{\mathcal{O}_U} v^*\mathcal{F} \\ \downarrow & & \downarrow \\ - \otimes_{\mathcal{O}_{\widetilde{U}}} (v\tilde{v})^*\mathcal{F} & \longrightarrow & (u\tilde{u})_* \otimes_{\mathcal{O}_X} \mathcal{F} \end{array} \tag{10}$$

whose horizontals and right vertical are the maps (3) and whose left vertical is induced by the canonical isomorphism $\tilde{v}^*v^* \xrightarrow{\cong} (v\tilde{v})^*$, and

$$\begin{array}{ccc} - \otimes_{\mathcal{O}_X} v_*\tilde{v}_*\mathcal{F} & \longrightarrow & u^*(-) \otimes_{\mathcal{O}_U} \tilde{v}_*\mathcal{F} \\ \downarrow & & \downarrow \\ (u\tilde{u})^*(-) \otimes_{\mathcal{O}_{\widetilde{U}}} \mathcal{F} & \xrightarrow{\cong} & \tilde{u}^*u^*(-) \otimes_{\mathcal{O}_{\widetilde{U}}} \mathcal{F} \end{array} \tag{11}$$

whose top horizontal and verticals are induced by (2), and whose bottom horizontal is induced by the canonical isomorphism $(u\tilde{u})^* \cong \tilde{u}^*u^*$, commutes.

Proof. Let $p : U \times Y \to U$ and $\tilde{p} : \widetilde{U} \times Y \to \widetilde{U}$ denote projections. In order to prove (10) commutes, it suffices to show that the diagram

(12)

$$\begin{array}{ccccc} \mathrm{pr}_1^*(u\tilde{u})_*(-) \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F} & \longrightarrow & v_*p^*\tilde{u}_*(-) \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F} & \longrightarrow & v_*(p^*\tilde{u}_*(-) \otimes_{\mathcal{O}_{U\times Y}} v^*\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow \\ (v\tilde{v})_*\tilde{p}^*(-) \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F} & \xrightarrow{=} & v_*\tilde{v}_*\tilde{p}^*(-) \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F} & \longrightarrow & v_*(\tilde{v}_*\tilde{p}^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^*\mathcal{F}) \\ \downarrow & & & & \downarrow \\ (v\tilde{v})_*(\tilde{p}^*(-) \otimes_{\mathcal{O}_{\widetilde{U}\times Y}} (v\tilde{v})^*\mathcal{F}) & & \longrightarrow & & v_*\tilde{v}_*(\tilde{p}^*(-) \otimes_{\mathcal{O}_{\widetilde{U}\times Y}} \tilde{v}^*v^*\mathcal{F}) \end{array}$$

whose top-left horizontal and top verticals are induced by basechange, whose top-right horizontal, middle-right horizontal and bottom verticals are induced by the projection formula, and whose bottom isomorphism is induced by the canonical isomorphism $(v\tilde{v})^* \xrightarrow{\cong} \tilde{v}^*v^*$, commutes.

The upper-right square of (12) commutes by the naturality of the projection formula. The fact that the upper-left square of (12) commutes follows from the commutivity of the diagram

$$\begin{tikzcd} \mathrm{pr}_1^*(u\tilde{u})_* \arrow[r, "="] \arrow[d, "\cong"'] & \mathrm{pr}_1^* u_*\tilde{u}_* \arrow[d, "\cong"] \\ (v\tilde{v})_*\tilde{p}^* \arrow[r, "\cong"'] & v_*p^*\tilde{u}_* \end{tikzcd} \tag{13}$$

whose non-trivial isomorphisms are induced by basechange. The commutivity of (13) can be checked affine locally and we omit the routine verification.

The commutivity of the bottom rectangle of (12) follows from the commutivity of the diagram

$$\begin{tikzcd} (v\tilde{v})_*(-) \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F} \arrow[r, "\cong"] \arrow[d, "\cong"'] & v_*(\tilde{v}_*(-) \otimes_{\mathcal{O}_{U\times Y}} v^*\mathcal{F}) \arrow[d, "\cong"] \\ (v\tilde{v})_*((-) \otimes_{\mathcal{O}_{\tilde{U}\times Y}} (v\tilde{v})^*\mathcal{F}) \arrow[r, "\cong"'] & v_*\tilde{v}_*((-) \otimes_{\mathcal{O}_{\tilde{U}\times Y}} \tilde{v}^*v^*\mathcal{F}) \end{tikzcd} \tag{14}$$

whose bottom horizontal is induced by the canonical isomorphism $(v\tilde{v})^* \xrightarrow{\cong} \tilde{v}^*v^*$ and whose other arrows are induced by the projection formula. The commutivity of (14) again follows from a routine affine computation, which we omit.

The proof that (11) commutes is similar and may be reduced to the commutivity of a diagram of the form (14) as well. We leave the details to the reader. □

Lemma 3.3. *Let $U_1, U_2 \subset X$ be affine open subschemes, let $U_{12} := U_1 \cap U_2$ with inclusions*

$$\begin{tikzcd} U_{12} \arrow[r, "u_{12}^1"] \arrow[d, "u_{12}^2"'] & U_1 \arrow[d, "u_1"] \\ U_2 \arrow[r, "u_2"'] & X. \end{tikzcd}$$

For $i = 1, 2$, let $v_i = u_i \times \mathrm{id}_Y$ and let $v_{12}^i = u_{12}^i \times \mathrm{id}_Y$. If $\mathcal{E}$ be an object of $\mathsf{Qcoh}U_1 \times Y$, then the diagram

$$\begin{tikzcd} u_1^*u_{2*}(-) \otimes_{\mathcal{O}_{U_1}} \mathcal{E} \arrow[r] & u_{12*}^1 u_{12}^{2*}(-) \otimes_{\mathcal{O}_{U_1}} \mathcal{E} \arrow[d] \\ u_{2*}(-) \otimes_{\mathcal{O}_X} v_{1*}\mathcal{E} \arrow[u] & u_{12}^{2*}(-) \otimes_{\mathcal{O}_{U_1}} v_{12}^{1*}\mathcal{E} \arrow[d] \\ - \otimes_{\mathcal{O}_{U_2}} v_2^*v_{1*}\mathcal{E} \arrow[u] \arrow[r] & - \otimes_{\mathcal{O}_{U_2}} v_{12*}^2 v_{12}^{1*}\mathcal{E} \end{tikzcd}$$

whose horizontals are induced by basechange, and whose verticals are (2) and (3), commutes.

Proof. By the naturality of units, counits and the morphisms (2) and (3), and by the commutivity of (4), it suffices to prove that the diagram

$$\begin{tikzcd}
u_{2*}(-)\otimes_{\mathcal{O}_X} v_{1*}v^1_{12*}v^{1*}_{12}\mathcal{E} \arrow[r] & u_1^*u_{2*}(-)\otimes_{\mathcal{O}_{U_1}} v^1_{12*}v^{1*}_{12}\mathcal{E} \arrow[r] & u^{1*}_{12}u_1^*u_{2*}(-)\otimes_{\mathcal{O}_{U_{12}}} v^{1*}_{12}\mathcal{E} \arrow[d] \\
-\otimes_{\mathcal{O}_{U_2}} v_2^*v_{1*}v^1_{12*}v^{1*}_{12}\mathcal{E} \arrow[u] & & u^{2*}_{12}u_2^*u_{2*}(-)\otimes_{\mathcal{O}_{U_{12}}} v^{1*}_{12}\mathcal{E} \arrow[d] \\
-\otimes_{\mathcal{O}_{U_2}} v_2^*v_{1*}\mathcal{E} \arrow[u] \arrow[r] & -\otimes_{\mathcal{O}_{U_2}} v^2_{12*}v^{1*}_{12}\mathcal{E} & u_2^*u_{2*}(-)\otimes_{\mathcal{O}_{U_2}} v^2_{12*}v^{1*}_{12}\mathcal{E} \arrow[l]
\end{tikzcd} \tag{15}$$

whose top horizontals and bottom-right vertical are induced by (2), whose middle and bottom horizontals are induced by basechange, whose bottom-left horizontal is induced by a counit, whose bottom-left vertical is induced by a unit, whose top-left vertical is induced by (3), and whose top-right vertical is induced by the canonical isomorphism

$$u^{1*}_{12}u_1^* \cong u^{2*}_{12}u_2^*$$

commutes. To this end, we note that, by the naturality of basechange, the diagram

$$\begin{tikzcd}
-\otimes_{\mathcal{O}_{U_2}} v_2^*v_{1*}v^1_{12*}v^{1*}_{12}\mathcal{E} & -\otimes_{\mathcal{O}_{U_2}} v^2_{12*}v^{1*}_{12}v^1_{12*}v^{1*}_{12}\mathcal{E} \arrow[l] \\
-\otimes_{\mathcal{O}_{U_2}} v_2^*v_{1*}\mathcal{E} \arrow[u] & -\otimes_{\mathcal{O}_{U_2}} v^2_{12*}v^{1*}_{12}\mathcal{E} \arrow[l] \arrow[u]
\end{tikzcd}$$

whose verticals are induced by units and whose horizontals are induced by basechange, commutes. Hence, to prove that (15) commutes, it suffices to prove that if $\mathcal{F} := v^{1*}_{12}\mathcal{E}$, then the diagram

$$\begin{tikzcd}
u_{2*}(-)\otimes_{\mathcal{O}_X} v_{1*}v^1_{12*}\mathcal{F} \arrow[r] & u_1^*u_{2*}(-)\otimes_{\mathcal{O}_{U_1}} v^1_{12*}\mathcal{F} \arrow[r] & u^{1*}_{12}u_1^*u_{2*}(-)\otimes_{\mathcal{O}_{U_{12}}} \mathcal{F} \arrow[d] \\
-\otimes_{\mathcal{O}_{U_2}} v_2^*v_{1*}v^1_{12*}\mathcal{F} \arrow[u] & & u^{2*}_{12}u_2^*u_{2*}(-)\otimes_{\mathcal{O}_{U_{12}}} \mathcal{F} \arrow[d] \\
-\otimes_{\mathcal{O}_{U_2}} v^2_{12*}v^{1*}_{12}v^1_{12*}\mathcal{F} \arrow[u] \arrow[r] & -\otimes_{\mathcal{O}_{U_2}} v^2_{12*}\mathcal{F} & u_2^*u_{2*}(-)\otimes_{\mathcal{O}_{U_2}} v^2_{12*}\mathcal{F} \arrow[l]
\end{tikzcd} \tag{16}$$

whose bottom-left vertical is induced by basechange, whose bottom-left horizontal is induced by a counit, and whose other maps are identical to the maps in (15), commutes.

The diagram (16) can be broken into the four subdiagrams

$$\begin{tikzcd}
-\otimes_{\mathcal{O}_{U_2}} v_2^*v_{1*}v^1_{12*}\mathcal{F} \arrow[r, "="] \arrow[d] & -\otimes_{\mathcal{O}_{U_2}} v_2^*v_{2*}v^2_{12*}\mathcal{F} \arrow[d] \\
-\otimes_{\mathcal{O}_{U_2}} v^2_{12*}v^{1*}_{12}v^1_{12*}\mathcal{F} \arrow[r] & -\otimes_{\mathcal{O}_{U_2}} v^2_{12*}\mathcal{F}
\end{tikzcd} \tag{17}$$

whose left vertical is induced by basechange and whose right vertical and bottom horizontal are counits,

$$
\begin{tikzcd}
u_{2*}(-) \otimes_{\mathcal{O}_X} v_{1*}v^1_{12*}\mathcal{F} \arrow[r] \arrow[d, "="'] & u_1^* u_{2*}(-) \otimes_{\mathcal{O}_{U_1}} v^1_{12*}\mathcal{F} \arrow[r] & u^{1*}_{12}u_1^*u_{2*}(-) \otimes_{\mathcal{O}_{U_{12}}} \mathcal{F} \arrow[d, "\cong"] \\
u_{2*}(-) \otimes_{\mathcal{O}_X} v_{2*}v^2_{12*}\mathcal{F} \arrow[r] & u_2^* u_{2*}(-) \otimes_{\mathcal{O}_{U_2}} v^2_{12*}\mathcal{F} \arrow[r] & u^{2*}_{12}u_2^*u_{2*}(-) \otimes_{\mathcal{O}_{U_{12}}} \mathcal{F}
\end{tikzcd}
\tag{18}
$$

whose horizontals are induced by (2) and whose right vertical is canonical,

$$
\begin{tikzcd}
- \otimes_{\mathcal{O}_{U_2}} v_2^* v_{2*}v^2_{12*}\mathcal{F} \arrow[d] & u_{2*}(-) \otimes_{\mathcal{O}_X} v_{2*}v^2_{12*}\mathcal{F} \arrow[l] \arrow[d] \\
- \otimes_{\mathcal{O}_{U_2}} v^2_{12*}\mathcal{F} & u_2^* u_{2*}(-) \otimes_{\mathcal{O}_{U_2}} v^2_{12*}\mathcal{F} \arrow[l]
\end{tikzcd}
\tag{19}
$$

whose top horizontal is (3), whose right vertical is (2) and whose other arrows are counits, and

$$
\begin{tikzcd}
u_{2*}(-) \otimes_{\mathcal{O}_X} v_{1*}v^1_{12*}\mathcal{F} \arrow[r, "="] & u_{2*}(-) \otimes_{\mathcal{O}_X} v_{2*}v^2_{12*}\mathcal{F} \\
- \otimes_{\mathcal{O}_{U_2}} v_2^* v_{1*}v^1_{12*}\mathcal{F} \arrow[r, "="] \arrow[u] & - \otimes_{\mathcal{O}_{U_2}} v_2^* v_{2*}v^2_{12*}\mathcal{F} \arrow[u]
\end{tikzcd}
\tag{20}
$$

whose verticals are induced by (3). It suffices to show that these subdiagrams all commute. The fact that diagram (17) commutes is left as an exercise to the reader. The fact that diagram (18) commutes follows from Lemma 3.2. The fact that diagram (19) commutes follows from Lemma 3.1, and the commutivity of (20) is trivial. □

4. Totally Global Functors

Our goal in this section is to define and study *totally global functors*. We begin with some terminology.

For the remainder of this section, an *affine open cover* of X is a set of pairs $\{(U_i, u_i)\}$ where $u_i : U_i \to X$ is inclusion of an affine open subset U_i of X such that every point of X is contained in some U_i.

Definition 4.1. We say $F \in \mathsf{Funct}_T(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ is *totally global* if for any affine open cover $\{(U_i, u_i)\}$ of X, $Fu_{i*} = 0$ for all i.

We note that this definition makes sense. For, $u_i : U_i \to X$ is affine since X is separated [3, II, ex. 3.2], so that u_i is quasi-compact and separated [3, II, ex. 5.17b]. Hence, u_{i*} takes values in the category of quasi-coherent $\mathcal{O}_X$-modules [3, II, Prop. 5.8c].

The following lemma explains the motivation behind the use of the term *totally global*.

Lemma 4.2. *Suppose X is noetherian. If F is totally global and $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module whose support lies in an affine open subset U of X (included via u), then $F\mathcal{M} = 0$.*

Proof. Since F commutes with direct-limits and X is noetherian, it suffices to prove that $F\mathcal{M} = 0$ for $\mathcal{M}$ coherent. Let $i : \operatorname{Supp}\mathcal{M} \to X$ and $i' : \operatorname{Supp}\mathcal{M} \to U$ denote inclusions, so that $i = ui'$. Since i is a closed immersion, the unit map $\mathcal{M} \to i_*i^*\mathcal{M}$ is an isomorphism. Thus,

$$\begin{aligned} F\mathcal{M} &\cong Fi_*i^*\mathcal{M} \\ &= F(ui')_*i^*\mathcal{M} \\ &= Fu_*i'_*i^*\mathcal{M} \\ &= 0. \end{aligned}$$

□

Example 4.3. Let W be a noetherian scheme. Then the functor $H^i(W, -)$ is totally global by [3, III, ex. 8.2].

Example 4.4. Let $\mathcal{F}$ be an object of $\mathsf{Qcoh}X \times Y$. We claim that if $F = - \otimes_{\mathcal{O}_X} \mathcal{F}$ is totally global then $F = 0$. To prove the claim, we let $u : U \to X$ denote an affine open immersion. The map (3) induces an isomorphism,

$$\begin{aligned} u_*(-) \otimes_{\mathcal{O}_X} \mathcal{F} &\xrightarrow{\cong} - \otimes_{\mathcal{O}_U} v^*\mathcal{F} \\ &\xrightarrow{=} q_*(p^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^*\mathcal{F}) \end{aligned}$$

where $v = u \times \mathrm{id}_Y$ and $p, q : U \times Y \to U, Y$ are projections. If F is totally global, then $q_*v^*\mathcal{F} = \mathrm{pr}_{2*}(v_*v^*\mathcal{F}) = 0$. Thus, if W is an affine open subset of Y, $v^*\mathcal{F}(U \times W) = 0$. Therefore, $v^*\mathcal{F} = 0$ since its sections on an affine open cover are 0. We conclude that if $p \in U \times Y$, then $\mathcal{F}_p = 0$. Since U is arbitrary, $\mathcal{F} = 0$ so $F = 0$ as desired.

We show in Proposition 4.5 that if $F \in \mathsf{Bimod}_T(X - Y)$ has the property that for *some* open cover $\{(U_i, u_i)\}$ of X, $Fu_{i*} = 0$ for all i, then F is totally global.

Proposition 4.5. *If $F \in \mathsf{Bimod}_T(X - Y)$ and $\{(U_i, u_i)\}$ is an affine open cover of X such that $Fu_{i*} = 0$ for all i, then F is totally global.*

Proof. We first prove that if X is affine, $\{(W_i, w_i)\}$ is an affine open cover of X, and $E \in \mathsf{Funct}_T(X - Y)$ such that $Ew_{i*} = 0$ for all i, then $E = 0$. Since X is affine, $E \cong - \otimes_{\mathcal{O}_X} \mathcal{E}$ for some object $\mathcal{E}$ of $\mathsf{Qcoh}X \times Y$ by Proposition 2.2. Thus, if $p, q : W_i \times Y \to W_i, Y$ are projections and $v_i = w_i \times \mathrm{id}_Y$, then by (3),

$$Ew_{i*} \cong q_*(p^* - \otimes_{\mathcal{O}_{W_i\times Y}} {v_i}^*\mathcal{E}).$$

The vanishing of Ew_{i*} for all i implies that $q_*{v_i}^*\mathcal{E} = 0$ for all i. Therefore, for all i and all $W \subset Y$ open affine,

$${v_i}^*\mathcal{E}(W_i \times W) = 0.$$

This implies that ${v_i}^*\mathcal{E} = 0$ for all i which implies that $\mathcal{E}$, and hence E, is 0.

