CHAPTER 32

MOVING FORWARD IN INTERNATIONAL MATHEMATICS EDUCATION RESEARCH

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> The need is as pressing as ever for a shared, scientific, non-ideological framework for empirical and theoretical research in mathematical learning and problem solving (Goldin, chapter 9, p. XX).

This final chapter revisits some of the key issues addressed by the authors and explores a selection of the many research issues that need attention in the advancement of our discipline. Specifically, we give consideration to the following questions:

1. What role can research play in illuminating the multidisciplinary debates on the powerful mathematical ideas required for the 21st century?
2. How can research support more equitable curriculum and learning access to powerful mathematical ideas?
3. How can research support the creation of learning environments that give learners better and more equitable access to powerful mathematical ideas?
4. How can research contribute to the kind of teacher education and teacher development programs that will be needed to facilitate student access to powerful mathematical ideas?

5. How can we assess the extent to which students have gained access to powerful mathematical ideas, and their abilities to make effective use of these ideas? How can research inform such assessment?
6. What are some of the latest research designs in mathematics education? How do we assess and improve research designs in mathematics education?
7. What is the nature of semiotic mediation and what is its role in mathematics education?

Illuminating the Multidisciplinary Debates on Powerful Mathematical Ideas for the 21st Century

We address this first question by considering the nature of mathematics, its roles in society, and what counts as important mathematical ideas for the 21st century. We address some of the psychological, epistemological, philosophical, and socio-cultural viewpoints on these issues.

The Nature of Mathematics

The way in which we conceive of mathematics has a major bearing on the mathematical ideas that we consider essential for today’s world (cf. Hersh's 1979 argument). Likewise, how we view knowledge, its development, and its application in the classroom and beyond also impacts on the mathematics curriculum we create for our students.

One prominent perspective on the nature of mathematics in recent years is that offered by Núñez (2007), and Lakoff and Núñez (1997, 2000), who see mathematics as structured by the human brain and limited by human mental capacities. Núñez (2007) argues that mathematics does not reside outside of human cognition, rather, “mathematics is a genuine creation of the abstraction and imagination of the human animal” (p. 152). To illustrate, Núñez (2007) noted that when one asks about the nature of the discipline of language or music education, the answers offered are usually quite different from those given for mathematics education. The former two present the disciplines as inherently human, in contrast to the mathematics education discipline, which is generally viewed as “an essentially pre-given dehumanized body of knowledge”, with associated negative implications for mathematics education (p. 127). Opposing the view that mathematics is essentially independent of human beings and the “only truly universal language” (“the romance of mathematics” view), Núñez (2007) maintains that mathematics education needs to de-emphasize its focus on formal definitions, axioms, algorithms, and reductionistic forms of logic and adopt a more “cognitive friendly” approach, one that is compatible with how the human mind actually functions (pp. 127-130).

The Roles of Cultural, Social, and Political Factors

The nature of mathematics and the direction of its growth are also considered by many to be shaped by a complex system of cultural, social, and political forces (e.g., D'Ambrosio, 1999, 2007; Greer & Mukhopadhyay, 2003; Mukhopadhyay & Greer, 2007; Secada, 1995; Skovsmose & Valero, chapter 17; Wilder, 1986). The role of social and political factors in shaping mathematical beliefs is illustrated nicely in Goldin’s chapter (9). He shows how the different mathematical beliefs held by various educational, social, and political groups have been fuelling debates on what should constitute mathematics education for today’s students. Despite these conflicting ideological perspectives, it is generally recognized that mathematics is a major element of all human cultures, whether they be ethnic, urban, rural, or indigenous.

We still have a long way to go to bring this human element to the fore, however (D’Ambrosio, 1999, 2007). When mathematics is considered to be intertwined with human contexts and practices, it follows that social accountability must be applied to the discipline (D’Ambrosio, 2007; Ernest, 1998; Greer & Mukhopadhyay, 2003; Mukhopadhyay & Greer, 2007; Gutstein, 2007). One potentially rich opportunity to address this issue lies in the increased emphasis on the inclusion of data handling and statistical reasoning, and mathematical modeling and applications in school curricula; here, there are numerous real-world examples where students can use mathematics to analyse socially and culturally relevant problems (Greer & Mukhopadhyay (2003). For example, Mukhopadhyay and Greer (2007) have outlined how the issue of gun violence, in particular as it impacts on students, can be analysed in relation to its socio-political contexts using mathematics as a critical tool.

Knowledge Building in the Mathematics Classroom

Given the rate at which our social and physical worlds are being mathematized (as noted over 20 years ago by Davis & Hersh, 1986), there is an increasing gap between our body of mathematical knowledge and what we can reasonably include in the school curriculum. Furthermore, it is questionable whether what we do include is intellectually stimulating. Greer and Mukhopadhyay (2003) commented that “the most salient features of most documents that lay out a K-12 program for mathematics education is that they make an intellectually exciting program boring”, a feature they refer to as “intellectual child abuse” (p. 4). Clearly, we need to make the mathematical experiences we include for our students more stimulating, authentic, and meaningful. Developing students’ abilities to work creatively with mathematical knowledge, as “distinct from working creatively on tasks that use knowledge” (Bereiter & Scardamalia, 2006, p. 695) is especially important in preparing our students for success in a knowledge-based economy. Establishing collaborative, knowledge-building communities in the mathematics classroom is a significant and challenging goal for the advancement of our discipline—a goal that is all the more important in today’s world with the rapid developments in digital technologies and the associated widespread digitization of learning materials, media, and conceptual learning tools, all of which impact directly on what students learn, how they learn, and what they value as important to learn (cf. sentiments of Lankshear & Knobel, 2003).

Mathematics in Work Place Settings

The advent of digital technologies also changes the world of work for our students. As Clayton (1999) and others (e.g., Er-sheng, 1999; Jenkins, Clinton, Purushotma, & Weigel, 2006; Lombardi & Lombardi, 2007; Pollak, 1997; Stevens, 1999; Roschelle, Kaput, & Stroup, 2000) have pointed out, the availability of increasingly sophisticated technology has led to changes in the way mathematics is being used in work place settings; these technological changes have led to both the addition of new mathematical competencies and the elimination of existing mathematical skills that were once part of the worker's toolkit.

Studies of the nature and role of mathematics used in the workplace and other everyday settings (e.g., nursing, engineering, grocery shopping, dieting, architecture, fish hatcheries) are important in helping us identify some of the powerful mathematical ideas for the 21st century. Such studies (e.g., de Abreu, chapter 15; Gainsburg, 2006; Hoyles, Bakker, Kent, & Noss, in press; Lave, 1997; Nunes, Schliemann, & Carraher, 1993; Roth, 2005) have revealed how the strategies and decision-making involved in these settings develop within, and thus become products of, the socio-cultural communities of these practices. For example, both Hoyles et al. (in press) and Roth (2005) studied the mathematical knowledge (in particular, the understanding and use of graphs) of employees in different industries (packaging, pharmaceuticals manufacturing, automotive manufacturing [Hoyles et al.] and a fish hatchery [Roth]). Both studies showed how the mathematics used in these practices is not simply an extension of school mathematics. Rather, practice-based mathematics tends to be situated in nature and “deeply integrated in the understanding of the local particulars; the ‘meaning’ of graphs and other mathematical inscriptions exists in their close integration into the work process”. (Roth, 2005; p. 75). These studies highlight the need to recognize the “reciprocal relationship between workplace and mathematical knowledge, and in particular, the ways in which each might come to shape an understanding of the other" (Hoyles et al., in press).

Research that has attempted to characterize the mathematics used in working practices and in other life activities has not been without obstacles, however. Hoyles et al. (2001) pointed out that in such research, most employees don't describe their duties in mathematical terms, and then often state that they use very little mathematics in their work. The basic problem here seems to be that the "mathematics of work is hidden beneath the surface of cultures and practices, so that any superficial classification of it in terms of school-mathematical knowledge will inevitably result in its reduction to simple measurement and arithmetic" (Hoyles et al., 2001, p. 5). However, Stevens' (1999) research has shown how mathematical practices can be considered as a part of another discipline, such as architectural design, rather than the discipline of mathematics itself. He argued that if we are to understand better how mathematics is a consequential part of the broader social world, we need to shift our attention from mathematics per se to mathematics within the disciplines.