Now we prove the lemma. Let F and $\{(U_i, u_i)\}$ be as in the statement of the Proposition. Let $\{(V_j, v_j)\}$ denote an affine open cover of X and let $w_{ij} : U_i \cap V_j \to V_j$ and $w'_{ij} : U_i \cap V_j \to U_i$ denote inclusions. Then $Fv_{j*}w_{ij*} = Fu_{i*}w'_{ij*} = 0$ for all i by hypothesis. But V_j is affine, $Fv_{j*} \in \mathsf{Bimod}_T(V_j \times Y)$ since v_{j*} is right-exact by the affineness of v_j [3, III, Cor. 8.2, prop. 8.1, and Remark 3.5.1], and $\mathfrak{W} := \{(U_i \cap V_j, w_{ij})\}_i$ is an affine open cover of V_j. Hence the argument of the first paragraph applies to the functor $E = Fv_{j*}$ and the open cover $\mathfrak{W}$ of V_j, so that $Fv_{j*} = 0$. Since j is arbitrary, the lemma follows. □

The following is elementary:

Lemma 4.6. *Let* TG *denote the full subcategory of* $\mathsf{Funct}_T(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ *consisting of totally global functors. Then* TG *is a Serre subcategory of the category* $\mathsf{Funct}_T(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$.

5. The Watts Functor

In this section we review the construction of an assignment

$$W : \mathsf{Bimod}_T(X - Y) \to \mathsf{Qcoh}X \times Y$$

sketched in [7, Lemma 3.1.1], and prove it is functorial (Subsection 5.2), left-exact (Proposition 5.1), and compatible with affine localization (Proposition 5.2). We will show in Corollary 6.5 that if $F \in \mathsf{Bimod}_T(X - Y)$ then $- \otimes_{\mathcal{O}_X} W(F)$ serves as a "best" approximation of F by tensoring with a bimodule. In order to prove this, we will need the fact, proven in Proposition 5.4, that if $F = - \otimes_{\mathcal{O}_X} \mathcal{F}$ then $W(F) \cong \mathcal{F}$.

The construction of W depends on a choice of finite affine open cover of X. If $\mathfrak{U}$ is a finite affine open cover of X, we denote the corresponding functor by $W_{\mathfrak{U}}$. We show in, Proposition 6.6, that $W(F)$ is independent of this choice in the following situation: if $F \in \mathsf{Bimod}_T(X - Y)$ and $\mathfrak{U}$ and $\mathfrak{V}$ are finite affine open covers of X such that $- \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ and $- \otimes_{\mathcal{O}_X} W_{\mathfrak{V}}(F)$ are right-exact, then

$$W_{\mathfrak{U}}(F) \cong W_{\mathfrak{V}}(F).$$

This holds, in particular, when X is affine, by Proposition 2.2.

5.1. **Preliminaries.** Before defining the functor W, we describe conventions we will employ throughout the rest of this paper.

Let $\{U_i\}_{i \in I}$ be a collection of open subschemes of X (we identify the underlying set of U_i with a subset of the underlying set of X). For any finite subset $\{i_1, \ldots, i_n\}$ of I, we let

$$U_{i_1 \cdots i_n} = U_{i_1} \cap \cdots \cap U_{i_n}$$

and we let

$$u_{i_1 \cdots i_n} : U_{i_1 \cdots i_n} \to X$$

denote the inclusion morphism. For any inclusion $\{j_1, \ldots, j_m\} \subset \{i_1, \ldots, i_n\}$ of finite subsets of I, we let

$$u^{j_1 \cdots j_m}_{i_1 \cdots i_n} : U_{i_1 \cdots i_n} \to U_{j_1 \cdots j_m}$$

denote the inclusion morphism. Similar conventions apply when the open subschemes are labelled $\{V_j\}$ or $\{W_k\}$, etc. We denote the open cover $\{U_i\}_{i \in I}$ by $\mathfrak{U}$.

For i, j with $j \neq i$, we let

$$\eta^i_{ij} : \mathrm{id}_{\mathsf{Qcoh}U_i} \to u^i_{ij*} u^{i*}_{ij}$$

denote the canonical unit of the adjoint pair (u^{i*}_{ij}, u^i_{ij*}).

5.2. **Definition of the Watts Functor.** Let F be an object in $\mathsf{Bimod}_T(X-Y)$. Our goal in this subsection is to associate to F an object $W(F) \in \mathsf{Qcoh} X \times Y$, and show the assignment $F \mapsto W(F)$ is functorial. To this end, we first choose a finite affine open cover of X, $\mathfrak{U} = \{U_i\}_{i \in I}$ with $I = \{1, \dots, n\}$. Recall that X is quasi-compact, so such an open cover exists.

For each $i \in I$, the proof of Proposition 2.2, gives us an $\mathcal{F}_i \in \mathsf{Qcoh} U_i \times Y$ and a canonical isomorphism $Fu_{i*} \xrightarrow{\cong} - \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i$.

Now let $V_i = U_i \times Y$. Recalling our conventions about open covers (of $X \times Y$), we claim that there exists a canonical isomorphism

$$\psi_{ij} : v_{ij}^{i*}\mathcal{F}_i \xrightarrow{\cong} v_{ij}^{j*}\mathcal{F}_j. \tag{21}$$

To prove the claim, we note that there are isomorphisms

$$\begin{aligned} - \otimes_{\mathcal{O}_{U_{ij}}} v_{ij}^{i*}\mathcal{F}_i &\xrightarrow{\cong} u_{ij*}^i(-) \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \\ &\xrightarrow{\cong} Fu_{i*}u_{ij*}^i \\ &\xrightarrow{=} Fu_{j*}u_{ij*}^j \\ &\xrightarrow{\cong} u_{ij*}^j(-) \otimes_{\mathcal{O}_{U_j}} \mathcal{F}_j \\ &\xrightarrow{\cong} - \otimes_{\mathcal{O}_{U_{ij}}} v_{ij}^{j*}\mathcal{F}_j, \end{aligned}$$

the first is the inverse of (3), the second is from the definition of $\mathcal{F}_i$ and the fourth and fifth are defined similarly. The map ψ_{ij} corresponds to the composition above under the equivalence from Proposition 2.2.

Next, for each pair $i, j \in I$ with $j > i$, we define a morphism

$$\phi_i^{ij} : v_{i*}\mathcal{F}_i \longrightarrow v_{i*}v_{ij*}^i v_{ij}^{i*}\mathcal{F}_i$$

induced by η_{ij}^i and a morphism

$$\phi_j^{ij} : v_{j*}\mathcal{F}_j \longrightarrow v_{i*}v_{ij*}^i v_{ij}^{i*}\mathcal{F}_i$$

as the composition of the morphism $v_{j*}\mathcal{F}_j \longrightarrow v_{j*}v_{ij*}^j v_{ij}^{j*}\mathcal{F}_j = v_{i*}v_{ij*}^i v_{ij}^{j*}\mathcal{F}_j$ induced by η_{ij}^j and the morphism $v_{i*}v_{ij*}^i v_{ij}^{j*}\mathcal{F}_j \longrightarrow v_{i*}v_{ij*}^i v_{ij}^{i*}\mathcal{F}_i$ induced by ψ_{ij}^{-1}.

Finally, since I is finite, in order to specify a morphism

$$\oplus_i v_{i*}\mathcal{F}_i \longrightarrow \oplus_{i<j} v_{i*}v_{ij*}^i v_{ij}^{i*}\mathcal{F}_i,$$

it suffices to define a morphism

$$\theta_i^{jk} : v_{i*}\mathcal{F}_i \to v_{j*}v_{jk*}^j v_{jk}^{j*}\mathcal{F}_j$$

for all $i, j, k \in I$ with $j < k$. We define such a morphism as

$$\theta_i^{jk} = \begin{cases} \phi_i^{ik} & \text{if } i = j, \\ -\phi_i^{ji} & \text{if } i = k, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

The morphisms $\{\theta_i^{jk}\}$ induce a morphism

$$\theta_F : \oplus_i v_{i*}\mathcal{F}_i \longrightarrow \oplus_{i<j} v_{i*}v_{ij*}^i v_{ij}^{i*}\mathcal{F}_i. \tag{23}$$

We define

$$W_{\mathfrak{U}}(F) := \ker \theta_F.$$

We next note that $W_{\mathfrak{U}}(F)$ is an object of $\mathsf{Qcoh}X \times Y$. For, since $v_{i*}v^i_{ij*} = v_{ij*}$ is an affine morphism, it is quasi-compact and separated by [3, II, ex. 5.17b]. Hence if $\mathcal{M}$ is an object of $\mathsf{Qcoh}U_{ij} \times Y$ then $v_{ij*}\mathcal{M}$ is an object of $\mathsf{Qcoh}X \times Y$ by [3, II, Prop. 5.8c].

We define $W_{\mathfrak{U}}$ on morphisms as follows. Let $\Delta : E \longrightarrow F$ be a morphism in $\mathsf{Bimod}_T(X - Y)$. The transformation Δ induces the horizontal composition of the natural transformations Δ and $\mathrm{id}_{u_{i*}}$, $\Delta * u_{i*} : Eu_{i*} \longrightarrow Fu_{i*}$. By the proof of Proposition 2.2, Steps 2,3,4 and 5, there are canonical isomorphisms $Eu_{i*} \longrightarrow - \otimes_{\mathcal{O}_{U_i}} \mathcal{E}_i$ and $Fu_{i*} \longrightarrow - \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i$. Hence, $\Delta * u_{i*}$ induces, via these isomorphisms, a morphism

$$- \otimes_{\mathcal{O}_{U_i}} \mathcal{E}_i \longrightarrow - \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i.$$

Hence, by Proposition 2.2, there is an induced morphism

$$\delta_i : \mathcal{E}_i \longrightarrow \mathcal{F}_i.$$

The fact that the maps $\{\delta_i\}_{i \in I}$ induce a morphism $\delta : W(E) \longrightarrow W(F)$ now follows from the naturality of η^i_{ij} and of ψ_{ij}. We leave it as a straightforward exercise for the reader to check that the naturality of the latter map follows from the naturality of (2) and (3).

We define

$$W_{\mathfrak{U}}(\Delta) := \delta.$$

It is straightforward to complete the verification that $W_{\mathfrak{U}}$ is a functor and we omit it.

5.3. **Properties of the Watts Functor.** The following result will not be used in the sequel.

Proposition 5.1. *The functor $W_{\mathfrak{U}} : \mathsf{Bimod}_T(X - Y) \to \mathsf{Qcoh}X \times Y$ is left-exact in the sense that if $F', F, F'' \in \mathsf{Bimod}_T(X - Y)$ are such that*

$$0 \to F' \xrightarrow{\Lambda} F \xrightarrow{\Xi} F'' \to 0 \tag{24}$$

is exact in $\mathsf{Funct}_T(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$, then

$$0 \to W_{\mathfrak{U}}(F') \xrightarrow{W_{\mathfrak{U}}(\Lambda)} W_{\mathfrak{U}}(F) \xrightarrow{W_{\mathfrak{U}}(\Xi)} W_{\mathfrak{U}}(F'')$$

is exact in $\mathsf{Qcoh}X \times Y$.

Proof. Exactness of (24) implies that, for all u_i,

$$0 \to F'u_{i*} \xrightarrow{\Lambda} Fu_{i*} \xrightarrow{\Xi} F''u_{i*} \to 0$$

is exact in $\mathsf{Funct}_T(U_i - Y)$. Thus, this sequence is exact in $\mathsf{Bimod}_T(U_i - Y)$. By Proposition 2.2, the induced sequence

$$0 \to \mathcal{F}'_i \to \mathcal{F}_i \to \mathcal{F}''_i \to 0$$

is exact in $\mathsf{Qcoh}U_i \times Y$. Therefore the induced sequences

$$0 \to \oplus_i v_{i*}\mathcal{F}'_i \to \oplus_i v_{i*}\mathcal{F}_i \to \oplus_i v_{i*}\mathcal{F}''_i \to 0$$

and

$$0 \to \oplus_{i<j} v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}'_i \to \oplus_{i<j} v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \to \oplus_{i<j} v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}''_i \to 0$$

are exact since v_i and v^i_{ij} are affine and v^i_{ij} is an open immersion. There is thus a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \oplus_i v_{i*}\mathcal{F}'_i \arrow[r] \arrow[d, "\theta_{F'}"'] & \oplus_i v_{i*}\mathcal{F}_i \arrow[r] \arrow[d, "\theta_F"'] & \oplus_i v_{i*}\mathcal{F}''_i \arrow[r] \arrow[d, "\theta_{F''}"] & 0 \\ 0 \arrow[r] & \oplus_{i<j} v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}'_i \arrow[r] & \oplus_{i<j} v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \arrow[r] & \oplus_{i<j} v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}''_i \arrow[r] & 0 \end{tikzcd}$$

Left-exactness of $W_{\mathfrak{U}}$ follows from the Snake Lemma. □

Proposition 5.2. *The functor $W_{\mathfrak{U}}$ is compatible with affine localization in the sense that if $\mathfrak{U}\cap U_k$ denotes the affine open cover $\{U_{ik}\}_{i\in I}$ of U_k, then*

$$W_{\mathfrak{U}\cap U_k}(Fu_{k*}) \cong v_k^* W_{\mathfrak{U}}(F)$$

naturally in F.

Proof. We prove the result in several steps.

Step 1: We note that the canonical basechange morphisms

$$u_k^* u_{i*} \longrightarrow u^k_{ik*}u^{i*}_{ik}$$

and

$$u^{i*}_{ik}u^i_{ij*} \longrightarrow u^{ik}_{ijk*}u^{ij*}_{ijk}$$

associated to the diagrams

$$\begin{tikzcd} U_{ik} \arrow[r, "u^i_{ik}"] \arrow[d, "u^k_{ik}"'] & U_i \arrow[d, "u_i"] \\ U_k \arrow[r, "u_k"'] & X \end{tikzcd}$$

and

$$\begin{tikzcd} U_{ijk} \arrow[r, "u^{ij}_{ijk}"] \arrow[d, "u^{ik}_{ijk}"'] & U_{ij} \arrow[d, "u^i_{ij}"] \\ U_{ik} \arrow[r, "u^i_{ik}"'] & U_i \end{tikzcd}$$

are isomorphisms. This follows from a routine affine computation, which we omit.

Step 2: We show that the composition

$$v^k_{ik*}v^{i*}_{ik} \to v^k_{ik*}v^{i*}_{ik}v^i_{ij*}v^{i*}_{ij} \to v^k_{ik*}v^{ik}_{ijk*}v^{ij*}_{ijk}v^{i*}_{ij} \to v^k_{ik*}v^{ik}_{ijk*}v^{ik*}_{ijk}v^{i*}_{ik}$$

whose left arrow is induced by the unit of the adjoint pair (v^{i*}_{ij}, v^i_{ij*}), whose middle arrow is the second basechange isomorphism from Step 1, and whose right arrow is induced from the canonical isomorphism

$$v^{ij*}_{ijk}v^{i*}_{ij} \xrightarrow{\cong} (v^i_{ij}v^{ij}_{ijk})^* = (v^i_{ik}v^{ik}_{ijk})^* \xrightarrow{\cong} v^{ik*}_{ijk}v^{i*}_{ik}, \tag{25}$$

is equal to the morphism induced by the unit of the adjoint pair $(v^{ik*}_{ijk}, v^{ik}_{ijk*})$. In order to prove this fact, it suffices to prove that the diagram

$$\begin{tikzcd} v^{i*}_{ik} \arrow[r] \arrow[d] & v^{i*}_{ik}v^i_{ij*}v^{i*}_{ij} \arrow[d] \\ v^{ik}_{ijk*}v^{ik*}_{ijk}v^{i*}_{ik} \arrow[r] & v^{ik}_{ijk*}v^{ij*}_{ijk}v^{i*}_{ij} \end{tikzcd}$$

whose top horizontal and left vertical are induced by canonical units and whose right vertical is induced by basechange isomorphisms, and whose bottom horizontal is induced by the inverse of (25), commutes. The verification of this fact follows from a routine affine computation, which we omit.

Step 3: Let F be an object of $\mathsf{Bimod}_T(X-Y)$ and consider the composition δ defined by

$$\begin{array}{rcl} \oplus_i v^k_{ik*}v^{i*}_{ik}\mathcal{F}_i & \xrightarrow{\cong} & \oplus_i v^*_k v_{i*}\mathcal{F}_i \\ & \xrightarrow{v^*_k\theta_F} & \oplus_{i<j} v^*_k v_{i*} v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \\ & \xrightarrow{\cong} & \oplus_{i<j} v^k_{ik*}v^{i*}_{ik}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \\ & \xrightarrow{\cong} & \oplus_{i<j} v^k_{ik*}v^{ik}_{ijk*}v^{ij*}_{ijk}v^{i*}_{ij}\mathcal{F}_i \\ & \xrightarrow{\cong} & \oplus_{i<j} v^k_{ik*}v^{ik}_{ijk*}v^{ik*}_{ijk}v^{i*}_{ik}\mathcal{F}_i \end{array}$$

whose first and third and fourth arrows are basechange isomorphisms from Step 1, and whose fifth arrow is induced by the canonical isomorphism (25). Let δ^{jl}_i denote the component of delta from the ith summand

$$v^k_{ik*}v^{i*}_{ik}\mathcal{F}_i$$

to the jlth summand

$$v^k_{jk*}v^{jk}_{jlk*}v^{jk*}_{jlk}v^{j*}_{jk}\mathcal{F}_j.$$

Then $\delta^{jl}_i = 0$ if i is not equal to j or l, δ^{ij}_i is induced by the unit of the adjoint pair $(v^{ik*}_{ijk}, v^{ik}_{ijk*})$, and δ^{ij}_j is equal to -1 times the composition

$$v^k_{jk*}v^{j*}_{jk}\mathcal{F}_j \longrightarrow v^k_{jk*}v^{jk}_{ijk*}v^{jk*}_{ijk}v^{j*}_{jk}\mathcal{F}_j \longrightarrow v^k_{ik*}v^{ik}_{ijk*}v^{ik*}_{ijk}v^{i*}_{ik}\mathcal{F}_i$$

whose left arrow is induced by the unit of the adjoint pair $(v^{jk*}_{ijk}, v^{jk}_{ijk*})$ and whose right arrow corresponds, under the equivalence of Proposition 2.2, to the composition of functors

$$\begin{array}{rl} - \otimes_{\mathcal{O}_{U_{ijk}}} v^{jk*}_{ijk}v^{j*}_{jk}\mathcal{F}_j & \xrightarrow{\cong} u^{jk}_{ijk*}(-) \otimes_{\mathcal{O}_{U_{jk}}} v^{j*}_{jk}\mathcal{F}_j \\ & \xrightarrow{\cong} u^j_{jk*}u^{jk}_{ijk*}(-) \otimes_{\mathcal{O}_{U_j}} \mathcal{F}_j \\ & \xrightarrow{\cong} Fu_{j*}u^j_{jk*}u^{jk}_{ijk*} \\ & \xrightarrow{=} Fu_{i*}u^i_{ik*}u^{ik}_{ijk*} \\ & \xrightarrow{\cong} - \otimes_{\mathcal{O}_{U_{ijk}}} v^{ik*}_{ijk}v^{i*}_{ik}\mathcal{F}_i \end{array} \tag{26}$$

whose first two arrows are (3), whose third arrow is the canonical isomorphism from the proof of Proposition 2.2, and whose last arrow is defined analogously. The fact that $\delta^{jl}_i = 0$ if i is not equal to j or l follows from the definition of θ_F. The assertion regarding δ^{ij}_i follows from Step 2.

It remains to verify the description of δ^{ij}_j. By Step 2, it suffices to prove that the diagram

$$\begin{tikzcd}
v_k^* v_{j*} v^j_{ij*} v^{j*}_{ij} \mathcal{F}_j \arrow[r] \arrow[d] & v_k^* v_{i*} v^i_{ij*} v^{i*}_{ij} \mathcal{F}_i \arrow[d] \\
v^k_{jk*} v^{j*}_{jk} v^j_{ij*} v^{j*}_{ij} \mathcal{F}_j \arrow[d] & v^k_{ik*} v^{i*}_{ik} v^i_{ij*} v^{i*}_{ij} \mathcal{F}_i \arrow[d] \\
v^k_{jk*} v^{jk}_{ijk*} v^{ij*}_{ijk} v^{j*}_{ij} \mathcal{F}_j \arrow[d] & v^k_{ik*} v^{ik}_{ijk*} v^{ij*}_{ijk} v^{i*}_{ij} \mathcal{F}_i \arrow[d] \\
v^k_{jk*} v^{jk}_{ijk*} v^{jk*}_{ijk} v^{j*}_{jk} \mathcal{F}_j \arrow[r] & v^k_{ik*} v^{ik}_{ijk*} v^{ik*}_{ijk} v^{i*}_{ik} \mathcal{F}_i
\end{tikzcd}$$

whose top horizontal is induced by the map ψ_{ji} defined in (21), whose bottom horizontal is induced by the morphism corresponding to (26) and whose verticals are induced by basechange isomorphisms from Step 1 and by canonical morphisms of the form (25), commutes. To that end, it suffices to prove that the diagram

$$\begin{tikzcd}
v_k^* v_{j*} v^j_{ij*} \arrow[r, "="] \arrow[d] & v_k^* v_{i*} v^i_{ij*} \arrow[d] \\
v^k_{jk*} v^{j*}_{jk} v^j_{ij*} \arrow[d] & v^k_{ik*} v^{i*}_{ik} v^i_{ij*} \arrow[d] \\
v^k_{jk*} v^{jk}_{ijk*} v^{ij*}_{ijk} \arrow[r, "="'] & v^k_{ik*} v^{ik}_{ijk*} v^{ij*}_{ijk}
\end{tikzcd} \tag{27}$$

whose verticals are basechange morphisms, and the diagram

$$\begin{tikzcd}
v^{ij*}_{ijk} v^{j*}_{ij} \mathcal{F}_j \arrow[r, "v^{ij*}_{ijk}\psi_{ji}"] \arrow[d] & v^{ij*}_{ijk} v^{i*}_{ij} \mathcal{F}_i \arrow[d] \\
v^{jk*}_{ijk} v^{j*}_{jk} \mathcal{F}_j \arrow[r] & v^{ik*}_{ijk} v^{i*}_{ik} \mathcal{F}_i
\end{tikzcd} \tag{28}$$

whose verticals are induced by canonical morphisms of the form (25) and whose bottom horizontal is the morphism corresponding to (26), commutes.

The commutivity of (27) follows from a straightforward affine computation, which we omit. To prove that (28) commutes, we first note that $v^{ij*}_{ijk}\psi_{ji}$ corresponds to the composition (26) by the naturality of (3). Hence, a straightforward computation shows that the commutivity of (28) follows from the commutivity of the four "corner" subdiagrams. The upper-left such diagram, for example, is the diagram

$$\begin{tikzcd}
- \otimes_{\mathcal{O}_{U_{ijk}}} v^{ij*}_{ijk} v^{j*}_{ij} \mathcal{F}_j \arrow[r] \arrow[d] & u^{ij}_{ijk*}(-) \otimes_{\mathcal{O}_{U_{ij}}} v^{j*}_{ij} \mathcal{F}_j \arrow[r] & u^j_{ij*} u^{ij}_{ijk*}(-) \otimes_{\mathcal{O}_{U_j}} \mathcal{F}_j \arrow[d, "="] \\
- \otimes_{\mathcal{O}_{U_{ijk}}} (v^j_{ij} v^{ij}_{ijk})^* \mathcal{F}_j \arrow[rr] & & (u^j_{ij} u^{ij}_{ijk})_*(-) \otimes_{\mathcal{O}_{U_j}} \mathcal{F}_j
\end{tikzcd}$$

whose horizontals are the isomorphisms (3) and whose left vertical is induced by the canonical isomorphism

$$v_{ijk}^{ij*}v_{ij}^{j*} \xrightarrow{\cong} (v_{ij}^{j}v_{ijk}^{ij})^*.$$

These corner subdiagrams commute by Lemma 3.2.

Step 4: Let $\mathcal{E}_i \in \mathsf{Qcoh} U_{ik} \times Y$ denote the object corresponding to the functor $Fu_{ik*} \in \mathsf{Bimod}_T(U_k - Y)$ in the proof of Proposition 2.2. We show that the diagram

$$\begin{array}{ccc} v_{ijk}^{jk*}v_{jk}^{j*}\mathcal{F}_j & \longrightarrow & v_{ijk}^{ik*}v_{ik}^{i*}\mathcal{F}_i \\ \Big\downarrow & & \Big\downarrow \\ v_{ijk}^{jk*}\mathcal{E}_j & \longrightarrow & v_{ijk}^{ik*}\mathcal{E}_{ik} \end{array}$$

whose top horizontal is the map (26), whose bottom horizontal is the map ψ_{ji} defined by (21) but corresponding to the functor Fu_{k*}, whose left vertical is induced by the composition

$$\begin{array}{rcl} - \otimes_{\mathcal{O}_{U_{ijk}}} v_{ijk}^{jk*}v_{jk}^{j*}\mathcal{F}_j & \xrightarrow{\cong} & u_{ijk*}^{jk}(-) \otimes_{\mathcal{O}_{U_{jk}}} v_{jk}^{j*}\mathcal{F}_j \\ & \xrightarrow{\cong} & u_{jk*}^{j}u_{ijk*}^{jk}(-) \otimes_{\mathcal{O}_{U_j}} \mathcal{F}_j \\ & \xrightarrow{\cong} & Fu_{j*}u_{jk*}^{j}u_{ijk*}^{jk} \\ & \xrightarrow{=} & Fu_{jk*}u_{ijk*}^{jk} \\ & \xrightarrow{\cong} & u_{ijk*}^{jk}(-) \otimes_{\mathcal{O}_{U_{jk}}} \mathcal{E}_j \\ & \xrightarrow{\cong} & - \otimes_{\mathcal{O}_{U_{ijk}}} v_{ijk}^{jk*}\mathcal{E}_j \end{array}$$

whose first, second, and sixth morphisms are (3), and whose third and fifth morphisms are the canonical ones constructed in Proposition 2.2, and whose right vertical is defined similarly, commutes. Upon expanding the rows and columns of the diagram, the proof is seen to follow from the trivial commutativity of the diagram

$$\begin{array}{ccc} Fu_{j*}u_{jk*}ju_{ijk*}^{jk} & \xrightarrow{=} & Fu_{i*}u_{ik*}^{i}u_{ijk*}^{ik} \\ {\scriptstyle =}\Big\downarrow & & \Big\downarrow{\scriptstyle =} \\ Fu_{jk*}u_{ijk*}^{jk} & \xrightarrow[=]{} & Fu_{ik*}u_{ijk*}^{ik}. \end{array}$$

Step 5: We complete the proof. We have

$$\begin{array}{rcl} v_k^* \ker \theta_F & \cong & \ker v_k^*\theta_F \\ & \cong & \ker \delta \\ & \cong & \ker \theta_{Fu_{k*}} \\ & = & W_{\mathfrak{U}\cap U_k}(Fu_{k*}) \end{array}$$

where the first isomorphism exists since u_k is an open immersion, the second isomorphism follows from Step 3, and the third isomorphism follows from Step 4. Naturality of this isomorphism in F follows from the naturality of each isomorphism above, which is easily verified. □

We now work towards a proof of Proposition 5.4. We begin by introducing some notation.

Let S be a scheme with finite open cover $\{W_i\}_{i\in I}$ where $I = \{1, \dots, n\}$ and let $\mathcal{F}$ be an object of $\mathsf{Qcoh}S$. Let

$$\psi_{ij} : w_{ij}^{i*} w_i^* \mathcal{F} \xrightarrow{\cong} (w_i w_{ij}^i)^* \mathcal{F} \xrightarrow{=} (w_j w_{ij}^j)^* \xrightarrow{\cong} w_{ij}^{j*} w_j^* \mathcal{F}$$

denote the canonical isomorphism, let

$$\phi_i^{ij} : w_{i*} w_i^* \mathcal{F} \longrightarrow w_{i*} w_{ij*}^i w_{ij}^{i*} w_i^* \mathcal{F}$$

be induced by the unit of (w_{ij}^{i*}, w_{ij*}^i), and let $\phi_j^{ij} = w_{ij*}\psi_{ji} \circ \phi_j^{ji}$. We define

$$\delta_{\mathcal{F}} : \oplus_i w_{i*} w_i^* \mathcal{F} \longrightarrow \oplus_{i<j} w_{i*} w_{ij*}^i w_{ij}^{i*} w_i^* \mathcal{F}$$

via its components

$$\delta_i^{jk} : w_{i*} w_i^* \mathcal{F} \longrightarrow w_{j*} w_{jk*}^j w_{jk}^{j*} w_j^* \mathcal{F}$$

as follows:

$$\delta_i^{jk} = \begin{cases} \phi_i^{ik} & \text{if } i = j, \\ -\phi_i^{ji} & \text{if } i = k, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

Lemma 5.3. *Retain the notation above. Then the map $\mathcal{F} \longrightarrow \oplus_i w_{i*} w_i^* \mathcal{F}$ induced by the units of (w_i^*, w_{i*}) is a kernel of $\delta_{\mathcal{F}}$.*

Proof. Let $\eta_{\mathcal{F}} : \mathcal{F} \to \oplus_i w_{i*} w_i^* \mathcal{F}$ be induced by the units of (w_i^*, w_{i*}) and let

$$\phi : \mathcal{N} \to \oplus_i v_{i*} v_i^* \mathcal{F}$$

be a morphism such that $\delta_{\mathcal{F}}\phi = 0$. We must show that there exists a unique $\psi : \mathcal{N} \to \mathcal{F}$ such that $\eta_{\mathcal{F}}\psi = \phi$. For each i, ϕ has a component $\phi_i : \mathcal{N} \to w_{i*} w_i^* \mathcal{F}$. By adjointness of (w_i^*, w_{i*}), there exists a morphism

$$\psi_i : w_i^* \mathcal{N} \to w_i^* \mathcal{F}$$

such that ϕ_i is the composition

$$\mathcal{N} \longrightarrow w_{i*} w_i^* \mathcal{N} \xrightarrow{w_{i*}\psi_i} w_{i*} w_i^* \mathcal{F}.$$

whose left map is the unit.

We claim there exists a unique $\psi : \mathcal{N} \to \mathcal{F}$ such that $w_i^* \psi = \psi_i$.

Suppose the claim holds. We first note that the diagram

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\eta_{\mathcal{N}}} & \oplus_i w_{i*} w_i^* \mathcal{N} \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \oplus_i w_{i*}\psi_i} \\ \mathcal{F} & \xrightarrow{\eta_{\mathcal{F}}} & \oplus_i w_{i*} w_i^* \mathcal{F} \end{array} \tag{30}$$

whose top horizontal is induced by unit morphisms, commutes. For, the application of w_i^* to (30) commutes for all i, hence the localization of (30) at any $p \in S$ is commutative, whence (30) is commutative.

We next note that if the claim above holds, then ψ is unique making (30) commute. For if $\gamma : \mathcal{N} \to \mathcal{F}$ makes (30) commute as well, then the commutivity of the diagram constructed by applying w_i^* to (30) and composing on the right with the counit $w_i^* w_{i*} \to \mathrm{id}_{\mathsf{Qcoh}S}$ implies that $w_i^* \gamma = \psi_i$. The claim tells us that ψ is unique with this property. Therefore $\gamma = \psi$.

Hence, to complete the proof of the lemma, it suffices to prove the claim. To prove the claim, it suffices, by the Main Theorem of Descent Theory [1, Section 6.1], to show that, for all pairs i, j, the diagram

$$\begin{tikzcd}
(w_i w^i_{ij})^*\mathcal{N} \arrow[r, "\cong"] \arrow[d, "="'] & w^{i*}_{ij} w^*_i \mathcal{N} \arrow[r, "w^{i*}_{ij}\psi_i"] & w^{i*}_{ij} w^*_i \mathcal{F} \arrow[r, "\cong"] & (w_i w^i_{ij})^*\mathcal{F} \arrow[d, "="] \\
(w_j w^j_{ij})^*\mathcal{N} \arrow[r, "\cong"'] & w^{j*}_{ij} w^*_j \mathcal{N} \arrow[r, "w^{j*}_{ij}\psi_j"'] & w^{j*}_{ij} w^*_j \mathcal{F} \arrow[r, "\cong"'] & (w_j w^j_{ij})^*\mathcal{F}
\end{tikzcd} \tag{31}$$

whose unlabelled arrows are canonical, commutes. To this end, we note that since $\delta_F\phi = 0$, the diagram

$$\begin{tikzcd}
\mathcal{N} \arrow[r] \arrow[d, "="'] & w_{i*}w^*_i\mathcal{N} \arrow[r, "w_{i*}\psi_i"] & w_{i*}w^*_i\mathcal{F} \arrow[r] & w_{i*}w^i_{ij*}w^{i*}_{ij}w_{i*}\mathcal{F} \arrow[d, "\cong"] \\
\mathcal{N} \arrow[r] & w_{j*}w^*_j\mathcal{N} \arrow[r, "w_{j*}\psi_j"'] & w_{j*}w^*_j\mathcal{F} \arrow[r] & w_{j*}w^j_{ij*}w^{j*}_{ij}w^*_j\mathcal{F}
\end{tikzcd} \tag{32}$$

whose right vertical is canonical and whose other unlabelled morphisms are units, commutes for all pairs i, j.

Applying w^*_{ij} to (32) yields the commutative diagram

$$\begin{tikzcd}
w^*_{ij}\mathcal{N} \arrow[r] \arrow[d, "="'] & w^*_{ij}w_{i*}w^*_i\mathcal{N} \arrow[r, "w^*_{ij}w_{i*}\psi_i"] & w^*_{ij}w_{i*}w^*_i\mathcal{F} \arrow[r] & w^*_{ij}w_{i*}w^i_{ij*}w^{i*}_{ij}w_{i*}\mathcal{F} \arrow[d, "\cong"] \\
w^*_{ij}\mathcal{N} \arrow[r] & w^*_{ij}w_{j*}w^*_j\mathcal{N} \arrow[r, "w^*_{ij}w_{j*}\psi_j"'] & w^*_{ij}w_{j*}w^*_j\mathcal{F} \arrow[r] & w^*_{ij}w_{j*}w^j_{ij*}w^{j*}_{ij}w^*_j\mathcal{F}.
\end{tikzcd} \tag{33}$$

It follows from a straightforward computation that the commutativity of (33) implies the commutativity of the diagram

$$\begin{tikzcd}
w^{i*}_{ij}w^*_iw_{i*}w^*_i\mathcal{N} \arrow[r, "w^{i*}_{ij}w^*_iw_{i*}\psi_i"] & w^{i*}_{ij}w^*_iw_{i*}w^*_i\mathcal{F} \arrow[r] \arrow[d] & w^{i*}_{ij}w^*_i\mathcal{F} \arrow[d, "="] \\
w^{i*}_{ij}w^*_i\mathcal{N} \arrow[u] \arrow[d, "\cong"'] & w^{i*}_{ij}w^*_iw_{i*}w^i_{ij*}w^{i*}_{ij}w^*_i\mathcal{F} \arrow[r] \arrow[d, "\cong"'] & w^{i*}_{ij}w^*_i\mathcal{F} \arrow[d, "\cong"] \\
w^{j*}_{ij}w^*_j\mathcal{N} \arrow[d] & w^{j*}_{ij}w^*_jw_{j*}w^j_{ij*}w^{j*}_{ij}w^*_j\mathcal{F} \arrow[r] & w^{j*}_{ij}w^*_j\mathcal{F} \arrow[d, "="] \\
w^{j*}_{ij}w^*_jw_{j*}w^*_j\mathcal{N} \arrow[r, "w^{j*}_{ij}w^*_jw_{j*}\psi_j"'] & w^{j*}_{ij}w^*_jw_{j*}w^*_j\mathcal{F} \arrow[r] \arrow[u] & w^{j*}_{ij}w^*_j\mathcal{F}
\end{tikzcd} \tag{34}$$

whose unadorned arrows are induced by units and counits, and whose unlabelled isomorphisms are canonical. As one can check, the outside circuit of this diagram equals (31). The claim, and hence the lemma, follows. □

Proposition 5.4. *If $\mathcal{F}$ is an object of $\mathsf{Qcoh}X \times Y$ and F is an object of $\mathsf{Bimod}_T(X-Y)$ such that $F = - \otimes_{\mathcal{O}_X} \mathcal{F}$, then $W_{\mathfrak{U}}(F) \cong \mathcal{F}$.*

Proof. We first construct an isomorphism $\psi_i : \mathcal{F}_i \longrightarrow v_i^*\mathcal{F}$ as corresponding, via Proposition 2.2, to the composition

$$\begin{aligned} - \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i &\xrightarrow{\cong} Fu_{i*} \\ &\xrightarrow{=} u_{i*}(-) \otimes_{\mathcal{O}_X} \mathcal{F} \\ &\xrightarrow{\cong} - \otimes_{\mathcal{O}_{U_i}} v_i^*\mathcal{F} \end{aligned}$$

whose first arrow is the canonical isomorphism from the proof of Proposition 2.2, and whose third arrow is (3).

By Lemma 5.3, it suffices to prove that the diagram

$$\begin{tikzcd} \oplus_i v_{i*}v_i^*\mathcal{F} \arrow[r, "\delta_F"] \arrow[d, "\oplus_i v_{i*}\psi_i^{-1}"'] & \oplus_{i<j} v_{i*}v^i_{ij*}v^{i*}_{ij}v_i^*\mathcal{F} \arrow[d, "\oplus_{i<j} v_{i*}v^i_{ij*}v^{i*}_{ij}\psi_i^{-1}"] \\ \oplus_i v_{i*}\mathcal{F}_i \arrow[r, "\theta_F"'] & \oplus_{i<j} v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \end{tikzcd}$$

commutes, where we specialize the notation for the definition of δ_F to our situation by setting $S = X \times Y$, $W_i = U_i \times Y$, and $w_i = v_i$.