Although we cannot simply list a number of mathematical competencies and assume these can be "unproblematically grafted onto workplace knowledge" (Hoyles et al., in press), there are a number of key understandings and processes that employers generally consider to be essential to productive outcomes (Anderson, 1999; Doerr & English, 2003; English, 2007a; Gainsburg, 2006; Langrall, Mooney, Nisbet, & Jones, chapter 6; Lesh & Zawojewski, 2007; Jenkins et al., 2006; Lappan, 1999). These are listed below. A major finding of Hoyles, Wolf, Molyneux-Hodgson and Kent's (2002) report on workplace mathematics was that basic numeracy is being displaced as the minimum required mathematical competence by an ability to apply a much wider range of mathematical concepts in using technological tools as part of working practice. More specifically, the following are some of the core competencies that have been identified as key elements of productive and innovative work place practices:

- problem solving, including working collaboratively on complex problems where planning, monitoring, and communicating are critical for success;
- applying numerical and algebraic reasoning in an efficient, flexible, and creative manner;
- generating, analysing, operating on, and transforming complex data sets;
- applying an understanding of core ideas from ratio and proportion, probability, rate, change, accumulation, continuity, and limit;
- constructing, describing, explaining, manipulating, and predicting complex systems (such as sophisticated buying, leasing, and loan plans);
- thinking critically and being able to make sound judgements (including being able to distinguish reliable from unreliable information sources);
- synthesizing, where an extended argument is followed across multiple modalities;
- engaging in research activity involving the searching, discovery, and dissemination of pertinent information in a credible manner;
- flexibility in working across disciplinary boundaries to generate innovative and effective solutions.

Despite the existing research on the mathematics needed for a productive society, there remain many issues in need of attention. These include what it means to analyze workplace settings from a mathematical perspective, the nature of the relationship between mathematical knowledge of the workplace and the mathematics taught formally in schools, and whether practical and formal mathematics are derived from different epistemologies (Hoyles et al., 2001; in press). These issues are critical in addressing whether the mathematical ideas experienced in formal settings are appropriate for students' success beyond the classroom. Every student, irrespective of gender, language, ethnicity, or disability, has the right of access to powerful mathematical ideas, how to think effectively with these ideas, and how to apply their mathematical understandings beyond the walls of the classroom.

Supporting More Equitable Curriculum Access to Powerful Mathematical Ideas

Equity in mathematics education is a multidimensional issue, with many forces working against students' democratic access to powerful mathematical ideas (see Rousseau Anderson, & Tate's discussion on this point in chapter 13). Given the complexity of the problem, it is not feasible to address the myriad factors that require attention; rather we consider a couple of the ways in which researchers are attempting to promote more equitable curriculum access to powerful mathematics. More in-depth discussion on this issue appears in the next section.

One of the problems that mathematics educators face is how to restructure the overcrowding in many existing curricula so that students from diverse backgrounds have access to the more sophisticated and complex mathematical ideas that society requires. Significant work is being done here by Roschelle and his colleagues in achieving their goal of teaching "much more mathematics to many more people" (Kaput & Roschelle, 1999, p. 161; Roschelle et al., 2000; Roschelle et al., 2007). Highlighting change (economic, social, and technological) as a central phenomenon of this century, Roschelle et al., (2000) expressed concern that the mathematics of rate, change, and variation is locked away in calculus courses, with the result that only a small percentage of students are gaining access to these important mathematical ideas — the very concepts that students need to both participate in their physical and social lives, and to make informed decisions in their personal and political lives.

In their recent report on “Scaling up SimCalc Project”, Roschelle et al. (2007) demonstrated how the power of technology can enhance the curriculum to deepen the mathematical learning of middle-grade students across diverse ethnic and economic settings. The longitudinal project focused on an intervention that integrated technology and curriculum to create opportunities for students to learn complex and conceptually difficult mathematics (with a focus on rate and proportionality). Their recent findings indicate that the up-scaled SimCalc approach was effective in a wide variety of seventh-grade Texas (USA) classrooms, where robust student gains were obtained irrespective of variation in gender, poverty, ethnicity, and prior achievement. The researchers argue that implementing their approach more widely could “boost diverse students along the pathway leading to algebra and calculus, a pathway that is widely seen as critical to increasing the number of students prepared to excel in science” (p. 7). A caveat is in order here, however. Roschelle and his colleagues have stressed that democractic access to the powerful mathematical ideas that innovations in computational media are providing is not simply a matter of choosing the right technology. Rather, it is imperative that conditions be created where students develop their abilities to understand and solve increasingly challenging and meaningful problems.

A recent special issue of *Mathematical Thinking and Learning* (8[3], 2006) was devoted to inequities in achievement and participation in mathematics for poor students and students of color. In particular, the articles explored urban parents’ perspectives on their children’s mathematics learning and issues of equity. In one of the studies, Remillard and Jackson (2006) found that, although African American parents in a low-income neighbourhood viewed themselves as “critical players in their children’s learning”, the implementation of reform-oriented curriculum tended to disempower them with regard to school mathematics. The parents’ limited understanding of the reform-based approaches meant their access to the discourse of reform was also limited. Their study highlights the need to examine the impact of curriculum reforms on parents, especially since their participation and support is increasingly sought in their children's education.

In a different study, Gutstein (2006) reported some interesting views of Latino parents who supported social justice mathematics curriculum for their children in a 7th-grade public school classroom in a low-income area in Chicago. As Gutstein noted, we know little about how parents in general, let alone Latino parents, think about social justice issues in a mathematics curriculum. His fascinating study described how the parents considered dealing with and resisting oppression as necessary parts of their existence and, at the same time, viewed mathematics as an integral and significant component of life:

> Because (mathematics) education should prepare one for life—and injustice, resistance, and mathematics were all interconnected parts of life—an education made sense if it prepared children to be aware of and respond to injustices that they faced as members of marginalized communities (Gutstein, 2006, p. 331).

In sum, Gutstein identified five themes from these parents' voices on teaching and learning mathematics for social justice, themes that hopefully will lessen the existing deficit notions towards communities of color and "uneducated" parents, and provide a platform for addressing inequities in curriculum access to powerful mathematics:

(a) oppression and resistance are dialectically related—and both are part of life;

(b) mathematics is a central part of life—utilitarian and critical views;

(c) education—including mathematics—should prepare students for life;

(d) education should be specifically about politics; and

(e) parents wanted their children to know more—not less—about the real world

(Gutstein, 2006, pp. 340-341).

Examining community and home contexts and parent perspectives is clearly crucial for improving the mathematics learning and participation of all students. However, many other avenues of action are needed for supporting more equitable curriculum access to powerful mathematical ideas. We explore some of these avenues in the next section.

Improving Learning Access to Powerful Mathematical Ideas

Much has been said in this volume about the powerful mathematical ideas that students need to learn; however, improving students' *learning access* to these powerful mathematical ideas continues to be both a complex and a vexed issue. More research is needed both to illuminate this issue and to design learning praxis in mathematics education that will generate genuine improvement in student access. Without such research, we will continue to see students avoid mathematics and begin to "tune out" before they even leave the elementary grades.

Studies investigating the learning of mathematics have long been the focal point of mathematics education research (Grouws, 1992; Lester, 2007). However, learning access goes beyond learning in that it involves issues relating to curriculum, culture, opportunity, equity and willingness to learn. Consequently we need to consider what kind of research is needed, not only to improve learning, but also to take cognizance of features like culture, curriculum, equity, and willingness to learn.