Let δ_i^{ij} denote the component of $\delta_{\mathcal{F}}$ from the ith summand to the i, jth summand, and let θ_i^{ij} be defined similarly. The verification that

$$v_{i*}v^i_{ij*}v^{i*}_{ij}\psi_i^{-1} \circ \delta_i^{ij} = \theta_i^{ij} \circ v_{i*}\psi_i^{-1}$$

is trivial, so that it remains to check that the diagram

$$\begin{tikzcd} v_{j*}\mathcal{F}_j \arrow[r] \arrow[d, "v_{j*}\psi_j"'] & v_{j*}v^j_{ij*}v^{j*}_{ij}\mathcal{F}_j \arrow[r, "v_{ij*}\psi_{ji}"] \arrow[d, "v_{j*}v^j_{ij*}v^{j*}_{ij}\psi_j"'] & v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \arrow[d, "v_{i*}v^i_{ij*}v^{i*}_{ij}\psi_i"] \\ v_{j*}v_j^*\mathcal{F} \arrow[r] & v_{j*}v^j_{ij*}v^{j*}_{ij}v_j^*\mathcal{F} \arrow[r, "\cong"'] & v_{i*}v^i_{ij*}v^{i*}_{ij}v_i^*\mathcal{F} \end{tikzcd}$$

whose unadorned arrows are induced by units, and whose unlabelled isomorphism is canonical, commutes. The left square commutes by naturality of units, while to prove the right square commutes, it suffices to prove that the square

$$\begin{tikzcd} v^{j*}_{ij}\mathcal{F}_j \arrow[r, "\psi_{ji}"] \arrow[d, "v^{j*}_{ij}\psi_j"'] & v^{i*}_{ij}\mathcal{F}_i \arrow[d, "v^{i*}_{ij}\psi_i"] \\ v^{j*}_{ij}v_j^*\mathcal{F} \arrow[r, "\cong"'] & v^{i*}_{ij}v_i^*\mathcal{F} \end{tikzcd} \tag{35}$$

whose unlabelled isomorphism is canonical, commutes. It suffices to prove that the diagram resulting in applying the functor $- \otimes_{\mathcal{O}_{U_{ij}}} (-)$ to (35) commutes, by Proposition 2.2. Upon expanding the resulting diagram, it is easy to check that the commutivity of (35) follows from the commutivity of the diagram

$$\begin{tikzcd} - \otimes_{\mathcal{O}_{U_{ij}}} v^{j*}_{ij}v_j^*\mathcal{F} \arrow[r, "\cong"] \arrow[d, "\cong"'] & u^j_{ij*}(-) \otimes_{\mathcal{O}_{U_j}} v_j^*\mathcal{F} \arrow[r, "\cong"] & u_{j*}u^j_{ij*}(-) \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[d, "="] \\ - \otimes_{\mathcal{O}_{U_{ij}}} v^{i*}_{ij}v_i^*\mathcal{F} \arrow[r, "\cong"'] & u^i_{ij*}(-) \otimes_{\mathcal{O}_{U_i}} v_i^*\mathcal{F} \arrow[r, "\cong"'] & u_{i*}u^i_{ij*}(-) \otimes_{\mathcal{O}_X} \mathcal{F} \end{tikzcd}$$

whose left vertical is canonical and whose horizontal isomorphisms are (3). This follows from Lemma 3.2. □

6. The Watts Transformation

Our goal in this section is to prove the generalization of Watts Theorem mentioned in the introduction (Theorem 1.4). Throughout this section, we assume that if $\mathcal{M}$ is an object of $\mathsf{Qcoh}X$, then there exists a flat object $\mathcal{L}$ in $\mathsf{Qcoh}X$ and an epimorphism $\mathcal{L} \to \mathcal{M}$. We begin by constructing, for each F in $\mathsf{Bimod}_T(X-Y)$, a natural transformation

$$\Gamma_F : F \longrightarrow - \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$$

which we show is natural in F.

It will follow readily from the construction that if F is exact, then $\ker \Gamma_F = 0$ (Proposition 6.1), if $- \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ is right-exact then $\operatorname{cok} \Gamma_F = 0$ (Proposition 6.2), if $F = - \otimes_{\mathcal{O}_X} \mathcal{F}$ for some object $\mathcal{F}$ in $\mathsf{Qcoh}X \times Y$ then Γ_F is an isomorphism (Proposition 6.4), and Γ is compatible with affine localization (Proposition 6.7). As a consequence, of these properties, we show that the kernel and cokernel of Γ_F are totally global (Corollary 6.8). It follows immediately that Γ_F is an isomorphism if X is affine. We also show that if $F \in \mathsf{Bimod}_T(X-Y)$ then $- \otimes_{\mathcal{O}_X} W(F)$ serves as a "best" approximation of F by tensoring with a bimodule (Corollary 6.5).

6.1. **Construction of the Watts Transformation.** For each non-flat object $\mathcal{M}$ of $\mathsf{Qcoh}X$, we fix a flat presentation

$$\mathcal{L}_1 \xrightarrow{\xi_1} \mathcal{L}_0 \xrightarrow{\xi_0} \mathcal{M}. \tag{36}$$

Let F be an object of $\mathsf{Bimod}_T(X-Y)$. We construct a natural transformation

$$\Gamma_F : F \longrightarrow - \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$$

and show it is natural in F.

Step 1: We note that for any morphism $\lambda : \mathcal{M} \longrightarrow \mathcal{N}$ in $\mathsf{Qcoh}X$, there is a canonical epimorphism

$$\pi_1 : F(\ker \lambda) \longrightarrow \ker F\lambda$$

which is natural in the sense that if

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\lambda"] \arrow[d] & \mathcal{N} \arrow[d] \\ \mathcal{M}' \arrow[r, "\lambda'"'] & \mathcal{N}' \end{tikzcd}$$

commutes, then the induced maps $\iota : F(\ker \lambda) \longrightarrow F(\ker \lambda')$ and $\iota' : \ker F\lambda \longrightarrow \ker F\lambda'$ make the diagram

$$\begin{tikzcd} F(\ker \lambda) \arrow[r] \arrow[d, "\iota"'] & \ker F\lambda \arrow[d, "\iota'"] \\ F(\ker \lambda') \arrow[r] & \ker F\lambda' \end{tikzcd}$$

whose horizontals are the canonical epimorphisms, commute. One can check that the natural epimorphism comes from the universal property of the kernel of a morphism.

Step 2: Let $\mathcal{L}$ be a flat object in $\mathsf{Qcoh}X$. We construct a morphism

$$\Gamma_{F\mathcal{L}} : F(\mathcal{L}) \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$$

in the category $\mathsf{Qcoh}Y$. Specialize the notation preceding Lemma 5.3 to the case that $S = X$ and $W_i = U_i$. By Lemma 5.3, the morphism

$$\mathcal{L} \longrightarrow \oplus_i u_{i*}u_i^*\mathcal{L}$$

induced by unit morphisms is a kernel of

$$\delta_{\mathcal{L}} : \oplus_i u_{i*}u_i^*\mathcal{L} \longrightarrow \oplus_i u_{i*}u^i_{ij*}u^{i*}_{ij}u_i^*\mathcal{L}.$$

By Step 1, there is an epimorphism

$$\pi : F(\mathcal{L}) \longrightarrow \ker F(\delta_{\mathcal{L}}),$$

where we abuse notation by commuting direct sums through F:

$$F(\delta_{\mathcal{L}}) : \oplus_i F u_{i*}u_i^*\mathcal{L} \longrightarrow \oplus_{i<j} F u_{i*}u^i_{ij*}u^{i*}_{ij}u_i^*\mathcal{L}.$$

Now, consider the composition

$$\gamma_i : F u_{i*}u_i^*\mathcal{L} \longrightarrow u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} v_{i*}\mathcal{F}_i$$

whose left arrow is the canonical isomorphism from Proposition 2.2 and whose right arrow is from (2), and the composition γ_{ij}:

$$\begin{aligned} F u_{i*}u^i_{ij*}u^{i*}_{ij}u_i^*\mathcal{L} &\xrightarrow{\cong} u^i_{ij*}u^{i*}_{ij}u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \\ &\xrightarrow{\cong} u^{i*}_{ij}u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_{ij}}} v^{i*}_{ij}\mathcal{F}_i \\ &\xrightarrow{\cong} u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_i}} v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \\ &\xrightarrow{\cong} \mathcal{L} \otimes_{\mathcal{O}_X} v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \end{aligned}$$

whose first arrow is from Proposition 2.2, whose second and fourth arrow are (3) and whose third arrow is (2).

We claim

$$(\oplus_{i<j}\gamma_{ij}) \circ F\delta_{\mathcal{L}} = \mathcal{L} \otimes_{\mathcal{O}_X} \theta_{\mathcal{F}} \circ (\oplus_i \gamma_i).$$

We prove the claim component-wise. To this end, we first show that

$$\gamma_{ij} \circ F\delta_i^{ij} = \mathcal{L} \otimes_{\mathcal{O}_X} \theta_i^{ij} \circ \gamma_i, \tag{37}$$

i.e. we show that the diagram

$$\begin{array}{ccc} F u_{i*}u_i^*\mathcal{L} & \longrightarrow & F u_{i*}u^i_{ij*}u^{i*}_{ij}u_i^*\mathcal{L} \\ \downarrow & & \downarrow \\ u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i & \longrightarrow & u^i_{ij*}u^{i*}_{ij}u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \\ \downarrow & & \downarrow \\ \mathcal{L} \otimes_{\mathcal{O}_X} v_{i*}\mathcal{F}_i & & u^{i*}_{ij}u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_{ij}}} v^{i*}_{ij}\mathcal{F}_i \\ \downarrow & & \downarrow \\ \mathcal{L} \otimes_{\mathcal{O}_X} v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i & \longrightarrow & u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_i}} v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \end{array}$$

whose two top horizontals and bottom-left vertical are induced by the units, whose top verticals are from Proposition 2.2, whose left-middle vertical is (2), whose right-middle vertical is (3), and whose right-bottom vertical and bottom horizontal is (2), commutes. The top square commutes by the naturality of the canonical isomorphism from Proposition 2.2, and the bottom rectangle and can split into two subdiagrams via the morphism

$$u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \longrightarrow u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_i}} v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i$$

induced by the unit of (v^{i*}_{ij}, v^i_{ij*}). The left subdiagram commutes by the naturality of (2), while the right subdiagram commutes by the commutivity of (4).

In order to complete the proof of the claim, we must prove

$$\gamma_{ij} \circ F\delta^{ij}_j = \mathcal{L} \otimes_{\mathcal{O}_X} \theta^{ij}_j \circ \gamma_j.$$

But since (37) holds when i and j are interchanged, it suffices to show that the diagram

$$\begin{array}{ccc}
Fu_{j*}u^j_{ij*}u^{j*}_{ij}u^*_j\mathcal{L} & \longrightarrow & Fu_{i*}u^i_{ij*}u^{i*}_{ij}u^*_i\mathcal{L} \\
\downarrow & & \downarrow \\
u^j_{ij*}u^{j*}_{ij}u^*_j\mathcal{L} \otimes_{\mathcal{O}_{U_j}} \mathcal{F}_j & & u^i_{ij*}u^{i*}_{ij}u^*_i\mathcal{L} \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \\
\downarrow & & \downarrow \\
u^{j*}_{ij}u^*_j\mathcal{L} \otimes_{\mathcal{O}_{U_{ij}}} v^{j*}_{ij}\mathcal{F}_j & & u^{i*}_{ij}u^*_i\mathcal{L} \otimes_{\mathcal{O}_{U_{ij}}} v^{i*}_{ij}\mathcal{F}_i \\
\downarrow & & \downarrow \\
u^*_j\mathcal{L} \otimes_{\mathcal{O}_{U_j}} v^j_{ij*}v^{j*}_{ij}\mathcal{F}_j & & u^*_i\mathcal{L} \otimes_{\mathcal{O}_{U_i}} v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \\
\downarrow & & \downarrow \\
\mathcal{L} \otimes_{\mathcal{O}_X} v_{j*}v^j_{ij*}v^{j*}_{ij}\mathcal{F}_j & \longrightarrow & \mathcal{L} \otimes_{\mathcal{O}_X} v_{j*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i
\end{array}$$

whose top horizontal is induced by the canonical isomorphism

$$u^{j*}_{ij}u^*_j \xrightarrow{\cong} (u_j u^j_{ij})^* \xrightarrow{=} (u_i u^i_{ij})^* \xrightarrow{\cong} u^{i*}_{ij}u^*_i$$

whose verticals are (3) and (2), and whose bottom horizontal is induced by

$$\psi_{ji} : v^{j*}_{ij}\mathcal{F}_j \xrightarrow{\cong} v^{i*}_{ij}\mathcal{F}_i$$

commutes. This follows trivially from the definition of ψ_{ji}.

It follows from the claim that there is an isomorphism

$$\pi_2 : \ker \mathcal{F}(\delta_{\mathcal{L}}) \xrightarrow{\cong} \ker(\mathcal{L} \otimes_{\mathcal{O}_X} \theta_F).$$

Finally, since $\mathcal{L}$ is flat and pr_{2*} is left-exact, there is a canonical isomorphism

$$\pi_3 : \ker(\mathcal{L} \otimes_{\mathcal{O}_X} \theta_F) \xrightarrow{\cong} \mathcal{L} \otimes_{\mathcal{O}_X} \ker \theta_F.$$

We define

$$\Gamma_{F\mathcal{L}} = \pi_3\pi_2\pi_1$$

where π_1 is the morphism constructed in Step 1.

Step 3: We show Γ_F is natural on flats, i.e. we show that if

$$\psi : \mathcal{L} \longrightarrow \mathcal{L}'$$

is a morphism of flat objects in $\mathsf{Qcoh}X$ *then the diagram*

$$\begin{tikzcd} F\mathcal{L} \arrow[r, "F\psi"] \arrow[d, "\Gamma_{F\mathcal{L}}"'] & F\mathcal{L}' \arrow[d, "\Gamma_{F\mathcal{L}'}"] \\ \mathcal{L} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \arrow[r, "\psi\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)"'] & \mathcal{L}' \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \end{tikzcd} \tag{38}$$

commutes. We leave it as an easy exercise for the reader to check that the diagram

$$\begin{tikzcd} \oplus_i F u_{i*}u_i^*\mathcal{L} \arrow[r, "\delta_{\mathcal{L}}"] \arrow[d] & \oplus_{i<j} F u_{i*}u^i_{ij*}u^{i*}_{ij}u_i^*\mathcal{L} \arrow[d] \\ \oplus_i F u_{i*}u_i^*\mathcal{L}' \arrow[r, "\delta_{\mathcal{L}'}"'] & \oplus_{i<j} F u_{i*}u^i_{ij*}u^{i*}_{ij}u_i^*\mathcal{L}' \end{tikzcd}$$

whose verticals are induced by ψ, commutes. Therefore, by Step 1, there exists a morphism $\psi' : \ker F\delta_{\mathcal{L}} \longrightarrow \ker F\delta_{\mathcal{L}'}$ such that the diagram

$$\begin{tikzcd} F\mathcal{L} = F\ker\delta_{\mathcal{L}} \arrow[r] \arrow[d, "F\psi"'] & \ker F\delta_{\mathcal{L}} \arrow[d, "\psi'"] \\ F\mathcal{L}' = F\ker\delta_{\mathcal{L}'} \arrow[r] & \ker F\delta_{\mathcal{L}'} \end{tikzcd}$$

whose horizontals are canonical, commutes. The remainder of the argument that (38) commutes is straightforward, and omitted.

Step 4: We construct, for $\mathcal{M}$ *an object in* $\mathsf{Qcoh}X$, *a morphism*

$$\gamma_{F\mathcal{M}} : F\mathcal{M} \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F).$$

Applying F to the flat presentation (36) yields a diagram

$$\begin{tikzcd} F\mathcal{L}_1 \arrow[r, "F\xi_1"] \arrow[d, "\Gamma_{F\mathcal{L}_1}"'] & F\mathcal{L}_0 \arrow[r, "F\xi_0"] \arrow[d, "\Gamma_{F\mathcal{L}_0}"] & F\mathcal{M} \\ \mathcal{L}_1 \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \arrow[r, "\xi_1\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)"'] & \mathcal{L}_0 \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \arrow[r, "\xi_0\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)"'] & \mathcal{M} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \end{tikzcd} \tag{39}$$

whose top right horizontal and verticals are epimorphisms, which commutes by Step 3. Thus, there exists a unique morphism

$$\gamma_{F\mathcal{M}} : F\mathcal{M} \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$$

making

$$\begin{tikzcd} F\mathcal{L}_0 \arrow[r, "F\xi_0"] \arrow[d, "\Gamma_{F\mathcal{L}_0}"'] & F\mathcal{M} \arrow[d, "\gamma_{F\mathcal{M}}"] \\ \mathcal{L}_0 \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \arrow[r, "\xi_0\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)"'] & \mathcal{M} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \end{tikzcd}$$

commute. We will show, in Step 6, that $\gamma_{F\mathcal{M}}$ is independent of presentation chosen.
Step 5: We show that if $\phi : \mathcal{M} \to \mathcal{N}$ is a morphism in $\mathsf{Qcoh} X$, then the diagram

$$\begin{tikzcd} F\mathcal{M} \arrow[r, "F\phi"] \arrow[d, "\gamma_{F\mathcal{M}}"'] & F\mathcal{N} \arrow[d, "\gamma_{F\mathcal{N}}"] \\ \mathcal{M} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}} \arrow[r, "\phi \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)"'] & \mathcal{N} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}} \end{tikzcd}$$

commutes. Suppose

$$\mathcal{L}_1' \longrightarrow \mathcal{L}_0' \xrightarrow{\pi'} \mathcal{N}$$

is the flat presentation chosen for $\mathcal{N}$. Then there exists a flat presentation

$$\mathcal{L} \longrightarrow \mathcal{L}_0 \oplus \mathcal{L}_0' \xrightarrow{\phi\pi \oplus \pi'} \mathcal{N}$$

for $\mathcal{N}$, with corresponding morphism $\gamma'_{F\mathcal{N}}$ constructed as in Step 4. Therefore, the outer circuit of the diagram

$$\begin{tikzcd} F(\mathcal{L}_0 \oplus \mathcal{L}_0') \arrow[r, "F(\pi \oplus \mathrm{id}_{\mathcal{L}_0'})"] \arrow[d, "\Gamma_{F(\mathcal{L}_0 \oplus \mathcal{L}_0')}"'] & F(\mathcal{M} \oplus \mathcal{L}_0') \arrow[r, "F(\phi \oplus \pi')"] \arrow[d, "\Gamma_{F(\mathcal{M} \oplus \mathcal{L}_0')}"'] & F\mathcal{N} \arrow[d, "\gamma'_{F\mathcal{N}}"] \\ (\mathcal{L}_0 \oplus \mathcal{L}_0') \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \arrow[r] & (\mathcal{M} \oplus \mathcal{L}_0') \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \arrow[r] & \mathcal{N} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \end{tikzcd}$$

whose left-bottom vertical is induced by $\pi \oplus \mathrm{id}_{\mathcal{L}_0'}$ and whose bottom-right vertical is induced by $\phi \oplus \pi'$, commutes. By Step 3 and the universal property of direct-limit, the outside of the diagram constructed by placing the diagram
(40)

$$\begin{tikzcd} F\mathcal{L}_0 \oplus F\mathcal{L}_0' \arrow[r, "F\phi \oplus F\,\mathrm{id}_{\mathcal{L}_0'}"] \arrow[d, "\Gamma_{F\mathcal{L}_0} \oplus \Gamma_{F\mathcal{L}_0'}"'] & F\mathcal{M} \oplus F\mathcal{L}_0' \arrow[d, "\gamma_{F\mathcal{M}} \oplus \Gamma_{F\mathcal{L}_0'}"'] \\ \mathcal{L}_0 \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \oplus \mathcal{L}_0' \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \arrow[r] & \mathcal{M} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \oplus \mathcal{L}_0' \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \end{tikzcd}$$

whose bottom horizontal is $\pi \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \oplus \mathrm{id}_{\mathcal{L}_0'} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$, to the left of the diagram

$$\begin{tikzcd} F\mathcal{M} \oplus F\mathcal{L}_0' \arrow[r, "F\phi \oplus F\pi'"] \arrow[d, "\gamma_{F\mathcal{M}} \oplus \Gamma_{F\mathcal{L}_0'}"'] & F\mathcal{N} \arrow[d, "\gamma'_{F\mathcal{N}}"] \\ \mathcal{M} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \oplus \mathcal{L}_0' \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \arrow[r] & \mathcal{N} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \end{tikzcd} \tag{41}$$

whose bottom horizontal is induced by $\phi \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ and $\pi' \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$, commutes. The commutivity of (40) follows from the construction of $\gamma_{F\mathcal{M}}$. Thus, since the upper horizontal in (40) is an epimorphism, (41) commutes as well. Restricting both routes of (41) to $F\mathcal{L}_0'$ and using the uniqueness of $\gamma_{F\mathcal{N}}$ implies that $\gamma_{F\mathcal{N}} = \gamma'_{F\mathcal{N}}$. On the other hand, restricting both routes of (41) to $F\mathcal{M}$ allows us to conclude that

$$\gamma_{F\mathcal{M}} \phi \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) = \gamma'_{F\mathcal{N}} F\phi.$$

Step 5 follows.
Step 6: We show that $\gamma_{F\mathcal{M}}$ is independent of presentation. Let $\gamma'_{F\mathcal{M}} : F\mathcal{M} \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ denote the morphism constructed in Step 4 using a flat presentation

$$\mathcal{L}_1' \longrightarrow \mathcal{L}_0' \longrightarrow \mathcal{M}.$$

Now apply Step 5 to conclude that the diagram

$$\begin{tikzcd} F\mathcal{M} \arrow[r, "F\,\mathrm{id}_{\mathcal{M}}"] \arrow[d, "\gamma_{F\mathcal{M}}"'] & F\mathcal{M} \arrow[d, "\gamma'_{F\mathcal{M}}"] \\ \mathcal{M}\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \arrow[r, "\mathrm{id}_{\mathcal{M}}\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)"'] & \mathcal{M}\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \end{tikzcd}$$

commutes. Step 6 follows.

We define

$$\Gamma_{F\mathcal{M}} := \gamma_{F\mathcal{M}}.$$

Step 7: We show that Γ_F *is natural in* $\mathcal{M}$. This follows from Step 5 in light of the definition of $\Gamma_{F\mathcal{M}}$ given in Step 6.

Step 8: We show Γ_F *is natural in* F. It suffices to check that if $\mathcal{L}$ is a flat object in $\mathsf{Qcoh}X$ and $\eta : F \to G$ is a morphism in $\mathsf{Bimod}_T(X-Y)$ then the diagram

$$\begin{tikzcd} F(\mathcal{L}) \arrow[r, "\eta_{\mathcal{L}}"] \arrow[d, "\gamma_{F\mathcal{L}}"'] & G(\mathcal{L}) \arrow[d, "\gamma_{G\mathcal{L}}"] \\ \mathcal{L}\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \arrow[r, "\mathcal{L}\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(\eta)"'] & \mathcal{L}\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(G) \end{tikzcd} \tag{42}$$

commutes. Sufficiency follows from the right-exactness of F. The proof that (42) commutes is straightforward, and we omit it.

6.2. **Properties of the Watts Transformation.**

Proposition 6.1. *If* $F \in \mathsf{Bimod}_T(X-Y)$ *is exact, then* $\ker\Gamma_F = 0$.

Proof. It suffices, by the naturality of Γ_F and the fact that every object of $\mathsf{Qcoh}X$ has a flat presentation, to prove that $\Gamma_{F\mathcal{L}}$ is an isomorphism for flat objects $\mathcal{L}$ in $\mathsf{Qcoh}X$. By the construction of $\Gamma_{F\mathcal{L}}$, it suffices to show that canonical map

$$\pi_1 : F(\mathcal{L}) = F(\ker\delta_{\mathcal{L}}) \longrightarrow \ker F\delta_{\mathcal{L}},$$

from Step 2 of the construction of Γ_F, is an isomorphism. This follows from the left-exactness of F. □

Proposition 6.2. *Let* F *be an object of* $\mathsf{Bimod}_T(X-Y)$. *If* $-\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ *is right-exact then* Γ_F *is epic in the category* $\mathsf{Funct}_T(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$.

Proof. Suppose $-\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ is right-exact, and let

$$\mathcal{L} \longrightarrow \mathcal{M}$$

be an epimorphism from a flat object $\mathcal{L}$. Then the induced diagram

$$\begin{tikzcd} F\mathcal{L} \arrow[r] \arrow[d, "\Gamma_{F\mathcal{L}}"'] & F\mathcal{M} \arrow[d, "\Gamma_{F\mathcal{M}}"] \\ \mathcal{L}\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \arrow[r] & \mathcal{M}\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \end{tikzcd}$$

has epic horizontals by hypothesis and epic left vertical by construction of $\Gamma_{F\mathcal{L}}$. Hence, $\Gamma_{F\mathcal{M}}$ is epic as well. □

Corollary 6.3. *Let* $F \in \mathsf{Bimod}_T(X-Y)$ *be a totally global, exact functor such that* $-\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ *is right-exact. Then* $F = 0$.

Proof. Since F is exact, $\ker \Gamma_F = 0$ by Proposition 6.1. Since, $- \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ is right-exact, $\operatorname{cok} \Gamma_F = 0$ by Proposition 6.2. Hence, $F \cong - \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$. Thus, since F is totally global, $F = 0$ by Example 4.4. □

Proposition 6.4. *If F is an object of $\mathsf{Bimod}_T(X - Y)$ such that $F = - \otimes_{\mathcal{O}_X} \mathcal{F}$ for some object $\mathcal{F}$ in $\mathsf{Qcoh}X \times Y$, then Γ_F is an isomorphism.*

Proof. By Proposition 5.4, $W_{\mathfrak{U}}(F) \cong \mathcal{F}$. Since F is right-exact, so is $- \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$. Hence, by Proposition 6.2, Γ_F is epic in $\mathsf{Funct}_T(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$.

It remains to prove that $\ker \Gamma_F = 0$. Since $- \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ is right-exact, it suffices to show that if $\mathcal{L}$ is flat, then $\ker \Gamma_{F\mathcal{L}} = 0$. By the construction of $\Gamma_{F\mathcal{L}}$ in Step 2, it suffices to show that the canonical map

$$F(\mathcal{L}) = F(\ker \delta_{\mathcal{L}}) \longrightarrow \ker F\delta_{\mathcal{L}}$$

is monic. To prove this, we note that the morphism

$$\mathcal{L} \longrightarrow \oplus_i u_{i*}u_i^*\mathcal{L}$$

induced by unit morphisms is a kernel, by Lemma 5.3. Thus, it suffices to prove that the morphism

$$F\mathcal{L} \longrightarrow \oplus_i F u_{i*}u_i^*\mathcal{L}$$

induced by unit morphisms, is monic. The diagram

$$\begin{tikzcd}
\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{L} \otimes_{\mathcal{O}_X} v_{i*}v_i^*\mathcal{F} \arrow[d] \\
u_{i*}u_i^*\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r] & u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_i}} v_i^*\mathcal{F}
\end{tikzcd}$$

whose top horizontal and left vertical are induced by unit morphisms, and whose right vertical and bottom horizontal are (3) and (2) respectively, commutes by Lemma 3.1. Hence, it suffices to prove that the morphism

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} \oplus_i v_{i*}v_i^*\mathcal{F} \tag{43}$$

induced by unit morphisms is monic. But since the morphism

$$\mathcal{F} \xrightarrow{\phi} \oplus_i v_{i*}v_i^*\mathcal{F}$$

induced by units is a kernel by 5.3, it is monic. Since $\mathcal{L}$ is flat, so is $\mathrm{pr}_1^*\mathcal{L}$. Hence $\mathrm{pr}_1^*\mathcal{L} \otimes_{\mathcal{O}_{X\times Y}} \phi$ is monic. Finally, since pr_{2*} is left-exact, it follows that (43) is monic. The proof follows. □

Corollary 6.5. *Let $\mathcal{F}'$ be an object of $\mathsf{Qcoh}X \times Y$ and suppose $F' := - \otimes_{\mathcal{O}_X} \mathcal{F}'$ is an object in $\mathsf{Bimod}_T(X-Y)$. If $\Phi : F \to F'$ is a morphism in $\mathsf{Bimod}_T(X-Y)$, then Φ factors through Γ_F. If $- \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ is right-exact, then Φ factors uniquely.*

Proof. Since Γ_G is natural in G, the diagram

$$\begin{tikzcd}
F \arrow[r, "\Phi"] \arrow[d, "\Gamma_F"'] & F' \arrow[d, "\Gamma_{F'}"] \\
- \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \arrow[r, "- \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(\Phi)"'] & - \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F')
\end{tikzcd}$$

commutes. Since $\Gamma_{F'}$ is an isomorphism by Proposition 6.4, the first assertion follows. The second assertion follows from the fact that if $- \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ is right-exact, then Γ_F is an epimorphism in the category $\mathsf{Funct}_T(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ by Proposition 6.2. $\square$

Proposition 6.6. *Let $F \in \mathsf{Bimod}_T(X-Y)$ and suppose $\mathfrak{U}$ and $\mathfrak{V}$ are two finite affine open covers of X such that $-\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ and $-\otimes_{\mathcal{O}_X} W_{\mathfrak{V}}(F)$ are right-exact. Then*

$$W_{\mathfrak{V}}(F) \cong W_{\mathfrak{U}}(F).$$

Proof. The natural transformation Γ depends on a choice of finite affine open cover. We let $\Gamma_F^{\mathfrak{V}}$ denote the natural transformation Γ_F constructed with respect to $\mathfrak{V}$. Applying Corollary 6.5 to the case $\Phi = \Gamma_F^{\mathfrak{V}}$ we get a map

$$\Psi : - \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \longrightarrow - \otimes_{\mathcal{O}_X} W_{\mathfrak{V}}(F).$$

By reversing the roles of $\mathfrak{U}$ and $\mathfrak{V}$ and using the uniqueness statement in Corollary 6.5, we have

$$- \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \cong - \otimes_{\mathcal{O}_X} W_{\mathfrak{V}}(F).$$

Applying the functor $W_{\mathfrak{U}}(-)$ to this isomorphism gives, by Proposition 5.4, the required isomorphism. $\square$

Proposition 6.7. *Let F be an object of $\mathsf{Bimod}_T(X-Y)$ and let $\Gamma_F * u_{k*}$ denote the horizontal composition of the natural transformations Γ_F and $\mathrm{id}_{u_{k*}}$. Then Γ_F is compatible with affine localization, i.e. the diagram*

$$\begin{tikzcd}
Fu_{k*} \arrow[r, "\Gamma_F * u_{k*}"] \arrow[d, "\Gamma_{Fu_{k*}}"'] & u_{k*}(-) \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \arrow[d] \\
- \otimes_{\mathcal{O}_{U_k}} W_{\mathfrak{U}\cap U_k}(Fu_{k*}) \arrow[r] & - \otimes_{\mathcal{O}_{U_k}} v_k^* W_{\mathfrak{U}}(F)
\end{tikzcd} \tag{44}$$

whose bottom horizontal is induced by the isomorphism constructed in Proposition 5.2 and whose right vertical is induced by the isomorphism (3), commutes for all k.

Proof. We prove the result in several steps.
Step 1: We show that it suffices to prove that (44) commutes when applied to flat objects of $\mathsf{Qcoh}U_k$. For, if $\pi : \mathcal{L} \to \mathcal{M}$ is an epimorphism in $\mathsf{Qcoh}U_k$ where $\mathcal{L}$ flat, then, since the arrows in (44) are natural, and since Fu_{k*} is right-exact, Step 1 follows from a standard diagram chase.
Step 2: We note that the diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[dd, "\Gamma_F"'] & \oplus_i F u_{i*} u_i^* \arrow[d] \\
 & \oplus_i u_i^*(-) \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \arrow[d] \\
- \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \arrow[r] & \oplus_i - \otimes_{\mathcal{O}_X} v_{i*}\mathcal{F}_i
\end{tikzcd} \tag{45}$$

whose top horizontal is induced by a unit, whose top vertical is induced by the canonical isomorphism from Proposition 2.2, whose bottom vertical is induced by (2), and whose bottom horizontal comes from the definition of $W_{\mathfrak{U}}(F)$ as a kernel, commutes. We first note that (45) commutes on flats by the definition of Γ_F. Now, if $\mathcal{M}$ is an object in $\mathsf{Qcoh}X$, there exists an epimorphism from a flat object $\mathcal{L}$ in $\mathsf{Qcoh}X$ to $\mathcal{M}$. This epimorphism induces a map from (45) applied to $\mathcal{L}$ to

(45) applied to $\mathcal{M}$. Since all the arrows in (45) are natural and the induced map $F\mathcal{L} \to F\mathcal{M}$ is an epimorphism, the commutivity of (45) applied to $\mathcal{M}$ follows from a routine diagram chase.

Step 3: We claim that the diagram

$$
\begin{array}{ccc}
Fu_{k*} & \longrightarrow & Fu_{k*}u^k_{ik*}u^{k*}_{ik} \\
\downarrow & & \downarrow = \\
Fu_{i*}u_i^*u_{k*} & \longrightarrow & Fu_{i*}u^i_{ik*}u^{k*}_{ik} \\
\downarrow & & \downarrow \\
u_i^*u_{k*}(-) \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i & \longrightarrow & u^i_{ik*}u^{k*}_{ik}(-) \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \\
\downarrow & & \downarrow \\
u_{k*}(-) \otimes_{\mathcal{O}_X} v_{i*}\mathcal{F}_i & & u^{k*}_{ik}(-) \otimes_{\mathcal{O}_{U_{ik}}} v^{i*}_{ik}\mathcal{F}_i \\
\downarrow & & \downarrow \\
- \otimes_{\mathcal{O}_{U_k}} v_k^*v_{i*}\mathcal{F}_i & \longrightarrow & - \otimes_{\mathcal{O}_{U_k}} v^k_{ik*}v^{i*}_{ik}\mathcal{F}_i
\end{array}
\tag{46}
$$

whose top horizontal and top-left vertical are unit morphisms, whose second verticals are from Propostion 2.2, whose second, third and fourth horizontal are induced by basechange, whose third left-vertical and bottom right-vertical are induced by (2), and whose bottom-left vertical and third right-vertical are induced by (3), commutes. The proof of the commutivity of the top square of (46) is routine and left to the reader. The commutivity of the middle square of (46) follows from the fact that the second verticals are induced by the same natural transformations. The fact that the bottom rectangle in (46) commutes follows from Lemma 3.3.

Step 4: Let $\mathcal{E}_i$ be the object of $\mathsf{Qcoh}U_{ik} \times Y$ such that there is a canonical isomorphism (as in Proposition 2.2) $Fu_{k*}u^k_{ik*} \to - \otimes_{\mathcal{O}_{U_{ik}}} \mathcal{E}_i$. Then the isomorphism $\rho : W_{\mathfrak{U}\cap U_k}(Fu_{k*}) \longrightarrow v_k^*W_{\mathfrak{U}}(F)$ constructed in Proposition 5.2 makes the diagram

$$
\begin{array}{ccccc}
W_{\mathfrak{U}\cap U_k}(Fu_{k*}) & \longrightarrow & \oplus_i v^k_{ik*}\mathcal{E}_i & \xrightarrow{\theta_{Fu_{k*}}} & \oplus_{i<j} v^k_{ik*}v^{ik}_{ijk*}v^{ik*}_{ijk}\mathcal{E}_i \\
 & & \downarrow & & \downarrow \\
\rho\downarrow & & \oplus_i v^k_{ik*}v^{i*}_{ik}\mathcal{F}_i & & \oplus_{i<j} v^k_{ik*}v^{i*}_{ik}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \\
 & & \downarrow & & \downarrow \\
v_k^*W_{\mathfrak{U}}(F) & \longrightarrow & \oplus_i v_k^*v_{i*}\mathcal{F}_i & \underset{v_k^*\theta_F}{\longrightarrow} & \oplus_{i<j} v_k^*v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i
\end{array}
\tag{47}
$$

whose top and bottom left horizontals are induced by inclusion, whose middle and right top verticals are induced by basechange, and whose middle and right bottom verticals are induced by the composition

$$
\begin{aligned}
- \otimes_{\mathcal{O}_{U_{ik}}} v^{i*}_{ik}\mathcal{F}_i & \xrightarrow{\cong} u^i_{ik*}(-) \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \\
& \xrightarrow{\cong} Fu_{i*}u^i_{ik*} \\
& \xrightarrow{=} Fu_{k*}u^k_{ik*} \\
& \xrightarrow{\cong} - \otimes_{\mathcal{O}_{U_{ik}}} \mathcal{E}_i
\end{aligned}
$$

whose first arrow is induced by (3), and whose second and fourth arrow are the canonical isomorphisms from Proposition 2.2, commutes. The fact that the right square commutes was checked in the proof of Proposition 5.2, and it follows that there exists a unique isomorphism, which is ρ, between the kernel of the top-right arrow and the kernel of the bottom-right arrow, making the left square of (47) commute.
Step 5: We complete the proof of the proposition. We first note that all squares in the diagram

$$\begin{tikzcd}
- \otimes_{\mathcal{O}_{U_k}} W_{\mathfrak{U}\cap U_k}(Fu_{k*}) \arrow[r] & \oplus_i - \otimes_{\mathcal{O}_{U_k}} v^k_{ik*}\mathcal{E}_i \arrow[r] & \oplus_i u^{k*}_{ik}(-) \otimes_{\mathcal{O}_{U_{ik}}} \mathcal{E}_i \arrow[d] \\
Fu_{k*} \arrow[u, "\Gamma_{Fu_{k*}}"] \arrow[r, "="'] \arrow[ddd, "\Gamma_{F^*u_{k*}}"'] & Fu_{k*} \arrow[r] \arrow[d] & \oplus_i Fu_{k*}u^k_{ik*}u^{k*}_{ik} \arrow[d, "="] \\
 & \oplus_i Fu_{i*}u_i^*u_{k*} \arrow[d] & \oplus_i Fu_{i*}u^i_{ik*}u^{k*}_{ik} \arrow[d] \\
 & \oplus_i u_i^*u_{k*}(-) \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \arrow[d] & \oplus_i u^i_{ik*}u^{k*}_{ik}(-) \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \arrow[d] \\
u_{k*}(-) \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \arrow[r] \arrow[d] & \oplus_i u_{k*}(-) \otimes_{\mathcal{O}_X} v_{i*}\mathcal{F}_i \arrow[d] & \oplus_i u^{k*}_{ik}(-) \otimes_{\mathcal{O}_{U_{ik}}} v^{i*}_{ik}\mathcal{F}_i \arrow[d] \\
- \otimes_{\mathcal{O}_{U_k}} v_k^*W_{\mathfrak{U}}(F) \arrow[r] & \oplus_i - \otimes_{\mathcal{O}_{U_k}} v_k^*v_{i*}\mathcal{F}_i \arrow[r] & \oplus_i - \otimes_{\mathcal{O}_{U_k}} v^k_{ik*}v^{i*}_{ik}\mathcal{F}_i
\end{tikzcd} \tag{48}$$

whose upper and middle-left rectangle are (45), whose lower-right rectangle is (46) and whose lower-left square has verticals induced by (3) and horizontals induced by the inclusion

$$W_{\mathfrak{U}}(F) \longrightarrow \oplus_i v_{i*}\mathcal{F}_i \tag{49}$$

commute. This follows from Step 2, Step 3, and the naturality of (3).