With respect to research that is needed to improve learning, an important starting point is the *primary intuitions* (Fischbein, 1975) that students bring to any learning situation. It is apparent that the literature is already replete with a rich volume of studies that exhibit the kind of informal knowledge and understandings that students bring in various areas of mathematics; moreover, this literature also suggests that students are capable of constructing new knowledge by reorganizing their own existing conceptual schemas. Broad and comprehensive reviews of this literature are available across the following topics: *whole number concepts, operations, and number sense* (Carpenter & Moser, 1984; Clements & Sarama, 2007; Fuson, 1992; Greer, 1992; Sowder, 1992; Verschaffel, Greer, & DeCorte, 2007), *rational numbers and proportional reasoning* (Behr, Harel, Post, & Lesh, 1992; Lamon, 2006; Lamon, 2007), *geometry and measurement* (Battista, 2007; Clements & Battista, 1992; Lehrer & Chazan, 1998), *algebra* (Bell & Lins, 2001; Carraher, 2007; Fey, 2003; Filloy & Sutherland, 1996; Kieran, 1992, 2007; Kaput, 1989), *probability and statistics* (Ben-Zvi & Garfield, 2004; Jones, 2005; Jones, Langrall, & Mooney, 2007; Shaughnessy, 1992, 2007; Shaughnessy, Garfield, and Greer, 1996; Watson, 2006) and *problem solving and mathematical modelling* (Lesh & Zawojewski, 2007; Lester, 1994; Lester & Kehle, 2003; Schoenfeld, 1992)

As valuable as these studies are, they pay scant attention to the mathematical thinking of socially and economically disadvantaged students and to students from different ethnic backgrounds. The studies represented above tend to focus largely on the *dominant* population (Diversity in Mathematics Education Center for Learning and Teaching [DiME], 2007) and, as a consequence, the researchers from DiME, pinpoint a much needed area for future research: “A charge to the field of mathematics education, then, is to conduct research within non-dominant populations of students who experience marginalization, and to attend to the positioning of these groups vis-à-vis their White[authors’ capitalization] counterparts… .”(p. 404). This charge clearly involves factors like culture, curriculum, and equity, because, as the DiME researchers point out, the practices and ways of reasoning of students from non-dominant populations are often marginalized or even ignored in traditional mathematics classrooms. This poignant call for research is not new; in fact, more than 12 years ago, Silver, Smith, and Nelson (1995) claimed that low levels of participation and performance in mathematics by females, ethnic minorities, and the poor were not primarily due to lack of ability or potential but rather to educational practices. The apparent lack of progress in this area of research simply highlights the definitive need for wide-ranging investigations on mathematical learning that begin to address issues of race, power, economic and social disadvantage and the multiple ways that students participate in mathematics. (e.g., Gutstein, 2003, in press; Ladson-Billings, 2006; Moses & Cobb, 2001)

The constructs, *participation* and *identity*, and the way they relate to *opportunities to learn* (DiME, 2007) provide a promising framework for research that could address mathematical learning from a cultural activity perspective. The notions of participation (e.g., Boaler, 1999; Boaler & Greeno, 2000; Cobb & Hodge, 2002; Gutierrez & Rogoff, 2003; Hand 2003) and identity (Sfard & Prusak, 2005; Wenger, 1998) are difficult to define precisely, but they are concerned with the so-called cultural resources that students bring to a mathematical learning environment and the access students have to mathematical engagement in that learning environment. More importantly, research that uses a participation-identity framework shifts the focus from assessing what an individual student knows to discerning what she or he has had an opportunity to learn (Greeno & Gresalfi, in press). Accordingly, future research on improving learning access to powerful mathematical ideas needs to pay greater cognizance to the social and political structures that shape learning environments. We say more about this in discussing learning environments and equitable access in the next section.

The need for studies that address mathematical learning from a cultural- activity lens brings into sharp relief cognitive research that pays attention to context and to social perspective. For these reasons, the growing field of distributed cognition (situated cognition) (e.g.. Brown, Collins, & Duguid, 1989; Lave, 1988; Lave & Wenger, 1991; Saxe, 1991, 1996) and its application to technological learning environments (Noss & Hoyles, 1996; Pratt, 2005) provide a valuable knowledge base for future researchers. In a similar way research that adopts a social perspective to learning (Afamasaga-Fuata'i, 2002; Bauersfeld, 1980; Civil & Andrade, 2002; Cobb, Stephan, McClain, & Gravemeijer, 2001; Ladson-Billings, 1997; Vygotsky, 1981; Zevenbergen, 2001) has special relevance to learning studies that deal with the social and political structures of classroom learning environments.

Finally, the field of research methodologies impinges on studies that would address socio-cultural aspects of mathematical learning. Although *teaching experiment* and *design experiment* methodologies, which address psychological and sociological aspects of classroom learning ([revisited later in this chapter] Brown, 1992; Cobb, 2000; Ball, 2000; Steffe, Thompson, & von Glasersfeld, 2000) have been around for some time now, it is only in recent years that their scope for dealing with cultural aspects like participation, identity, and assessment of opportunity to learn have become more apparent (Bartolini-Bussi & Bazzini, 2003; Clarke, 2001; Cobb & Hodge, 2002; Cobb, Stephan, McClain, & Gravemeijer, 2001; Gravemeijer, 1998). In particular, Cobb and Hodge (2002) carried out a study that brings into sharp relief "the relations between the specifically mathematical practices in which students participate in the classroom and the practices of out-of-school communities of which the students are members" (p. 251). The authors claimed that these relations are the locus of the successes and inequities that arise in mathematics classrooms.

In essence, Cobb and Hodge's (2002) conclusion highlights the challenge for future research on learning access to powerful mathematical ideas. If opportunity to learn and equity issues are to be addressed using teaching experiments or other related methodologies, learning activities and actions need to be viewed through a wide-ranging sociocultural lens.

Supporting Learning Environments that Provide Learners with More Equitable Access to Powerful Mathematical Ideas

In this section, we consider how research might inform the establishment of learning environments that address opportunity to learn and equitable access to powerful mathematics. Although research on instructional environments is the major focus, there is of necessity a strong emphasis on learning and the reflexivity among teaching, learning, and curriculum. We direct our attention to research on effective practice, teaching and learning models, the role of technology, and classroom research methodologies. We also look briefly at the kind of research that shows promise in clarifying and explicating these issues.

In the first edition of this handbook, we made reference to the precedence of research in the 20th century on predominant models of teaching mathematics that were expected to have salience for all teachers and all students. More specifically, we discussed four models of teaching identified by Kuhs and Ball (1986) as being distinctive and dominant during the latter part of last century: *learner-focused* (mathematics teaching that focuses on the learner's personal construction of mathematics); *content-focused with an emphasis on conceptual understanding* (mathematics teaching that is driven by the content itself but emphasizes conceptual understanding); *content-focused with an emphasis on performance* (mathematics teaching that emphasizes student performance and mastery of mathematical rules and procedures); and *classroom-focused* (mathematics teaching based on knowledge of effective classrooms). In responding to these models of teaching from a futuristic perspective we stated that although they were useful in providing broad characterizations of classroom practice, they were not sensitive enough to address the complexities of mathematics teaching that has as its goal the provision of equitable access for all children. In essence we were claiming that although key ideas like students' construction of mathematics, conceptual understanding, mathematical performance, and effective classroom practice are and will continue to be critical for research on mathematical learning environments, they don't go far enough especially with respect to the dimension "opportunity to learn mathematics".