We next note that the outside of the diagram formed by placing

$$\begin{tikzcd}
- \otimes_{\mathcal{O}_{U_k}} W_{\mathfrak{U}\cap U_k}(Fu_{k*}) \arrow[r, "="] \arrow[dddd, "- \otimes_{\mathcal{O}_{U_k}} \rho"'] & - \otimes_{\mathcal{O}_{U_k}} W_{\mathfrak{U}\cap U_k}(Fu_{k*}) \\
 & Fu_{k*} \arrow[u, "\Gamma_{Fu_{k*}}"'] \arrow[d, "\Gamma_{F^*u_{k*}}"] \\
 & u_{k*}(-) \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \arrow[dd] \\
 & \\
- \otimes_{\mathfrak{O}_{U_k}} v_k^*W_{\mathfrak{U}}(F) \arrow[r, "="'] & - \otimes_{\mathfrak{O}_{U_k}} v_k^*W_{\mathfrak{U}}(F)
\end{tikzcd} \tag{50}$$

whose bottom-right vertical is induced by (3), to the left of (48) commutes by Step 4. Since (50) equals (44), and since the map

$$- \otimes_{\mathcal{O}_{U_k}} v_k^*W_{\mathfrak{U}}(F) \longrightarrow \oplus_i(-) \otimes_{\mathcal{O}_{U_k}} v_k^*v_{i*}\mathcal{F}_i$$

induced by (49) is monic on flat objects, we conclude, by a straightforward diagram chase on the diagram constructed by placing (50) to the left of (48), that (44) commutes on flat objects. The proposition follows from Step 1. □

Corollary 6.8. *If F is an object of $\mathsf{Bimod}_T(X-Y)$ then $\ker\Gamma_F$ and $\operatorname{cok}\Gamma_F$ are totally global. In particular, if X is affine, then Γ_F is an isomorphism.*

Proof. By Proposition 4.5, it suffices to show that $(\ker\Gamma_F)u_{i*}$ and $(\operatorname{cok}\Gamma_F)u_{i*}$ equal 0 for all i. To this end, we compute

$$\begin{aligned}(\ker\Gamma_F)u_{i*} &= \ker(\Gamma_F * u_{i*}) \\ &\cong \ker\Gamma_{Fu_{i*}} \\ &= 0\end{aligned}$$

where the second equality is from Proposition 6.7, and the third follows from the fact that since $Fu_{i*} \cong - \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i$ by Proposition 2.2, $\Gamma_{Fu_{i*}}$ is an isomorphism by Proposition 6.4.

A similar proof establishes the fact that $\operatorname{cok}\Gamma_F$ is totally global.

The last statement follows from the fact that $\mathsf{TG} \equiv 0$ when X is affine. □

From now on, we fix a finite affine open cover $\mathfrak{U}$ of X and write W for $W_{\mathfrak{U}}$.

By Lemma 4.6, we may form the quotient category

$$\mathsf{Funct}_T(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)/\mathsf{TG}.$$

Let

$$\pi : \mathsf{Funct}_T(\mathsf{Qcoh}X, \mathsf{Qcoh}Y) \to \mathsf{Funct}_T(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)/\mathsf{TG}$$

denote the quotient functor. The exactness of π together with Corollary 6.8 implies the following naïve version of Watts Theorem:

Corollary 6.9. *If $F \in \mathsf{Bimod}_T(X-Y)$, then $\pi(\Gamma_F)$ is an isomorphism, i.e.*

$$\pi F \cong \pi(- \otimes_{\mathcal{O}_X} W(F))$$

in $\mathsf{Funct}_T(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)/\mathsf{TG}$.

7. Application to Functors from Smooth Curves

Let k be an algebraically closed field. In this section, we assume $T = k$ and X is a smooth curve. We show that if F is an object of $\mathsf{Bimod}_k(X-Y)$, then $\operatorname{cok}\Gamma_F = 0$ (Proposition 7.5), so that there exists an exact sequence

$$0 \longrightarrow \ker\Gamma_F \longrightarrow F \xrightarrow{\Gamma_F} - \otimes_{\mathcal{O}_X} W(F) \longrightarrow 0.$$

We also show that if F is exact on vector-bundles, then Γ_F is an isomorphism (Corollary 7.8). We begin with some preliminary facts.

Lemma 7.1. *If S is a finite subset of closed points of X, then there exists an affine open subset $U \subset X$ such that S is contained in U.*

Proof. There exists a finite, hence affine, morphism $f : X \to \mathbb{P}^1$. Since $f(S)$ is a finite set of closed points of $\mathbb{P}^1$, there exists a copy of the affine line, $\mathbb{A}^1 \subset \mathbb{P}^1$ such that $f(S) \subset \mathbb{A}^1$. Therefore, $S \subset f^{-1}(\mathbb{A}^1)$. Since f is affine, the result follows. □

Lemma 7.2. *Let $\mathcal{T}$ be a coherent torsion module over X. Then there exists an affine open subset $U \subset X$ such that $\operatorname{Supp} \mathcal{T} \subset U$ and such that the unit morphism $\mathcal{T} \longrightarrow u_* u^* \mathcal{T}$ is an isomorphism.*

Proof. Since $\mathcal{T}$ is coherent, the set $\operatorname{Supp} \mathcal{T}$ is closed. Hence, to show $\operatorname{Supp} \mathcal{T}$ is finite, it suffices to find a closed point $q \in X$ such that $\mathcal{T}_q = 0$. To this end, let $\operatorname{Spec} A = W \subset X$ be an open affine subset with inclusion map $w : W \to X$. Since $\mathcal{T}$ is torsion, $w^* \mathcal{T}$ is torsion as well. Since A is a Dedekind domain,

$$\Gamma(W, w^* \mathcal{T}) = \oplus_i A/p_i^{n_i}$$

where p_i is maximal, $n_i > 0$ and the sum is finite. Since A has infinitely many maximal ideals, there exists a maximal ideal $q \in A$ such that $w^* \mathcal{T}_q = 0$. Hence, $\mathcal{T}_q = 0$, so that $\operatorname{Supp} \mathcal{T}$ is finite, and by Lemma 7.1, there exists an affine open subset $u : U \to X$ such that $\operatorname{Supp} \mathcal{T} \subset U$.

The proof of the second statement is routine, and we omit it. □

The straightforward proof of the following result is left to the reader.

Lemma 7.3. *If*

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact in $\mathsf{Funct}_T(\mathsf{Qcoh} X, \mathsf{Qcoh} Y)$ *and A and B commute with direct limits, then C commutes with direct limits.*

Corollary 7.4. *The functor* $\operatorname{cok} \Gamma_F$ *commutes with direct limits.*

Proof. There is an exact sequence

$$F \xrightarrow{\Gamma_F} - \otimes_{\mathcal{O}_X} W(F) \longrightarrow \operatorname{cok} \Gamma_F \longrightarrow 0.$$

in $\mathsf{Funct}_k(\mathsf{Qcoh} X, \mathsf{Qcoh} Y)$. The proof follows from Lemma 7.3, since both F and $- \otimes_{\mathcal{O}_X} W(F)$ commute with direct limits. □

Proposition 7.5. *Suppose F is an object of* $\mathsf{Bimod}_k(X - Y)$*. Then* $\operatorname{cok} \Gamma_F = 0$.

Proof. We first show that if $\mathcal{M}$ is a coherent object of $\mathsf{Qcoh} X$, then $\operatorname{cok} \Gamma_F(\mathcal{M}) = 0$. To this end, we note that $\mathcal{M} \cong \mathcal{F} \oplus \mathcal{T}$ where $\mathcal{F}$ is a vector-bundle and $\mathcal{T}$ is a coherent torsion module over X [5, Lemma 5.2.2]. By Lemma 7.2, there exists an open affine inclusion $u : U \to X$ such that the unit morphism $\mathcal{T} \to u_* u^* \mathcal{T}$ is an isomorphism. Thus

$$\begin{aligned} \operatorname{cok} \Gamma_F(\mathcal{M}) &\cong \operatorname{cok} \Gamma_F(\mathcal{F} \oplus \mathcal{T}) \\ &\cong \operatorname{cok} \Gamma_F(\mathcal{F} \oplus u_* u^* \mathcal{T}) \\ &\cong \operatorname{cok} \Gamma_F(\mathcal{F}) \oplus \operatorname{cok} \Gamma_F(u_* u^* \mathcal{T}) \\ &\cong \operatorname{cok} \Gamma_F(\mathcal{F}) \end{aligned}$$

where the second isomorphism is induced by the unit, and the fourth isomorphism follows from Corollary 6.8. Now, since $\mathcal{F}$ is flat, $\Gamma_{F\mathcal{F}}$ is epic. Thus, $\operatorname{cok} \Gamma_F \mathcal{F} = 0$, so that $\operatorname{cok} \Gamma_F \mathcal{M} = 0$.

Now suppose $\mathcal{N}$ is an object of $\mathsf{Qcoh}X$. Since X is noetherian, $\mathcal{N} \cong \varinjlim \mathcal{N}_i$ with $\mathcal{N}_i$ coherent. Therefore

$$\begin{aligned} \operatorname{cok}\Gamma_F(\mathcal{N}) &\cong \operatorname{cok}\Gamma_F(\varinjlim \mathcal{N}_i) \\ &\cong \varinjlim \operatorname{cok}\Gamma_F(\mathcal{N}_i) \\ &= 0 \end{aligned}$$

where the second isomorphism follows from Corollary 7.4 and the equality is from the first part of the proof. The result follows. □

Lemma 7.6. *Let $\mathcal{F}$ be a vector-bundle on X, let I be a finite set, and let $u_i : U_i \to X$ denote affine open inclusions. Then every coherent submodule of $\oplus_i u_{i*}u_i^*\mathcal{F}$ is a vector-bundle.*

Proof. We first prove the result in the case that $|I| = 1$. In this case let $u = u_1$. Suppose there exists a coherent torsion submodule $\mathcal{T} \subset u_*u^*\mathcal{F}$. Localizing at $p \in \operatorname{Supp}\mathcal{T}$ gives a monic

$$\phi : \mathcal{O}_p/m_p^n \to (u_*u^*\mathcal{F})_p.$$

with $n > 0$. It follows that $\mathcal{O}_p/m_p^n$ is an $\mathcal{O}_p$-submodule of K^m for some $m > 0$, where K is the field of fractions of $\mathcal{O}_p$. This contradicts the fact that K is a domain.

To prove the general case, we note that a finite direct sum is a finite product, hence if $\oplus_i u_{i*}u_i^*\mathcal{F}$ contains a non-zero coherent torsion, submodule, then $u_{k*}u_k^*\mathcal{F}$ contains a non-zero coherent submodule for some $k \in I$. By the first part of the proof, this is impossible. □

Corollary 7.7. *Let $\mathcal{F}$ be a vector-bundle on X. Then the exact sequence*

$$0 \longrightarrow \mathcal{F} = \ker\delta_{\mathcal{F}} \longrightarrow \oplus_i u_{i*}u_i^*\mathcal{F} \longrightarrow \operatorname{im}\delta_{\mathcal{F}} \longrightarrow 0 \tag{51}$$

is a direct limit of short exact sequences of the form

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

where $\mathcal{F}_i$ is a vector-bundle.

Proof. Let $\mathcal{C} \subset \oplus_i u_{i*}u_i^*\mathcal{F}$ be coherent. Since X is noetherian, there is a short exact sequence of coherent $\mathcal{O}_X$-modules

$$0 \longrightarrow \ker(\delta_{\mathcal{F}}|_{\mathcal{C}}) \longrightarrow \mathcal{C} \xrightarrow{\delta_{\mathcal{F}}|_{\mathcal{C}}} \operatorname{im}\delta_{\mathcal{F}}|_{\mathcal{C}} \longrightarrow 0.$$

Let $\{\mathcal{C}_\alpha\}_{\alpha \in A}$ denote the collection of coherent submodules of $\oplus_i u_{i*}u_i^*\mathcal{F}$.

We first claim that the direct system $\{\ker\delta_{\mathcal{F}}|_{\mathcal{C}_\alpha}\}$ induced by inclusion induces an isomorphism

$$\varinjlim_{\alpha \in A} \ker\delta_{\mathcal{F}}|_{\mathcal{C}_\alpha} \cong \ker\delta_{\mathcal{F}}.$$

To prove this claim, we note that if $\mathcal{K} \subset \oplus_i u_{i*}u_i^*\mathcal{F}$ is contained in the kernel of $\delta_{\mathcal{F}}$ and is coherent, then the system above includes $\mathcal{K} = \ker\delta_{\mathcal{F}}|_{\mathcal{K}}$. Thus, the direct limit is taken over all coherent subsheaves of $\ker\delta_{\mathcal{F}}$.

We now take the direct-limit of the sequences

$$0 \longrightarrow \ker(\delta_{\mathcal{F}}|_{\mathcal{C}_\alpha}) \longrightarrow \mathcal{C}_\alpha \xrightarrow{\delta_{\mathcal{F}}|_{\mathcal{C}_\alpha}} \operatorname{im}\delta_{\mathcal{F}}|_{\mathcal{C}_\alpha} \longrightarrow 0. \tag{52}$$

By the universal property of direct-limits, there is a morphism from this sequence to (51). By the first claim, the induced map

$$\lim_{\rightarrow} \ker(\delta_{\mathcal{F}}|_{\mathcal{C}_\alpha}) \longrightarrow \ker \delta_{\mathcal{F}}$$

is an isomorphism. By hypothesis, the induced map

$$\lim_{\rightarrow} \mathcal{C}_\alpha \longrightarrow \oplus_i u_{i*} u_i^* \mathcal{F}$$

is an isomorphism. Hence, the induced map

$$\lim_{\rightarrow} \operatorname{im} \delta_{\mathcal{F}}|_{\mathcal{C}_\alpha} \longrightarrow \operatorname{im} \delta_{\mathcal{F}}$$

is an isomorphism as well. Hence, (51) is isomorphic to the direct-limit of the sequences (52).

The corollary now follows from Lemma 7.6 in light of the fact that $\operatorname{im} \delta_{\mathcal{F}} \subset \oplus_{i<j} u_{ij*} u_{ij}^* \mathcal{F}$. □

Corollary 7.8. *If X is a smooth curve and F is an object of $\mathsf{Bimod}_k(X-Y)$ which is exact on vector-bundles, the Γ_F is an isomorphism so that*

$$F \cong - \otimes_{\mathcal{O}_X} W(F).$$

Proof. Let F be exact on vector-bundles. It suffices, by Propostion 7.5, to show that $\ker \Gamma_F = 0$. By Corollary 7.7, F is exact on

$$0 \longrightarrow \ker \delta_{\mathcal{F}} \longrightarrow \oplus_i u_{i*} u_i^* \mathcal{M} \longrightarrow \operatorname{im} \delta_{\mathcal{F}} \longrightarrow 0.$$

Hence, the canonical map $F(\ker \delta_{\mathcal{F}}) \to \ker F(\delta_{\mathcal{F}})$ is an isomorphism. Thus, $\ker \Gamma_F$ vanishes on flat objects. Hence, Γ_F is an isomorphism on flat objects. It follows from a straightforward diagram chase that Γ_F is monic on arbitrary objects, so that $\ker \Gamma_F = 0$. The result follows. □

8. A Structure Theorem for Objects in $\mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$

The purpose of this section is to compute the structure of certain objects in the category $\mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$ when k is algebraically closed.

Throughout this section, we let $T = k$ where k is an algebraically closed field, we assume X and Y are noetherian, and we let

$$\mathsf{funct}_k(\mathsf{Qcoh} X, \mathsf{Qcoh} Y)$$

denote the category of k-linear functors from $\mathsf{Qcoh}X$ to $\mathsf{Qcoh}Y$ which take coherent objects to coherent objects. If F is an object of $\mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$, we let $F|_{\mathsf{coh}X}$ denote the restriction of F to full subcategory of $\mathsf{Qcoh}X$ consisting of coherent objects.

In order to simplify the exposition, we introduce the concept of an *admissible* functor.

Definition 8.1. Suppose X is a projective variety with very ample invertible sheaf $\mathcal{O}(1)$. A nonzero object F in $\mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ is called an *admissible functor* if it

1. is totally global
2. is half-exact on vector-bundles,
3. commutes with direct-limits, and
4. has the property that whenever $F\mathcal{O}(n) \neq 0$ and $m < n$, $F\alpha$ is epic for all nonzero $\alpha \in \operatorname{Hom}(\mathcal{O}(m), \mathcal{O}(n))$.

Our first substantial result in this section (Proposition 8.6) is that an admissible functor $F \in \mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$ admits a monic

$$\Delta : H^1(\mathbb{P}^1, (-)(i)) \longrightarrow F$$

for some $i \in \mathbb{Z}$. We then prove (Corollary 8.10) that $\overline{\Delta}$ is split. This allows us to prove (Corollary 8.12) that every admissible functor in $\mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$ is a direct sum of cohomologies. This result, along with Proposition 7.5, allows us to compute the structure of those objects F of $\mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$ such that $\ker \Gamma_F$ is right-exact. Finally, we give necessary and sufficient conditions for $\ker \Gamma_F$ to be right-exact.

8.1. **Subfunctors of Admissible Functors.** In this subsection we prove that if $F \in \mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$ is admissible, it has a subfunctor isomorphic to $H^1(\mathbb{P}^1, (-)(i))$ for some integer i. We begin with some preliminary results.

Lemma 8.2. *Let X be a projective variety with very ample invertible sheaf $\mathcal{O}(1)$ such that for all $n > 0$, we have*

$$\dim_k \Gamma(X, \mathcal{O}(n)) > 1.$$

Suppose $F \in \mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}\mathbb{P}^0)$ has the property that whenever $F\mathcal{O}(n) \neq 0$ for some n then $F\alpha$ is epic for all nonzero $\alpha \in \operatorname{Hom}(\mathcal{O}(m), \mathcal{O}(n))$ and all $m < n$. Then

$$\dim_k F\mathcal{O}(m) > \dim_k F\mathcal{O}(n)$$

for all $m < n$.

The hypothesis on F is satisfied if F is right-exact and vanishes on coherent torsion modules.

Proof. We first prove that if F is right-exact, vanishes on coherent torsion modules and $F\mathcal{O}(n) \neq 0$, then $F\alpha$ is epic for all nonzero α in $\operatorname{Hom}(\mathcal{O}(m), \mathcal{O}(n))$ whenever $m < n$.