Much of the research on supportive mathematical learning environments since the first edition of this handbook bears out our advocacy for a more socially pervasive and equity-driven approach to research on so-called models of teaching. The longitudinal studies of Boaler (1997, 1998, 2002, 2003, 2006a, 2006b) have provided a distinctive direction to research "on classroom mathematical practices that serve to level the playing field in classrooms of students from diverse backgrounds" (DiME, p. 411). Starting out with research in 1997 that focused on critical elements of mathematical instruction, espoused in the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989), Boaler found that reform mathematics classrooms were more likely to engage students in effective mathematical practices and discourse than traditional approaches. Further research (Boaler & Greeno, 2000) that activated and explored various social aspects in mathematical classroom activity revealed that students in these settings came to view mathematics as a meaningful part of their lives. This latter study and related studies (e.g., Boaler, 2002) lead to the conclusion that the nature of a classroom's mathematical activity is an important factor in providing opportunities for more students to see themselves as mathematical learners. Boaler (2006b) has taken the social aspect of mathematical activity even further in her research and development at the Railside Mathematics Department. The teachers at this school have employed heterogeneous grouping and a multidimensional approach to mathematical instruction: one that was designed to counter student status differences and to encourage students to be responsible for each other. This approach, referred to by Boaler (2006a) as "relational equity," has not only generated equitable mathematical achievement but has also helped students to learn to act in more equitable ways in the classroom and school environments.

Related research on mathematical learning environments and opportunity to learn has focused on a number of critical issues*: ethnic diversity and the achievement gap* (Bol & Berry, 2005; Lee & Wong, 2004; Moses-Snipes & Snipes, 2005; Rumberger & Palardy, 2005; Woodward & Brown, 2006); *ability grouping* (Ireson, Hallam, & Hurley, 2005); *equitable learning environments* (Berry, 2004, 2005; Gutstein et al, 2005; Matthews, 2005; Rousseau & Powell, 2005; Rubel & Meyer, 2005), *participation in ancillary programs* (Craig & Kacer, 2000; Cohen, 2003; Seaton & Carr, 2005) and *creating a culture of academic success* (Buxton, 2005). Boaler's studies and those listed above have already documented substantial information on the complexity of mathematical learning environments especially in relation to topics and concerns like opportunity to learn and equity. It is already clear from this research that there are no simplistic models of teaching that will address the litany of diversity and equity issues pertaining to mathematical learning environments. However, there is an opportunity, through this research, to document more and more socio mathematical and pedagogical content knowledge for teachers as they pursue the goal of equitable mathematics learning. As the researchers from DiME(2007) advocate, "Additional research needs to be conducted on the application of these principles [design principles fostering equity] across a range of schools and classrooms, and the broader social and structural barriers that shape, support, and constrain their proper implementation" (p. 411).

Technological learning environments deserve special attention in the continuing research agenda. If, as seems very reasonable, we assume that governments have the financial resources to provide computer access for every child in the near future, technological learning environments possess the potential to provide viable and equitable access to powerful mathematical ideas for both individual students and groups of students. Moreover, their unique characteristics are well suited to the diversity of background and learning styles that students bring to instruction.

As early as 1996, Noss and Hoyles claimed that technology can be used to develop distinctive learning environments by providing a window for viewing children's construction of meaning. Moreover, they have also claimed that technology, such as microworlds (Pappert, 1980; Pratt, 2005), are capable of producing a language through which meanings can be externalized and emerging knowledge can be expressed, changed, and explored. Certainly recent research in this area (Johnson, 2001; Noss & Hoyles, 1996; Parker, 2003; Pratt, 2005; Steffe & Olive, 2002; Moyer, Niezgoda, & Stanley, 2005) suggests that technological learning environments like microworlds enable students to build their own mathematical ideas and to do it naturally. Moreover, large-scale technological projects (e.g., The Learning Federation, 2007) that deliver mathematical learning objects on a *national* basis to classrooms from Preparatory though Year 12, are beginning to emerge and to create new learning environments and opportunities for all children.

In spite of the ubiquitous promise of technological learning environments, Langrall, Nisbet, Mooney and Jones (chapter 6) have reported that the acceptance and use of technology in mathematics classrooms continues to be limited. They cite data from the 2003 Trends in International Mathematics Science Study [TIMSS] (Mullis, Martin, Gonzales, & Chrostowski, 2004) that exposed minimal use of computers even in countries with high availability. The reluctance of schools to use technology or even encourage students to engage in mathematical learning via technology is a critical issue for future research in mathematics education. It raises questions concerning the match between technology and classroom practice and in what Dick

(2007) refers to as *pedagogical fidelity*: the extent to which teachers and students believe that a technological tool allows them to act mathematically in ways that correspond to the nature of mathematical learning reflected in the teacher's practice (p. 1187). Zbiek, Heid, Blume and Dick (2007) claimed that pedagogical fidelity is a key construct for future research and they also advocate the need for researchers to document relationships between teacher and technology and students and technology, and to characterize differences between technological and non-technological learning environments. Clearly the next decade must endeavour to produce research on classroom environments that addresses the indifference or is it hostility that still prevails with respect to technologically-oriented mathematical learning.

Research methodologies constitute a critical area for future research into supportive and equal-opportunity teaching and learning environments. Hiebert and Grouws (2007) have recently observed that the task of documenting evidence about the effects of teaching on learning is methodologically difficult. They underscore three special methodological difficulties: accounting for relevant factors, creating appropriate measures, and, dealing with individual differences of students. Of these difficulties most progress seems to have been made in recent years in the area of creating appropriate measures. In particular, there are now extensive and expanding methodologies for measuring the plethora of activities that characterize classroom learning environments and teacher and student actions.

Starting with classroom observations (Dunkin & Biddle, 1974) and teacher survey questionnaires (Rowan, Correnti, & Miller, 2002), Hiebert and Grouws (2007) refer to *video observation* (Stigler, Gonzales, Kawanaka, Knoll, & Serano, 1999), *large-scale carefully targeted questionnaires* (Ross, McDougall, Hogaboam-Gray, & LaSage, 2003), *combination questionnaire-case-studies* (Stecher & Borko, 2002), *teacher logs* (Rowan, Harrison, & Hayes,

2004) and *teacher and student artefacts* (Borko, Steecher, Alonzo, Moncure, & McClam, 2005). To those above may be added methodologies from the previous decade that continue to enable researchers not only to make multiple observations of classroom practice and processes, but also to integrate research on teaching with research on learning, to focus on issues of equity, and to investigate teacher characteristics. Examples of research reflecting these latter methodologies include: *Cognitively Guided Instruction* (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996); *classroom teaching experiments* (Cobb, 2000; Kelly & Lesh, 2000); *educational development and developmental research* (Gravemeijer, 1998); *"first-person" research or working from the inside* (Ball, 2000; Lampert, 1998); *mathematics knowledge as a collective enterpise* (Bartolini-Bussi, 1990); *recherches en didactique des mathematiques* (Brousseau, 1992); *the "open approach" method* (Nohda, 2000) and *teacher development experiments* (Simon, 2000). The combined list of methodologies is impressive and given the phenomenal growth of observation technology and technology to capture classroom discourse, there is considerable cause for optimism. However, as Hiebert and Grouws also note, many of these methodologies are currently in the development and testing phase; consequently they contend that much more research is needed to evaluate these methodologies and to extend them to the stage where that they are capable of accounting for relevant classroom and instructional factors, of supporting research on individual differences and ipso facto of examining equitable access to powerful mathematical ideas.

In summary, our examination of the ways in which research can support the creation of more powerful learning environments, has highlighted some of the promising research endeavours in recent years. These include research that has focused on equity-driven learning environments, research into technologically-supported learning environments that provide unique opportunities for all children to learn, and research methodologies that have the potential to illuminate learning environments vis-a-vis access and opportunity for learners from diverse backgrounds. The research base on *learning environments* is extensive and incredibly complex. What is needed in the next decade are carefully-orchestrated reviews that tease out key instructional patterns such as discourse, norms, and relationships (Franke, Kazemi, & Battey, 2007), major connections with issues such as equity (Boaler & Greeno, 2000) , and more general theories of teaching that stimulate and sustain future research (Hiebert & Grouws, 2007). We give greater consideration to teaching issues in the next section, where we consider the domain of mathematics teacher education—a key plank in propelling our discipline forward.