Let $m < n$ and suppose $\alpha \in \operatorname{Hom}(\mathcal{O}(m), \mathcal{O}(n))$ is nonzero. We first show that the kernel of α must be zero. If not, pick an affine open cover over which both $\mathcal{O}(m)$ and $\mathcal{O}(n)$ are free. Over one of these sets, U, $\ker \alpha$ is nonzero. Since $\alpha(U)$ is just multiplication by some element of $\mathcal{O}(U)$, and since X is integral, $\alpha(U)$ must be the zero map. Therefore, $U \subset \operatorname{Supp} \ker \alpha$. On the other hand, since $\ker \alpha$ is coherent, its support is closed in X. Since X is integral, the support of $\ker \alpha$ must equal X. But the support of $\ker \alpha$ is disjoint from the set of points $p \in X$ such that $\alpha_p \neq 0$, since this map is just multiplication by a nonzero element of a domain. We conclude that the kernel of α equals 0.

The cokernel of α_p is a torsion $\mathcal{O}_{X,p}$-module for all p. We conclude that the cokernel of α is torsion. Therefore, there is an exact sequence

$$0 \to \mathcal{O}(m) \xrightarrow{\alpha} \mathcal{O}(n) \to \mathcal{T} \to 0$$

with $\mathcal{T}$ torsion. Hence $\dim F\mathcal{O}(m) \geq \dim F\mathcal{O}(n)$ by the right-exactness of F and by the fact that $F\mathcal{T} = 0$.

Next, let n be such that $F\mathcal{O}(n) \neq 0$ and suppose that for all nonzero $\alpha \in \operatorname{Hom}(\mathcal{O}(m), \mathcal{O}(n))$ with $m < n$ we have $F\alpha$ epic. To prove the assertion, we must exclude the possibility that there exists some $m < n$ such that $\dim_k F\mathcal{O}(m) = \dim_k F\mathcal{O}(n)$. Suppose to the contrary that for some $m > n$, $\dim_k F\mathcal{O}(m) = d =$

$\dim_k F\mathcal{O}(n) \neq 0$. Then, for all nonzero $\alpha \in \operatorname{Hom}(\mathcal{O}(m), \mathcal{O}(n))$, $F\alpha$ is an isomorphism. Pick a basis $\alpha_0, \ldots, \alpha_r$ for $\operatorname{Hom}(\mathcal{O}(m), \mathcal{O}(n))$ and let $x_0, \ldots, x_r$ denote indeterminates. Note that by hypothesis, $r > 0$. Since

$$\det(x_0 F\alpha_0 + \cdots + x_r F\alpha_r)$$

is a homogeneous polynomial of degree $d > 0$ in $k[x_0, \ldots, x_r]$, it has a non-trivial zero which then gives a non-zero α such that $F\alpha$ is not invertible. This is a contradiction. □

Corollary 8.3. *Let X be a smooth curve. If $F \in \mathsf{Funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}\mathbb{P}^0)$ is totally global and $\mathcal{M} \in \mathsf{Qcoh}X$ is coherent torison, then $F\mathcal{M} = 0$.*

Proof. The lemma follows from Lemma 4.2 and Lemma 7.2. □

Lemma 8.4. *If $F \in \mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$ is non-zero and totally global, then F is admissible.*

Proof. Since $F \in \mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$ is totally global, F vanishes on coherent torsion modules by Corollary 8.3. Therefore, F is admissible by Lemma 8.2. □

The following lemma will be invoked in the proof of Proposition 8.6. Its straightforward proof is omitted.

Lemma 8.5. *Suppose $F_1, F_2 \in \mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ preserve direct limits.*

If $\underline{\Delta} : F_1|_{\mathsf{coh}X} \longrightarrow F_2|_{\mathsf{coh}X}$ is a natural transformation, then $\underline{\Delta}$ extends uniquely to a natural transformation $\Delta : F_1 \longrightarrow F_2$. If $\underline{\Delta}$ is monic, i.e. if $\underline{\Delta}_{\mathcal{M}}$ is monic for all coherent objects $\mathcal{M}$ in $\mathsf{Qcoh}X$, then Δ is monic in $\mathsf{Funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$. If $\underline{\Delta}$ is epic, then Δ is epic in $\mathsf{Funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$.

We introduce notation which will be used in the proof of Proposition 8.6: let $A = k[x_0, x_1]$ denote the polynomial ring in 2 variables with its usual grading. Let $f_i : A[-(n+1)] \to A[n]$ and $g_i : A[-(n+2)] \to A[-(n+1)]$ denote multiplication by x_i. Then we have a short exact sequence in $\mathsf{Gr}A$:

$$0 \longrightarrow A[-(n+2)] \overset{(g_1, -g_0)}{\longrightarrow} A[-(n+1)]^{\oplus 2} \overset{f_0 + f_1}{\longrightarrow} A[-n] \longrightarrow T[-n] \longrightarrow 0.$$

This induces the short exact sequence

$$0 \longrightarrow \mathcal{O}(-(n+2)) \overset{(\phi_1, -\phi_0)}{\longrightarrow} \mathcal{O}(-(n+1))^{\oplus 2} \overset{\psi_0 + \psi_1}{\longrightarrow} \mathcal{O}(-n) \longrightarrow 0. \tag{53}$$

in $\mathsf{coh}\mathbb{P}^1_k$.

Proposition 8.6. *Suppose $F \in \mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$ is admissible. Then the set*

$$\{i \in \mathbb{Z} | F\mathcal{O}(i) \neq 0\}$$

has a maximum, r, and there is a monic morphism

$$\Delta : H^1(\mathbb{P}^1, (-)(-2-r)) \to F$$

in $\mathsf{Funct}_k(\mathsf{Qcoh}\mathbb{P}^1_k, \mathsf{Qcoh}\mathbb{P}^0)$.

Proof. We first show that r is well defined. Since F is non-zero and totally global, $F\mathcal{O}(n) \neq 0$ for some n. Then $\dim F\mathcal{O}(n) > \dim F\mathcal{O}(n+1)$ by Lemma 8.2, so $F\mathcal{O}(i) = 0$ for all $i >> 0$. Hence, the set $\{i \in \mathbb{Z} | F\mathcal{O}(i) \neq 0\}$ indeed has a maximum.

Define $H = H^1(\mathbb{P}^1, (-)(-2-r))$. Note that $H\mathcal{O}(r) = H^1(\mathbb{P}^1, \mathcal{O}(-2)) \cong k$. We first define a natural transformation $\underline{\Delta} : H|_{\mathsf{coh}\mathbb{P}^1} \to F|_{\mathsf{coh}\mathbb{P}^1}$ on coherent objects of

$\mathsf{Qcoh}X$ by defining $\underline{\Delta}_{\mathcal{F}}$ for each indecomposable coherent sheaf $\mathcal{F}$. If $\mathcal{F}$ is torsion or isomorphic to $\mathcal{O}(i)$ with $i > r$ we define $\underline{\Delta}_{\mathcal{F}} = 0$, and we define $\underline{\Delta}_{\mathcal{O}(r)} : H\mathcal{O}(r) \to F\mathcal{O}(r)$ to be any nonzero map. Now suppose we have defined $\underline{\Delta}_{\mathcal{O}(i)}$ for all $i > m$ such that each such $\underline{\Delta}_{\mathcal{O}(i)}$ is injective and such that

$$\begin{array}{ccc} H\mathcal{O}(j) & \xrightarrow{H\psi} & H\mathcal{O}(j+1) \\ {\scriptstyle \underline{\Delta}_{\mathcal{O}_j}}\downarrow & & \downarrow{\scriptstyle \underline{\Delta}_{\mathcal{O}_{j+1}}} \\ F\mathcal{O}(j) & \xrightarrow{F\psi} & F\mathcal{O}(j+1) \end{array}$$

commutes for $j \geq i$ and $\psi \in \operatorname{Hom}_{\mathbb{P}^1}(\mathcal{O}(j), \mathcal{O}(j+1))$. We construct an injective homomorphism $\theta : H\mathcal{O}(m) \to F\mathcal{O}(m)$ such that

$$\begin{array}{ccc} H\mathcal{O}(m) & \xrightarrow{H\phi} & H\mathcal{O}(m+1) \\ {\scriptstyle \theta}\downarrow & & \downarrow{\scriptstyle \underline{\Delta}_{\mathcal{O}_{m+1}}} \\ F\mathcal{O}(m) & \xrightarrow{F\phi} & F\mathcal{O}(m+1) \end{array} \tag{54}$$

commutes for $\phi = \phi_0, \phi_1$ (see 53 for a definition of these maps). To this end, we apply both H and F to the exact sequence (53) with $n := -m-2$ to get a diagram

$$\begin{array}{ccccc} H\mathcal{O}(m) & \xrightarrow{(H\phi_1, -H\phi_0)} & H\mathcal{O}(m+1)^{\oplus 2} & \xrightarrow{H\psi_0 + H\psi_1} & H\mathcal{O}(m+2) \\ & & \downarrow{\scriptstyle \underline{\Delta}^{\oplus 2}_{\mathcal{O}(m+1)}} & & \downarrow{\scriptstyle \underline{\Delta}_{\mathcal{O}(m+2)}} \\ F\mathcal{O}(m) & \xrightarrow{(F\phi_1, -F\phi_0)} & F\mathcal{O}(m+1)^{\oplus 2} & \xrightarrow{F\psi_0 + F\psi_1} & F\mathcal{O}(m+2) \end{array} \tag{55}$$

with exact rows whose right square commutes.

To construct θ, choose a basis $u_1, \ldots, u_{r-m+1}$ for $H\mathcal{O}(m)$. Now,

$$(H\phi_1, -H\phi_0)(u_i) \in \ker(H\psi_0 + H\psi_1).$$

Thus, by the commutivity of the right-hand square of (55),

$$(\underline{\Delta}_{\mathcal{O}(m+1)}, \underline{\Delta}_{\mathcal{O}(m+1)})(H\phi_1(u_i), -H\phi_0(u_i))$$

is in the image of $(F\phi_1, -F\phi_0)$. Hence, there exists a $v_i \in F\mathcal{O}(m)$ such that $F\phi_j(v_i) = \underline{\Delta}_{\mathcal{O}(m+1)} H\phi_j(u_i)$ for $i = 1, \ldots, r-m+1$ and $j = 0, 1$. We define $\theta(u_i) = v_i$. Since F is k-linear, we conclude that (54) commutes for all

$$\phi \in \operatorname{Hom}_{\mathbb{P}^1}(\mathcal{O}(m), \mathcal{O}(m+1)).$$

We define $\underline{\Delta}_{\mathcal{O}(m)} := \theta$.

Next, we define $\underline{\Delta}_{\mathcal{F}}$ when $\mathcal{F}$ is isomorphic to $\mathcal{O}(n)$. Let $\alpha : \mathcal{F} \to \mathcal{O}(n)$ be an isomorphism. Define

$$\underline{\Delta}_{\mathcal{F}} := (F\alpha)^{-1} \circ \underline{\Delta}_{\mathcal{O}(n)} \circ H\alpha.$$

If $\beta : \mathcal{F} \to \mathcal{O}(n)$ is another isomorphism, then $\beta = \lambda\alpha$ for some $0 \neq \lambda \in k$, whence $(F\beta)^{-1} = \lambda^{-1}(F\alpha)^{-1}$ and $H\beta = \lambda H\alpha$; thus the definition of $\delta_{\mathcal{F}}$ does not depend on the choice of α.

We now define $\underline{\Delta}_{\mathcal{F}}$ for arbitrary $\mathcal{F}$ by writing $\mathcal{F}$ as a direct sum of indecomposables, say $\mathcal{F} = \oplus \mathcal{F}_i$, and defining $\underline{\Delta}_{\mathcal{F}} := \oplus \underline{\Delta}_{\mathcal{F}_i}$.

To show that $\underline{\Delta}$ is a natural transformation we must show that

$$\begin{tikzcd} H\mathcal{F} \arrow[r, "Hf"] \arrow[d, "\underline{\Delta}_{\mathcal{F}}"'] & H\mathcal{G} \arrow[d, "\underline{\Delta}_{\mathcal{G}}"] \\ F\mathcal{F} \arrow[r, "Ff"'] & F\mathcal{G} \end{tikzcd} \tag{56}$$

commutes for all $\mathcal{F}$ and $\mathcal{G}$ and all maps $f : \mathcal{F} \to \mathcal{G}$. It suffices to check this when $\mathcal{F}$ and $\mathcal{G}$ are indecomposable. The diagram commutes when $\mathcal{G}$ is torsion because $F\mathcal{G} = 0$ then. If $\mathcal{G}$ is torsion-free and $\mathcal{F}$ torsion, then $f = 0$ so the diagram commutes. Thus, the only remaining case is that when $\mathcal{F} \cong \mathcal{O}(i)$ and $\mathcal{G} \cong \mathcal{O}(j)$ with $i \leq j$.

Write $f = \beta^{-1} g \alpha$ were $\alpha : \mathcal{F} \to \mathcal{O}(i)$ and $\beta : \mathcal{G} \to \mathcal{O}(j)$ are isomorphisms and $0 \neq g : \mathcal{O}(i) \to \mathcal{O}(j)$. We can write $g = \psi_j \psi_{j-1} \cdots \psi_{i+1}$ where each $\psi_l : \mathcal{O}(l-1) \to \mathcal{O}(l)$ is monic. Now

$$\underline{\Delta}_{\mathcal{O}(j)} \circ Hg = \underline{\Delta}_{\mathcal{O}(j)} \circ H\psi_j \circ \cdots \circ H\psi_{i+1} = F\psi_j \circ \cdots \circ F\psi_{i+1} \circ \underline{\Delta}_{\mathcal{O}(i)} = Fg \circ \underline{\Delta}_{\mathcal{O}(i)}.$$

Therefore,

$$\begin{aligned} \underline{\Delta}_{\mathcal{G}} \circ Hf &= F\beta^{-1} \circ \underline{\Delta}_{\mathcal{O}(j)} \circ H\beta \circ Hf \\ &= F\beta^{-1} \circ \underline{\Delta}_{\mathcal{O}(j)} \circ Hg \circ H\alpha \\ &= F\beta^{-1} \circ Fg \circ \underline{\Delta}_{\mathcal{O}(i)} \circ H\alpha \\ &= Ff \circ F\alpha^{-1} \circ \underline{\Delta}_{\mathcal{O}(i)} \circ H\alpha \\ &= Ff \circ \underline{\Delta}_{\mathcal{F}}. \end{aligned}$$

This shows that (56) commutes and so completes the proof that Δ is natural.

Finally, $\underline{\Delta}_{\mathcal{F}}$ is monic for all indecomposable coherent $\mathcal{F}$ and hence for all coherent $\mathcal{F}$. It follows from Lemma 8.5 that $\underline{\Delta}$ extends to a monic natural transformation

$$\Delta : H \longrightarrow F.$$

□

8.2. **The Structure of Admissible Functors in $\mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$.** In this subsection, we work towards a proof, realized in Corollary 8.10, that the monic Δ constructed in Proposition 8.6 is split. We assume, throughout the subsection, that X and Y are projective schemes, $\mathcal{F}, \mathcal{G}, \mathcal{M} \in \mathsf{Qcoh}X$ are coherent, and F is an object of $\mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$.

We first define a natural transformation

$$\Phi_F : F|_{\mathsf{coh}X} \longrightarrow \operatorname{Hom}(-, \mathcal{G})^*|_{\mathsf{coh}X} \otimes_k F\mathcal{G}$$

which will be used to split the monic Δ constructed in Proposition 8.6. To this end, we let

$$\eta_{\mathcal{F},\mathcal{G}} : k \longrightarrow \operatorname{Hom}_X(\mathcal{F}, \mathcal{G})^* \otimes_k \operatorname{Hom}_X(\mathcal{F}, \mathcal{G})$$

be defined as follows: $\eta_{\mathcal{F},\mathcal{G}}(a) := a(\sum_i f_i^* \otimes f_i)$ where $\{f_1, \ldots, f_m\}$ is a basis for $\operatorname{Hom}_X(\mathcal{F}, \mathcal{G})$. We next note that the functor F induces a map

$$\phi_{\mathcal{F},\mathcal{G}} : \operatorname{Hom}_X(\mathcal{F}, \mathcal{G}) \otimes_k F\mathcal{F} \longrightarrow F\mathcal{G} \tag{57}$$

as follows: if U is an open set in Y, and $s \in F\mathcal{F}(U)$, we define (57) over U to be the map

$$f \otimes s \mapsto F(f)(U)(s).$$

We define the natural transformation

$$\Phi_F : F|_{\mathsf{coh}X} \longrightarrow \mathrm{Hom}(-,\mathcal{G})^*|_{\mathsf{coh}X} \otimes_k F\mathcal{G}. \tag{58}$$

as follows:

$$\Phi_{F\mathcal{F}} : F\mathcal{F} \longrightarrow \mathrm{Hom}_X(\mathcal{F},\mathcal{G})^* \otimes_k F\mathcal{G}$$

is defined by be the composition of

$$\eta_{\mathcal{F},\mathcal{G}} \otimes_k F\mathcal{F} : F\mathcal{F} \longrightarrow \mathrm{Hom}_X(\mathcal{F},\mathcal{G})^* \otimes_k \mathrm{Hom}(\mathcal{F},\mathcal{G}) \otimes_k F\mathcal{F}$$

with

$$\mathrm{Hom}_X(\mathcal{F},\mathcal{G})^* \otimes_k \phi_{\mathcal{F},\mathcal{G}} : \mathrm{Hom}_X(\mathcal{F},\mathcal{G})^* \otimes_k \mathrm{Hom}_X(\mathcal{F},\mathcal{G}) \otimes_k F\mathcal{F} \longrightarrow \mathrm{Hom}_X(\mathcal{F},\mathcal{G})^* \otimes_k F\mathcal{G}.$$

The proof that Φ_F is natural is straightforward and we omit it.

Lemma 8.7. *If $\mathcal{N}$ is a coherent object of $\mathsf{Qcoh}Y$, $\mathcal{G}$ is an invertible $\mathcal{O}_X$-module and*

$$F = \mathrm{Hom}_X(-,\mathcal{G})^* \otimes_k \mathcal{N},$$

then the morphism Φ_F is an isomorphism.

Proof. Let $\{f_1,\dots,f_m\}$ be a basis for $\mathrm{Hom}_X(\mathcal{F},\mathcal{G})$ and let U be open in Y. Then

$$\Phi_{F\mathcal{F}}(U) : F\mathcal{F}(U) \longrightarrow \mathrm{Hom}_X(\mathcal{F},\mathcal{G})^* \otimes_k F\mathcal{G}(U)$$

sends $s \in F\mathcal{F}(U)$ to $\sum_{i=1}^m f_i^* \otimes F(f_i)(U)(s)$.

Suppose s is a simple tensor, so

$$s = \delta \otimes t \in \mathrm{Hom}_X(\mathcal{F},\mathcal{G})^* \otimes_k \mathcal{N}(U).$$

We describe $F(f_i)(U)(\delta \otimes t)$. The map $f_i : \mathcal{F} \longrightarrow \mathcal{G}$ induces the map

$$- \circ f_i : \mathrm{Hom}_X(\mathcal{G},\mathcal{G}) \longrightarrow \mathrm{Hom}_X(\mathcal{F},\mathcal{G}).$$

Dualizing gives a map

$$\mathrm{Hom}_X(\mathcal{F},\mathcal{G})^* \longrightarrow \mathrm{Hom}_X(\mathcal{G},\mathcal{G})^*$$

which sends δ to $\delta \circ (-\circ f_i)$. Hence, $F(f_i)(U)(\delta \otimes t) = \sum_{i=1}^m f_i^* \otimes \delta \circ (-\circ f_i) \otimes t$.