A Search for Identity: A Key Stage in the Never-Ending Dialogue between Research and Practice in Mathematics Teacher Education

One of the four questions that were raised at the beginning of this chapter was: "How can research contribute to the kind of teacher education and teacher development programs that will be needed to facilitate student access to powerful mathematical ideas?" In our attempt to address this issue in the first edition of this Handbook, we focused mainly on teacher change, arguing that this is a relatively young area of study and that in the coming years, research on teacher change will undoubtedly evolve to produce substantial knowledge. We further envisioned that new models of teacher education and of professional development programs will be generated and explored and that new types of collaboration between researchers and teachers will emerge. We concluded by stating that "This body of knowledge could significantly contribute to our understanding of fundamental, controversial issues related to mathematics teaching, including: (a) What do mathematics teachers need to know? (b) How do mathematics teachers come to know? and, (c)

What are the relationships between teacher knowledge and classroom practices, including how teachers assess students' complex achievements?" (English, 2002, p. 798).

The vision presented in the first edition was substantially fulfilled during the relatively short period of time that has passed since then. The domain of mathematics teacher education has witnessed a dramatic flourishing, as can be seen from the significant increase (50% per year), from 2005, in the publication output of the *Journal for Mathematics Teacher Education* as well as from ICME's (International Committee of Mathematics Education) survey on "Professional Development of Mathematics Education", which reported that "while still relatively young, mathematics teacher education (MTE) has mushroomed in the past five years in particular with multiple approaches and initiatives evident" (Adler, Ball, Krainer, Lin, & Novotna, 2005). A third indication is the theme of the recently conducted ICMI 15 study, co-chaired by Even and Ball on "The Professional Education and Development of Teachers of Mathematics" (the study conference was held in May, 2005, in Brazil, and related publications are at final stage of preparation).

This flourishing highlights the importance of dealing with substantial issues regarding the relationship between research and practice in mathematics teacher education. We deal here with only one such question, namely, *What is the nature of research in mathematics teacher education?* It is clearly impossible to encompass, in this part of the concluding chapter, all the issues related to this significant question, let alone to refer to other, substantial issues in mathematics teacher education. Hence we reflect here only on some aspects of this question, and raise some issues for further consideration.

The partial response that is provided here to the question: "What is the nature of the *current* research in mathematics teacher education?" is mainly based on Adler et al.'s (2005) survey on professional development in mathematics education. This survey addressed 282 articles, published in 1999-2003, in international mathematics education journals, international handbooks of mathematics education, international mathematics education conference proceedings and some regional sources from various parts of the world. On the basis of these publications, Adler et al. formulated four main claims regarding the current nature of mathematics teacher education research:

Claim 1: *Small-scale qualitative research predominates*. Most studies (more than 60%) focus on a single teacher or on small groups of teachers ($N < 20$) within individual programs or courses.

Claim 2: *Most teacher education research is conducted by teacher educators studying the teachers with whom they are working*. At least 72% of the articles that were examined for the survey were of this type.

Claim 3: *Research in countries where English is the national language dominates the literature*. This predominance is mainly reflected in mathematics education journals (about 75%), and less so, but still evident, in the major proceedings of international, mathematics education conferences (i.e., 43% in PME).

Adler et al. clarify that these first three claims provide valuable information regarding the nature of the emerging field of mathematics teacher education: *Who is writing* (mainly teacher educators), *from and in what setting* (countries where English is a national language) and *how* (a predominance of small- scale qualitative studies). They argue that these characterizations (*who*, *how* and *where*) have important consequences for the *what*, that is, the core questions that are currently studied in mathematics teacher education.

Claim 4: *Some questions have been studied, not exhaustively, but extensively, while other important questions remain unexamined.* Adler et al. (2005) reported that many of the published articles involve efforts to show that particular programs of teacher education work. A large number of papers deal with reform processes (e.g., studies of teachers learning or relearning mathematics, teachers learning about students' thinking, their language, their orientations, and pedagogical practices).

Towards the end of their article, Adler, et al, (2005) listed five issues that they classify as notably missing in the research: teachers learning outside of reform contexts; teachers learning from experience; teachers learning to directly address inequality and diversity in their teaching of mathematics; comparisons of different opportunities to learn; and "Scaling up" (studying what it means to extend a program that has worked in one setting to another setting).

Lists of issues ("missing links") that are presented as essentially in need of research in mathematics teacher education can be found in contemporary as well as in more dated, related chapters and other documents (e.g., Cooney, Grouws, & Jones, 1988; Even & Ball, 2005; Jaworski & Gellert, 2003; Philipp, 2007; Sowder, 2007; Tirosh & Graeber, 2003). Typically, these publications report on the current stage of research in some sub-themes of this domain (e.g., teachers' mathematical knowledge, teachers' change, teaching and classroom practice, teachers' beliefs, teachers' values), often acknowledging that the existing research is still minute, and thus they raise issues for further research. These recommendations include substantial issues, each of which deserves a massive, long-scale study (e.g., What kind of mathematics knowledge is needed for teaching? What are the characteristics of effective, teacher education programs? Do mathematics educators integrate research on teaching into their own teaching?).

The wide range of issues currently under study on mathematics teacher education and the diversity of recommended issues for further research highlight the importance of reflection on the field's identity. Such an attempt, much like the search for identity of mathematics education as a research domain (Sierpinska & Kilpatrick, 1998) will probably not lead to overall agreement of the type: “This is what research in mathematics teacher education is (or should) be.” Still, we argue that the time is ripe for the emerging field of mathematics teacher education to address the crucial question: “What is research in *mathematics teacher education*, and what are its results?” We conclude by suggesting that the five questions that were discussed with respect to mathematics education (Sierpinska & Kilpatrick, 1998) could serve as a guide for such a reflection (these questions are listed here, with a slight, but meaningful adjustment from mathematics education to mathematics teacher education):

- What is the specific object of study in mathematics teacher education?
- What are the aims of research in mathematics teacher education?
- What are the specific research questions or problematic of research in mathematics teacher education?
- What are the results of research in mathematics teacher education?
- What criteria should be used to evaluate the results of research in mathematics teacher education?

One of the major challenges facing mathematics teacher education is how to assess effectively students’ mathematical achievements. Assessing students' complex achievements is one of the most important and difficult tasks teachers face, as we indicate in the next section.

Emerging Issues in Assessing Students’ (and Teachers’) Complex Achievements

Not only is effective assessment of students’ achievements a critical issue to address, but also the assessment of teachers’ professional growth and the assessment of instructional programs. Undertaking such assessments raise many contentious issues that will continue to demand substantial research. In our own field, we must attend to the development of tools for documenting, assessing, and (in some cases) measuring the kind of complex achievements that we hope to elicit from students, teachers, and programs of instruction (Doerr & Lesh, 2003; Lesh & Clark, 2000). Yet, even though, in some topic areas, mathematics educators have made enormous progress to clarify the nature of students' developing mathematical knowledge and abilities, assessment instruments tend to be based on assumptions that are poorly aligned with modern views about the nature of mathematics, problem solving, learning, and teaching.

Educators have tended to think about the nature of mathematical knowledge (and the mind) as though it is similar to our most sophisticated technology (Kelly & Lesh, 2000). Consequently, there has been a gradual transition:

- Away from analogies based on *hardware* - where teachers are led to believe that the construction of mathematical knowledge in a child's mind is similar to the process of assembling a robot, or an automobile, or a string of condition-action rules in a computer program. That is, whole systems are considered to be no more than the sum of their parts.
- Beyond analogies based on *software* - where silicone-based electronic circuits often involve layers of recursive interactions that may lead to emergent phenomena at higher levels, which cannot be derived from characteristics of phenomena at lower levels.
- Toward analogies based on *wetware* – where neurochemical interactions may involve "logics" that are fuzzy, partly redundant, partly inconsistent, and unstable.

As an age of biotechnologies gradually supersedes an age of electronic technologies, the systems that are priorities for mathematics educators to explain, create, modify, predict, or influence are no longer assumed to be inert. They are complex, dynamic, continually adapting and self-regulating systems; and, their behaviors and "ways of thinking" cannot be described in superficial ways using simple-minded input-output rules (or simple algebraic or probabilistic equations) that ignore emergent phenomena and feedback loops in which second-order effects often outweigh first-order effects (Lesh, 2006). Often, they are not simply given in nature; instead, their existence is partly the result of human constructions. Furthermore, it may not be possible to isolate them because their entire nature may change if they are separated from complex holistic systems in which they are embedded. Or, they may not be observable directly, but may be knowable only by their effects. Or, when they are observed, changes may be induced in the “subjects” being investigated. So, the researchers and their assessment instruments become an integral part of the system being measured.

Regardless of whether attention is focused on students, groups, teachers, or other learners or problem solvers, if we want to assess something more than low-level factual and procedural knowledge, attention generally needs to focus on models and conceptual systems that the “subjects” develop to interpret their experiences. But, most modern theories of teaching and learning believe that the way learning and problem solving experiences are interpreted is influenced by *both* (internal) conceptual systems and by (external) systems that are encountered. So, the interpretations that learners and problem solvers construct involve interactions between (internal) structuring capabilities and (external) structured environments — Consequently, the following kinds of assessment issues arise: When different problem solvers are expected to interpret a single problem solving situation in fundamentally different ways, what does it mean to speak about “standardized” questions? When different learners are expected to interpret a given learning experience in fundamentally different ways, what does it mean to speak about “the same treatment” being given to two different participants? When assessments are integral parts of the systems being investigated, what does it mean to speak of “detached objectivity” for “outside” assessments?

Consider the case of high stakes standardized tests. It is widely recognized that such tests are powerful leverage points that influence (for better or worse) both *what* is taught and *how* it is taught. In particular, when such tests are used to clarify (or define) the goals of instruction, they go beyond being neutral *indicators* of learning *outcomes*; they become powerful *components* of the *initiatives* themselves. Far from being passive indicators of non-adapting systems, these tests have powerful positive or negative effects, depending on whether they support or subvert efforts to address desirable objectives. Therefore, when they are poorly aligned with the standards for instruction, they often create serious impediments to curriculum reform, teacher development, and student achievement.

- *For students*: If standardized tests focus on only narrow and shallow conceptions of the mathematical knowledge and abilities that are needed for success (beyond school) in a technology-based society, then such tests are likely to recognize and reward only students whose profiles of abilities fit this biased conception. On the other hand, if new understandings and abilities are emphasized in simulations of “real life” problem solving situations, than a broader range of students naturally emerge as having great potential (Lesh & Doerr, 2003). Similarly, under the preceding circumstances, our research has shown that most students are able to invent better mathematics and science ideas than those associated with their prior failure experiences with traditional textbooks, tests, and teaching (Lesh & Yoon, 2006; Lesh, Hole, Hoover, Kelly, & Post, 2000).
- *For teachers*: Even though gains in student achievement surely should be one factor to consider when documenting the accomplishments of teachers (or programs), it is foolish to assume that great teachers always produce larger student learning gains than their less great colleagues. What would happen if a great teacher chose to deal with only difficult

students or difficult circumstances? What would happen if a great teacher chose to never deal with difficult students or difficult circumstances? No teacher can be expected to be "good" in "bad" situations (such as when students do not want to learn, or when there is no support from parents and administrators). Not everything "experts" do is effective, and not everything "novices" do is ineffective. Furthermore, no teacher is equally effective across all grade levels (from kindergarten through calculus), with all types of students (from the gifted to those with physical, social, or mental handicaps), and in all types of settings (from those dominated by inner-city minorities to those dominated by the rural poor). In fact, characteristics that lead to success in one situation often turn out to be counterproductive in other situations. Consequently, it is naïve to make comparisons among teachers using only a single number on a "good-bad" scale (without identifying profiles of strengths and weaknesses, and without giving any attention to the conditions under which these profiles have been achieved, or the purposes for which the evaluation was made). Nonetheless, if tests include *thought-revealing activities* that teachers can use to learn about their students ways of thinking, then a variety of positive results may occur (Lesh & Yoon, 2006; Lesh, Hole, Hoover, Kelly, & Post, 2000).

- *For programs:* When evaluating large and complex program innovations, it is misleading to label them "successes" or "failures" (as though everything that successful programs did was effective, and everything unsuccessful programs did was not effective). All programs have profiles of strengths and weaknesses. Most programs "work" for achieving some types of results but "don't work" for others: and, most are effective for some students (or teachers, or situations) but are not for others. In other words, most programs "work" some of the time, for some purposes, and in some circumstances. But, none “work” all of the time, for all purposes, in all circumstances. For example:

- When the principal of a school doesn’t understand or support the objectives of a program, the program seldom succeeds. Therefore, when programs are evaluated, the characteristics and roles of key administrators also should be assessed; and, these assessments should not take place in a neutral fashion. Attempts should be made to optimize understanding and support from administrators (as well as parents, school board members, and other leaders from business and the community). Then, during the process of optimization, documentation should be gathered to produce a simple-yet-high-fidelity trace of continuous progress.

    The success of a program depends on how much and how well it is implemented. For example, if only half of a program is implemented, or if it is only implemented in a half-hearted way, then 100% success can hardly be expected. Powerful innovations usually need to be introduced gradually over periods of several years. So, when programs are evaluated, the quality and extent of the implementation should be assessed; and, this assessment should aim toward systemic validity. It should not pretend to be done in a neutral fashion. Optimization and documentation are not incompatible processes. The goal is to improve performance, not just audit. So, assessment should be longitudinal and recursive (Fredericksen & Collins, 1989)

- A systemically valid test (or item, or report) is one that induces in the education system curricular and instructional changes that foster the development of achievements that the test is designed to measure.

How can we assess the extent to which students, teachers, and programs achieve deeper and higher-order goals of instruction? How can these assessments avoid biases related to cultural, ethnic, and gender differences? How can they provide information that is as useful as possible to a variety of different decision makers – ranging from students, to parents, to teachers, to policy makers? To begin to address such questions, it is important to emphasize that assessment is not the same as evaluation. Whereas the goal of *evaluation* is to assign a value to the subjects being inspected, the goal of assessment is to provide information for decision makers - perhaps by describing the subject's location (and recent history of progress) in some landscape of ideas and abilities that it is desirable for them to learn.

To create assessment programs in which the preceding kinds of factors are taken into account, two distinct types of assessment designs are useful to sort out: *pretest-posttest designs* (referred to here as *mechanistic*) and *continuous (systemic) documentation, monitoring, and feedback*.

- *Mechanistic pretest-post designs* are intended to prove that "it" works; but they often promote conditions that minimize the chances of success by measuring progress in terms of deficiencies with respect to simplistic conceptions of success.
- *Systemic documentation, monitoring, and feedback* is intended to (simultaneously) document progress as well as encourage continuous development in directions that are increasingly "better" without using a simplistic definition of "best" to define the next steps.

Table 32.1 displays some of the key features of these two types of assessment.

INSERT TABLE 32.1 ABOUT HERE

The important points to note in Table 32.1 include the following. (a) It is possible to *examine* students closely, without relying on non-productive ordeals. (b) It is possible to *document* achievements and abilities without reducing relevant information to a single-number "score" (or letter grade). (c) It is possible to *assess* "where students are" and "where they need to go" without *comparing* students with one another along a simplistic "good-bad" scale. (d) It is possible to address the decision-making needs of administrators without neglecting the decision-making needs of students, parents, and teachers. (e) It is foolish to say that a student (or teacher, or program) is "good" without saying "Good for what?" or "Good under what conditions?"

The preceding observations suggest that there should be close connections among assessments of achievement for students, teachers, programs, program administrators, and program implementations. Regardless of whether the entity being studied is a student, or a teacher, or a program, the assessment should be expected to be wildly naive if: (a) it takes information from only a single source, (b) it results in only simpleminded comparisons among individuals on a uni-dimensional "good-bad" scale, (c) it ignores conditions under which alternative profiles of achievement occur, and (d) achievement is assessed using tests which reduce expertise to simplistic lists of "condition-action rules" (e.g., Given…, the student will… .).