Next, we note that the map $\delta \circ (-\circ f_i) \in \mathrm{Hom}_X(\mathcal{G},\mathcal{G})^* \cong k$ sends multiplication by α to multiplication by $\alpha\delta(f_i)$. Let us denote this map by $\mu_{\alpha\delta(f_i)}$.

Now, the function

$$\mathrm{Hom}_X(\mathcal{F},\mathcal{G})^* \longrightarrow \mathrm{Hom}_X(\mathcal{F},\mathcal{G})^* \otimes_k \mathrm{Hom}_X(\mathcal{G},\mathcal{G})^*$$

defined by sending δ to $\sum_i f_i^* \otimes_k \mu_{\delta(f_i)}$ is injective and k-linear, hence an isomorphism of vector spaces. Therefore, since

$$\Phi_{F\mathcal{F}}(U) : \mathrm{Hom}_X(\mathcal{F},\mathcal{G})^* \otimes_k \mathcal{N}(U) \longrightarrow \mathrm{Hom}_X(\mathcal{F},\mathcal{G})^* \otimes_k \mathrm{Hom}_X(\mathcal{G},\mathcal{G})^* \otimes_k \mathcal{N}(U)$$

is the identity on $\mathcal{N}(U)$, $\Phi_{F\mathcal{F}}(U)$ is an isomorphism. Therefore Φ_F is an isomorphism. □

Lemma 8.8. *Let $\Theta : F' \longrightarrow F$ be a natural transformation between elements of $\mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$. Then the diagram*

$$\begin{tikzcd} F \arrow[r, "\Phi_F"] & \mathrm{Hom}_X(-,\mathcal{G})^* \otimes_k F\mathcal{G} \\ F' \arrow[u, "\Theta"] \arrow[r, "\Phi_{F'}"'] & \mathrm{Hom}_X(-,\mathcal{G})^* \otimes_k F'\mathcal{G} \arrow[u] \end{tikzcd}$$

whose right vertical is induced by Θ*, commutes on coherent objects.*

Proof. From the definition of Φ, it suffices to show that the diagram

$$\begin{tikzcd} F \arrow[r] & \operatorname{Hom}_X(\mathcal{M},\mathcal{G})^* \otimes_k \operatorname{Hom}_X(\mathcal{M},\mathcal{G}) \otimes_k F \\ F' \arrow[r] \arrow[u, "\Theta"] & \operatorname{Hom}_X(-,\mathcal{G})^* \otimes_k \operatorname{Hom}_X(-,\mathcal{G}) \otimes_k F' \arrow[u] \end{tikzcd} \tag{59}$$

whose right vertical is induced by Θ and whose horizontals are induced by the unit map $k \longrightarrow \operatorname{Hom}_X(-,\mathcal{G})^* \otimes_k \operatorname{Hom}_X(-,\mathcal{G})$ commutes, and that the diagram

$$\begin{tikzcd} \operatorname{Hom}_X(\mathcal{M},\mathcal{G}) \otimes_k F\mathcal{M} \arrow[r] & F\mathcal{G} \\ \operatorname{Hom}_X(\mathcal{M},\mathcal{G}) \otimes_k F'\mathcal{M} \arrow[r] \arrow[u] & F'\mathcal{G} \arrow[u, "\Theta_{\mathcal{G}}"'] \end{tikzcd} \tag{60}$$

whose left vertical is induced by Θ and whose horizontals are induced by evaluation, commutes. It is straightforward to check (59) commutes. We now study the commutivity of (60). The top route of (60) evaluated on the open set $U \subset Y$ sends $f \otimes x$ to $F(f)(U)(\Theta_{\mathcal{M}}(U)(x))$ while the bottom route of (60) sends $f \otimes x$ to $\Theta_{\mathcal{G}}(U)F'(f)(U)(x)$. These values are equal by the naturality of Θ. $\square$

Lemma 8.9. *If* $F \in \mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}\mathbb{P}^0)$*, is such that there exists an invertible* $\mathcal{G} \in \mathsf{Qcoh}X$ *and a monomorphism*

$$\Psi : \operatorname{Hom}_X(-,\mathcal{G})^* \longrightarrow F$$

in $\mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}\mathbb{P}^0)$*, then the restriction of* Ψ *to coherents,*

$$\underline{\Psi} : \operatorname{Hom}_X(-,\mathcal{G})^*|_{\mathsf{coh}X} \longrightarrow F|_{\mathsf{coh}X},$$

splits.

Proof. Let $\psi : F\mathcal{G} \longrightarrow \operatorname{Hom}_X(\mathcal{G},\mathcal{G})^*$ be a splitting of $\Psi_{\mathcal{G}}$. Consider the diagram

$$\begin{tikzcd} F \arrow[r, "\Phi_{F\mathcal{M}}"] & \operatorname{Hom}_X(\mathcal{M},\mathcal{G})^* \otimes_k F(\mathcal{G}) \\ \operatorname{Hom}_X(\mathcal{M},\mathcal{G})^* \arrow[r, "\Phi_{\operatorname{Hom}_X(-,\mathcal{G})^*\mathcal{M}}"'] \arrow[u, "\Psi_{\mathcal{M}}"] & \operatorname{Hom}_X(\mathcal{M},\mathcal{G})^* \otimes_k \operatorname{Hom}_X(\mathcal{G},\mathcal{G})^* \arrow[u] \end{tikzcd}$$

whose right vertical is induced by $\Psi_{\mathcal{G}}$. The bottom horizontal is an isomorphism by Lemma (8.7), and the diagram commutes by Lemma 8.8. It follows that the diagram

$$\begin{tikzcd} F \arrow[r, "\Phi_{F\mathcal{M}}"] & \operatorname{Hom}_X(\mathcal{M},\mathcal{G})^* \otimes_k F(\mathcal{G}) \arrow[d] \\ \operatorname{Hom}_X(\mathcal{M},\mathcal{G})^* \arrow[r, "\Phi_{\operatorname{Hom}_X(-,\mathcal{G})^*\mathcal{M}}"'] \arrow[u, "\Psi_{\mathcal{M}}"] & \operatorname{Hom}_X(\mathcal{M},\mathcal{G})^* \otimes_k \operatorname{Hom}_X(\mathcal{G},\mathcal{G})^* \end{tikzcd}$$

whose right vertical is induced by ψ, commutes. $\square$

Corollary 8.10. *If* $F \in \mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$ *is admissible, then the monic*

$$\Delta : H^1(\mathbb{P}^1, (-)(-2-r)) \longrightarrow F$$

constructed in Proposition 8.6 splits.

Proof. The monic Δ restricts to a monic

$$\underline{\Delta}: H^1(\mathbb{P}^1, (-)(-2-r))|_{\mathsf{coh}\mathbb{P}^1} \longrightarrow F|_{\mathsf{coh}\mathbb{P}^1}.$$

By Serre duality, $\underline{\Delta}$ induces a monic

$$\underline{\Delta}': \operatorname{Hom}_{\mathbb{P}^1}(-, \mathcal{O}(r))^*|_{\mathsf{coh}\mathbb{P}^1} \longrightarrow F|_{\mathsf{coh}\mathbb{P}^1}$$

which by Lemma 8.9, admits a splitting

$$\underline{\Psi}': F|_{\mathsf{coh}\mathbb{P}^1} \longrightarrow \operatorname{Hom}_{\mathbb{P}^1}(-, \mathcal{O}(r))^*|_{\mathsf{coh}\mathbb{P}^1}.$$

The map $\underline{\Psi}'$ induces, by Serre duality again, a splitting

$$\underline{\Psi}: F|_{\mathsf{coh}\mathbb{P}^1} \longrightarrow H^1(\mathbb{P}^1, (-)(-2-r))|_{\mathsf{coh}\mathbb{P}^1}.$$

of $\underline{\Delta}$. We claim that $\underline{\Psi}$ extends to a splitting

$$\Psi: F \longrightarrow H^1(\mathbb{P}^1, (-)(-2-r)) \tag{61}$$

of Δ. To this end, Lemma 8.5 implies that $\underline{\Psi}$ has a unique extension (61). We also know that $\Psi\Delta$ restricts on coherent objects to the map $\underline{\Psi}\ \underline{\Delta} = \mathrm{id}_{H^1(\mathbb{P}^1,(-)(-2-r))|_{\mathsf{coh}\mathbb{P}^1}}$. But by Lemma 8.5, $\underline{\Psi}\ \underline{\Delta}$ extends uniquely to a natural transformation

$$H^1(\mathbb{P}^1, (-)(-2-r)) \longrightarrow H^1(\mathbb{P}^1, (-)(-2-r)).$$

Thus, $\Psi\Delta = \mathrm{id}_{H^1(\mathbb{P}^1,(-)(-2-r))}$, whence the Corollary. □

We omit the straightforward proof of the following

Lemma 8.11. *Suppose $F \in \mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$ is admissible.*

If $F \cong A \oplus B$ in $\mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$, and if A is non-zero, then A is admissible as well.

Corollary 8.12. *If $F \in \mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$ is admissible, then there exist integers $m, n_i \geq 0$ such that*

$$F \cong \oplus_{i=-m}^{\infty} H^1(\mathbb{P}^1, (-)(i))^{\oplus n_i}.$$

Proof. Consider the set C of pairs (H, Ψ) where

$$0 \neq H = \oplus_{i=-m}^{\infty} H^1(\mathbb{P}^1, (-)(i))^{\oplus n_i}$$

with $n_i \geq 0$ for all i, $n_{-m} > 0$ and where $\Psi: H \longrightarrow F$ is a split monomorphism. By Corollary 8.10, there exits a split monomorphism

$$\Lambda: H^1(\mathbb{P}^1, (-)(j)) \longrightarrow F$$

for some $j \in \mathbb{Z}$. Thus, the set C is nonempty.

We next note that the set

$$\{m | (\oplus_{i=-m}^{\infty} H^1(\mathbb{P}^1, (-)(i))^{\oplus n_i}, \Psi) \in C \text{ with } n_{-m} > 0\}$$

has a maximum, which we call M. If not, the dimension of $F\mathcal{O}$ would be infinite, contradicting the fact that $F \in \mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$.

Similarly, for each $i \geq -M$, the set

$$\{n_i | (\oplus_{i=-m}^{\infty} H^1(\mathbb{P}^1, (-)(i))^{\oplus n_i}, \Psi) \in C\}$$

has a maximum, which we call N_i.

We next claim that there exists a split monomorphism

$$\Theta: H = \oplus_{-M}^{\infty} H^1(\mathbb{P}^1, (-)(i))^{\oplus N_i} \longrightarrow F.$$

We define Θ as follows: let (H_i, Ψ_i) be an element of C whose first component has a summand of the form $H^1(\mathbb{P}^1, (-)(i))^{\oplus N_i}$ and let Θ be induced by the compositions Θ_i:

$$H^1(\mathbb{P}^1, (-)(i))^{\oplus N_i} \longrightarrow H_i \xrightarrow{\Psi_i} F$$

whose left composite is the canonical inclusion. Let Υ_i denote the splitting of Θ_i induced by the splitting of Ψ_i.

In order to define a splitting of Θ, we first define a natural map

$$\underline{\Upsilon} : F|_{\mathsf{coh}\mathbb{P}^1} \longrightarrow H|_{\mathsf{coh}\mathbb{P}^1}$$

as follows. Since $H^1(\mathbb{P}^1, \mathcal{M}(i)) = 0$ for all $i >> 0$, there exists an integer N such that

$$H(\mathcal{M}) = \oplus_{i=-M}^{N} H^1(\mathbb{P}^1, \mathcal{M}(i))^{\oplus N_i}.$$

We define

$$\underline{\Upsilon}_{\mathcal{M}} : F(\mathcal{M}) \longrightarrow H(\mathcal{M})$$

as $\oplus_{i=-M}^{N} \Upsilon_{i\mathcal{M}}$.

It remains to prove that $\underline{\Upsilon}$ is natural. But this follows from the naturality of Υ_i since, for coherent $\mathcal{F}$ and $\mathcal{M}$, there exists an integer N such that if

$$H' := \oplus_{i=-M}^{N} H^1(\mathbb{P}^1, (-)(i))^{\oplus N_i},$$

then $H(\mathcal{F}) = H'(\mathcal{F})$ and $H(\mathcal{M}) = H'(\mathcal{M})$. Thus, $\underline{\Upsilon}$ is natural. Hence, by Lemma 8.5, $\underline{\Upsilon}$ extends to a morphism

$$\Upsilon : F \to H.$$

As in the proof of Corollary 8.10, $\Upsilon\Theta = \mathrm{id}_H$. Therefore, Θ is a split monomorphism, as claimed.

To complete the proof, we claim Θ is an isomorphism. For, since Θ is split, we have

$$F \cong H \oplus F'.$$

If $F' \neq 0$, then F' is admissible by Lemma 8.11. Hence, by Corollary 8.10 $F' \cong H^1(\mathbb{P}^1, (-)(j)) \oplus F''$ for some integer j. Therefore, there exists a split monomorphism

$$H \oplus H^1(\mathbb{P}^1, (-)(j)) \longrightarrow F.$$

It follows that N_j is not maximal, which is a contradiction. □

8.3. **The Structure of Objects in $\mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$ with $\ker \Gamma_F$ Right-exact.** Our structure theorem for objects of $\mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$ is the following

Corollary 8.13. *Suppose $F \in \mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$. If $\ker \Gamma_F$ is right-exact, then for some $m, n_i \geq 0$, there is an exact sequence*

$$0 \longrightarrow \oplus_{i=-m}^{\infty} H^1(\mathbb{P}^1, (-)(i))^{\oplus n_i} \longrightarrow F \xrightarrow{\Gamma_F} - \otimes_{\mathcal{O}_X} W(F) \longrightarrow 0$$

in $\mathsf{Funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$.

Proof. By Proposition 7.5, there is an exact sequence

$$0 \longrightarrow \ker \Gamma_F \longrightarrow F \xrightarrow{\Gamma_F} - \otimes_{\mathcal{O}_X} W(F) \longrightarrow 0 \tag{62}$$

in $\mathsf{Funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$. By Corollary 6.8, $\ker \Gamma_F$ is totally global. Since the functors F and $- \otimes_{\mathcal{O}_X} W(F)$ commute with direct-limits, so does $\ker \Gamma_F$. Therefore, $\ker \Gamma_F$ is an object of $\mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$. Hence, the result follows from Corollary 8.12. □

Lemma 8.14. *If F is an object of $\mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$ then*

(1) *the functor $\ker \Gamma_F$ is half-exact on short exact sequences of vector-bundles, and*
(2) *the map $\ker F(\psi) \longrightarrow \ker(\psi \otimes_{\mathcal{O}_X} W(F))$ induced by Γ_F is epic for $0 \neq \psi \in \operatorname{Hom}(\mathcal{O}(i), \mathcal{O}(i+1))$ iff $\ker \Gamma_F(\psi)$ is epic.*

Proof. Let

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0 \tag{63}$$

be a short-exact sequence of vector-bundles. Then by Proposition 7.5, there is an exact sequence

$$0 \longrightarrow \ker \Gamma_F \longrightarrow F \xrightarrow{\Gamma_F} - \otimes_{\mathcal{O}_X} W(F) \longrightarrow 0 \tag{64}$$

and we apply it to (63) to get a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker \Gamma_F(\mathcal{F}_1) \arrow[r] \arrow[d] & \ker \Gamma_F(\mathcal{F}_2) \arrow[r] \arrow[d] & \ker \Gamma_F(\mathcal{F}_3) \arrow[d] & \\
 & F(\mathcal{F}_1) \arrow[r] \arrow[d] & F(\mathcal{F}_2) \arrow[r] \arrow[d] & F(\mathcal{F}_3) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_1 \otimes_{\mathcal{O}_X} W(F) \arrow[r] \arrow[d] & \mathcal{F}_2 \otimes_{\mathcal{O}_X} W(F) \arrow[r] \arrow[d] & \mathcal{F}_3 \otimes_{\mathcal{O}_X} W(F) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact columns and exact second and third row. The half-exactness of $\ker \Gamma_F$ now follows from a standard diagram chase.

We now prove the second assertion. There is a short-exact sequence

$$0 \longrightarrow \mathcal{O}(n) \xrightarrow{\psi} \mathcal{O}(n+1) \longrightarrow \mathcal{O}_p \longrightarrow 0 \tag{65}$$

where $p \in \mathbb{P}^1$ is closed. We apply the exact sequence (64) to (65). We get a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \ker \Gamma_F(\mathcal{O}(n)) \arrow[r, "\ker \Gamma_F(\psi)"] \arrow[d] & \ker \Gamma_F(\mathcal{O}(n+1)) \arrow[d] & 0 \arrow[d] & \\
\ker F(\psi) \arrow[r] \arrow[d] & F(\mathcal{O}(n)) \arrow[r, "F(\psi)"] \arrow[d] & F(\mathcal{O}(n+1)) \arrow[r] \arrow[d] & F(\mathcal{O}_p) \arrow[r] \arrow[d, "\cong"] & 0 \\
\ker(\psi \otimes_{\mathcal{O}_X} W(F)) \arrow[r] & \mathcal{O}(n) \otimes_{\mathcal{O}_X} W(F) \arrow[r, "\psi \otimes_{\mathcal{O}_X} W(F)"] \arrow[d] & \mathcal{O}(n+1) \otimes_{\mathcal{O}_X} W(F) \arrow[r] \arrow[d] & \mathcal{O}_p \otimes_{\mathcal{O}_X} W(F) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

whose columns and second row is exact. The result now follows from the serpent lemma on the two middle columns. □

Corollary 8.15. *Let F be a nonzero object of* $\mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$. *Then* $\ker \Gamma_F$ *is right-exact if and only if the condition of Lemma 8.14(2) is satisfied for all* $\psi \in \operatorname{Hom}(\mathcal{O}(i), \mathcal{O}(i+1))$ *and all* i. *In this case, there exist integers* $m, n_i \geq 0$ *such that*

$$\ker \Gamma_F \cong \oplus_{i=-m}^{\infty} H^1(\mathbb{P}^1, (-)(i))^{\oplus n_i}. \tag{66}$$

Proof. If $\ker \Gamma_F$ is right-exact and $0 \neq \psi \in \operatorname{Hom}(\mathcal{O}(i), \mathcal{O}(i+1))$, then $\ker \Gamma_F(\psi)$ is epic since $\ker \Gamma_F$ is totally global by Corollary 6.8. It follows that the condition of Lemma 8.14(2) holds for all i.

Conversely, $\ker \Gamma_F$ commutes with direct-limits since the functors F and $- \otimes_{\mathcal{O}_X} W(F)$ commute with direct-limits. Therefore, by Lemma 8.14(1) and Corollary 6.8, $\ker \Gamma_F$ is admissible. Hence, Corollary 8.13 applies to $\ker \Gamma_F$, and (66) follows. In particular, $\ker \Gamma_F$ is right-exact. □

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