Finally, assessment is about generating information that is useful to decision makers; and, in education, these decision makers range from students, to teachers, to parents or guardians, to program administrators, to policy makers. Decisions may range from high stakes decisions that are irreversible decisions to low stakes decisions where timeliness may be more important than high levels of precision or accuracy. In any case, it is foolish for educators to adopt a "one size fits all" policy that is static and that treats all “subjects” as if their abilities were accurately characterized by a single point on a number line extending from bad to good.

In an age when many applied sciences are using a variety of graphics-oriented and interactive global information systems to display complex information about complex systems, it is remarkable that educators continue to limit themselves to single-number descriptions of students, teaching effectiveness, and programs of instruction. For example, if the mathematics curriculum is visualized as a three-dimensional topographical map where the mountains represent “big ideas” and surrounding valleys represent related lower level skills, a given student’s progress report might look similar to a map in a historical atlas that shows areas that have been conquered by an invading army — In the context of such a map, a simple arrow labeled “you are here” might provide a great deal of information about directions for future progress (Lesh & Lamon, 1993).

Mathematics educators need to design better tools for generating information about deeper and higher-order achievements of students, teachers, and programs. Developments in research designs in mathematics education provide promising opportunities for improving such tools, although we still have a good way to go here.

Assessing and Improving Research Designs in Mathematics Education

Mathematics education is still in its “infancy” as a field of scientific inquiry. This is evident in the fact that the first journals devoted purely to research only started appearing in the 1960s, prominent among which were the *Zentralblatt für Didaktik der Mathematik* (ZDM) and *Educational Studies in Mathematics*(ESM). In the early 1970s, there was an explosion of new journals devoted to research - including *the Journal for Research in Mathematics Education* (JRME) and the *Journal für MathematikDidaktik* (JMD). Until this time period we had no professional organization for researchers; and, we had few sharable tools to facilitate research.

Arguably there were journals such as *the l'Enseignment Mathematique* (founded in 1899 in Geneva), *The Mathematics Teacher* (founded by NCTM in 1901), the *School Science and Mathematics Journal* (founded in 1901 by the SSMA) and *The Mathematical Gazette* (founded in 1894 in the UK), all of which were supposed to address the teaching and learning of mathematics. However, a survey of the papers appearing in these journals suggests that few were aimed at advancing what is known about mathematics problem solving, leaning, or teaching. For the purpose of our discussion, research as we mean it today only started in the 1960s and depended mainly on theory borrowing (from other fields such as developmental psychology or cognitive science). We really had no stable research community – with a distinct identity in terms of theory, methodologies, tools, or coherent and well-defined collections of priority problems to be addressed. Only recently have we begun to clarify the nature of research methodologies that are distinctive to our field (Biehler et al., 1994; Bishop et al., 2003; Kelly & Lesh, 2000; English, 2002); and, in general, assessment instruments have not been developed to measure most of the constructs that we believe to be important. These facts tend to be somewhat alarming to those who were not firsthand witnesses to the birth of our field or familiar with its history - and whose training seldom prepares them to think in terms of growing a new field of inquiry (Lesh & Sriraman, 2005a; Schoenfeld, 1999).

On the positive side, new research designs have been developed that are based on new ways of thinking about the nature of students' mathematical knowledge, problem-solving, learning, and teaching – and that involve closer and more meaningful working relationships between many levels and types of both researchers and practitioners. Examples of these research designs include the following.

- *Action Research* in which distinctions between researchers and practitioners often are blurred – as teachers participate as co-researchers and/or as researchers participate as teachers or as designers of instructional activities.
- *Carefully Structured Clinical Interviews* in which relevant data includes more than isolated pieces of information – and also includes patterns of behavior across iterative sequences of tasks.
- *Videotape Analyses* where decisions about *whom* to observe, *when* to observe, and *what* to observe may radically influence the nature of apparent results – and where many interpretation cycles may be needed in order to recognize relevant patterns of behavior.
- *Ethnographic Observations*, which serve to identify the types of mathematical understandings needed for success in various out-of-school contexts; such observations usually need triangulation techniques to compensate for the fact that multiple theoretical perspectives often result in significantly different interpretations of people, places, and practices that are exhibited.
- *New Approaches to Assessment* that focus on deeper and higher-order understandings and abilities – and that focus on complex performances that must be classified in ways that are far more sophisticated than simply "correct" and "not correct."
- *Teaching Experiments* that go beyond investigating *typical* development in *natural* environments to focus on *induced* development in *carefully controlled* and *mathematically enriched* environments – and that investigate the interacting development of students, teachers, and others (e.g., parents, policy makers).

Teaching experiment methodologies (Cobb, 2000; Confrey & Lachance, 2000; Lesh & Kelly, 2000; Steffe, Thompson, & von Glasersfeld, 2000) offer considerable promise in providing more micro and dynamic evidence of students' learning, in contrast to research that offers only a series of discrete snapshots of students' mathematical thinking. In a teaching experiment, an on-going series of teaching episodes is undertaken with individuals, groups, or complete classes of students in such a way that the planning of each exploratory teaching episode is based upon the teacher's and students' thinking and actions in prior teaching episodes. Through these exploratory teaching episodes the teacher/researcher gains first-hand access to a continuous film of students' learning and attempts to build dynamic models of "students' mathematics" (Steffe, Thompson & von Glasersfeld, 2000, p. 268) These models, built through the researchers' retrospective analysis of the teaching experiment, incorporate both a psychological and a sociological perspective (Cobb, 2000, p. 309). As such the models are expected to be able to justify the students' mathematical language and social actions both individually and collectively. In essence the researcher formulates an image of the students' mental operations and an itinerary of what students might learn and how they might learn it.

- *The principled design experiment* (Hawkins, 1997; diSessa & Abelson, 1986) is a potent methodology that explores novel possibilities for learning in what are high-risk situations for software developers. In some sense, principled design experiments are the technological counterpart of the teaching experiment because they provide opportunities for observing students' mathematical thinking in situations that take the students to the brink. In essence, they use software that has the potential not merely to amplify students' cognition but to fundamentally change it (Doerfler, 1993). The Boxer design space (diSessa & Abelson, 1996) is a reconceptualization of Logo that is said to be at least a generation removed from incremental variations of Logo currently available commercially. While this kind of research is in its infancy, it is clear that we need to pursue it with vigor in the early part of this century.

The negative side of the preceding research designs is that the development of widely recognized standards for assessing (and improving) the quality of research designs has not kept pace with the recognition of new levels and types of problems, new theoretical perspectives, and new approaches to the collection, analysis, and interpretation of data. Consequently, there is a growing concern that future progress may be impeded unless criteria and procedures become clearer for optimizing the usefulness, sharability, and cumulativeness of results that are produced by the preceding kinds of innovative research designs.

Research is not just about learning a cluster of "accepted" techniques for gathering data, analyzing data, and reporting results in some standard accepted form. Research is about the development of knowledge; and, in particular, it is about the development of *shared* constructs (models, prototypes, principles, and conceptual systems), which provide *useful* ways to think about problems that are priorities for practitioners in the field to address (Lesh, Hamilton, & Kaput, 2007; Lesh, Lovitts, & Kelly, 2000). Consequently, the design of research involves the development of a coherent chain of reasoning that is *powerful* and *auditable*, and that should be both *meaningful* and *persuasive* to practitioners, researchers, and sceptics. — It cannot be reduced to a step-by-step formula-based process.

Two of the most important factors that determine what kind of information to seek, and what kind of interpretative frameworks to use, involve being clear about: (i) assumptions concerning the nature of the “subjects” being investigated, and (ii) assumptions concerning the “products” that are intended to be produced.

The nature of the “subjects” being investigated. Mathematics educators have come to recognize that students, teachers, classrooms, courses, curricula, learning tools, and minds are complex systems, taken singly, let alone in combination. Dealing with complexity in a disciplined way is the essence of research design (Kelly & Lesh, 2000). In general, the systems we investigate are dynamic, interacting, self-regulating, and continually adapting. Their behaviors cannot be described in superficial ways using simple input-output rules (or simple algebraic or probabilistic functions), which ignore emergent phenomena and second-order effects that result from recursive feedback loops (e.g., when a factor X influences factor Y, which influences factor Z, which in turn, influences factor X).

The nature of the "products" being investigated. If we recognize that research is about the development of sharable and re-useable knowledge, then it is clear that not all knowledge consists of tested hypotheses and answered quantitative questions. For example, in the history of more mature fields of science, many of the most important products that research produces have consisted of tools for creating, observing, classifying, or measuring "subjects" that are considered to be important. Or, they are models or conceptual systems for constructing, describing, or explaining complex systems. Or, they are demonstrated possibilities involving existence proofs in special circumstances. Similarly, in mathematics education, using iterative research techniques of the type emphasized by Kelly and Lesh (2000), participants (which may include researchers as well as students and/or teachers) are challenged to express their current ways of thinking in forms that must be tested and revised repeatedly. Thus, after a series of such testing-and-revising cycles, auditable trails of documentation are produced that enable retrospective analyses to reveal the nature of significant developments that occur.

Investigations that involve these kinds of testing-and-revising cycles often are referred to as *design studies* – because the products that participants generate consist of models or conceptual tools that must be developed and refined using iterative testing-and-revising cycles, similar to those commonly used by engineers and other scientists when they design complex systems ranging from spacecraft to artificial ecosystems (Lesh & Sriraman, 2005a,b). Design studies are particularly beneficial to mathematics education – because the goal is for participants (whose ways of thinking are being investigated) to design thought-revealing artifacts using a process that involves a series of iterative testing-and-revising cycles. Thus, as participants' ways of thinking evolve and become more clear, byproducts of this design process include auditable trails of documentation that reveal important aspects about developments that occur. Numerous examples from the history and philosophy of science and mathematics (e.g., Lesh & Sriraman, 2005a,b; Moreno & Sriraman, 2005) and ongoing findings of Models & Modeling research (e.g., Lesh & Doerr, 2003; Lesh & English, 2005; Lesh, Hamilton & Kaput, 2007) illustrate this notion of mathematics education research as a design science.

Two factors have emerged as having especially strong influences on the kind of research designs that have been pioneered by mathematics educators. First, most of these research designs have been intended to radically increase the relevance of research to practice - often by involving practitioners in the identification and formulation of problems to be addressed, or in the interpretation of results, or in other key roles in the research process. Second, there has been a growing realization that, regardless of whether researchers focus on the developing capabilities of students, or groups of students, or teachers, or schools, or other relevant learning communities, the evolving ways of thinking of each of these "problem solvers" involve complex systems that are not simply inert and waiting to be stimulated. Instead, they are dynamic, living, interacting, self-regulating, and continually adapting systems whose competencies generally cannot be reduced to simple-minded checklists of condition-action rules.

Furthermore, among the most important systems that mathematics educators need to investigate and understand: (a) many do not occur naturally (as givens in nature) but instead are products of human construction, (b) many cannot be isolated because their entire nature may change if they are separated from complex holistic systems in which they are embedded, (c) many may not be observable directly but may be knowable only by their effects, and (d) rather than simply lying dormant until they are acted upon, most initiate actions; and, when they are acted upon, they act back. In particular, when they're observed, changes may be induced that make researchers integral parts of the systems being investigated. So, there may be no such thing as an immaculate perception (See Part II, Kelly & Lesh, 2000).

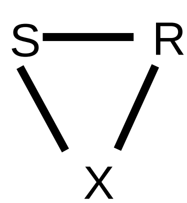
For the preceding kinds of reasons, in mathematics education, just as in more mature modern sciences, it has become necessary to move beyond machine-based metaphors and factory-based models to explain patterns and regularities in the behaviors of relevant complex systems. In particular, it has become necessary to move beyond the assumption that the behaviors of these systems can be described using simple linear combinations of one-directional cause-and-effect mechanisms that are described using closed-form equations from elementary algebra or statistics.

Design science ideas are intended as a framework and do not constitute a "grand" theory (Lesh & English, 2005; Lester, 2005; Michelsen & Sriraman, 2007). That is, they provide a framework (a system of thinking together with accompanying concepts, language, methodologies, tools, and so on) that provides structure to help mathematics education researchers develop both models and theories, which encourage diversity and emphasize Darwinian processes such as: (a) selection (rigorous testing), (b) communication (so that productive ways of thinking spread throughout relevant communities), and (c) accumulation (so that productive ways of thinking are not lost and get integrated into future developments).

In the remaining issue addressed in this chapter, we review the nature and role of semiotic mediation in mathematics education. Semiotic mediation is a popular theoretical construct used (and misused and abused) in the current literature on mathematics education. Given that some of the chapters in this Handbook explicitly deal with semiotic mediation, we consider it worthwhile to compare the perspectives they offer. We do so without covering all the possible issues addressed by the various authors.

The Nature and Role of Semiotic Mediation in Mathematics Education

In recent decades, Vygotsky (1978) has become a major reference in all forms of constructivism (for a critical review of some trends see Vianna & Stetsenko, 2006). The most famous (and cryptic) quotation about semiotic mediation is the following, taken from Vygotsky (1978):

> Every elementary form of behavior presupposes direct reaction to the task set before the organism (which can be expressed with the simple S - R formula). But the structure of sign operations requires an intermediate link between the stimulus and the response. This intermediate link is a second order stimulus (sign) that is drawn into the operation where it fulfills a special function; it creates a new relation between S and R. The term 'drawn into' indicates that an individual must be actively engaged in establishing such a link.
>
> The sign also possesses the important characteristic of reverse action (that is, it operates on the individual, not the environment). Consequently, the simple stimulus-response process is replaced by a complex, mediated act, which we picture as:
>
> 
>
>
> In this new process the direct impulse to react is inhibited, and an auxiliary stimulus that facilitates the completion of the operation by indirect means is incorporated. Careful studies demonstrate that this type of organization is basic to all higher psychological processes, although in much more sophisticated forms than that shown above. The

> intermediate link in this formula is not simply a method of improving the previously existing operation, nor is a mere additional link in an S-R chain. Because this auxiliary stimulus possesses the specific function of reverse action, it transfers the psychological operation to higher and qualitatively new forms and permits humans, by the aid of extrinsic stimuli, to control their behavior from the outside. The use of signs leads humans to a specific structure of behavior that breaks away from biological development and creates new forms of a culturally-based psychological process (Vygotsky, 1978, pp.XXX)

This isolated excerpt is not transparent at all, yet it may be interpreted by means of the following vignette (see Fig. 32.1).

INSERT FIGURE 32.1ABOUT HERE

The vignette, based on actual classroom data (Bartolini Bussi & Boni, 2003), shows what has been called in this Handbook, the *semiotic potential of abacus* (Bartolini Bussi & Mariotti, chapter 28). The pupil uses the abacus, first under the teacher's guide and later by him/herself, to accomplish the given task and constructs personal meanings of positional representation of numbers. The abacus inhibits his/her direct reaction, which comes through the common mistake in transcoding (Power & Dal Martello, 1990) from number words to Arabic numerals, when the zeroes on the right (as in 100) are not overwritten by tens and units (e.g., 10013 instead of 113). In this process, however, he /she does not simply 'improve' his/her performance (i. e., learn how to write correctly particular numbers, which could also be obtained by means of a systematic training), but also learns to 'draw' into this operation a physical (first) and a mental (later) abacus, which may be used with larger and larger numbers and is, potentially, unlimited. Hence the abacus is not simply a technical tool to solve a particular task but a psychological tool that shifts to the plane of pupil’s consciousness the meaning of the positional representation of numbers. Because of this double relationship the artefact may function as a tool of semiotic mediation under the teacher’s guide.

This process has been illustrated by Bartolini Bussi and Mariotti in chapter 28 by means of the following scheme:

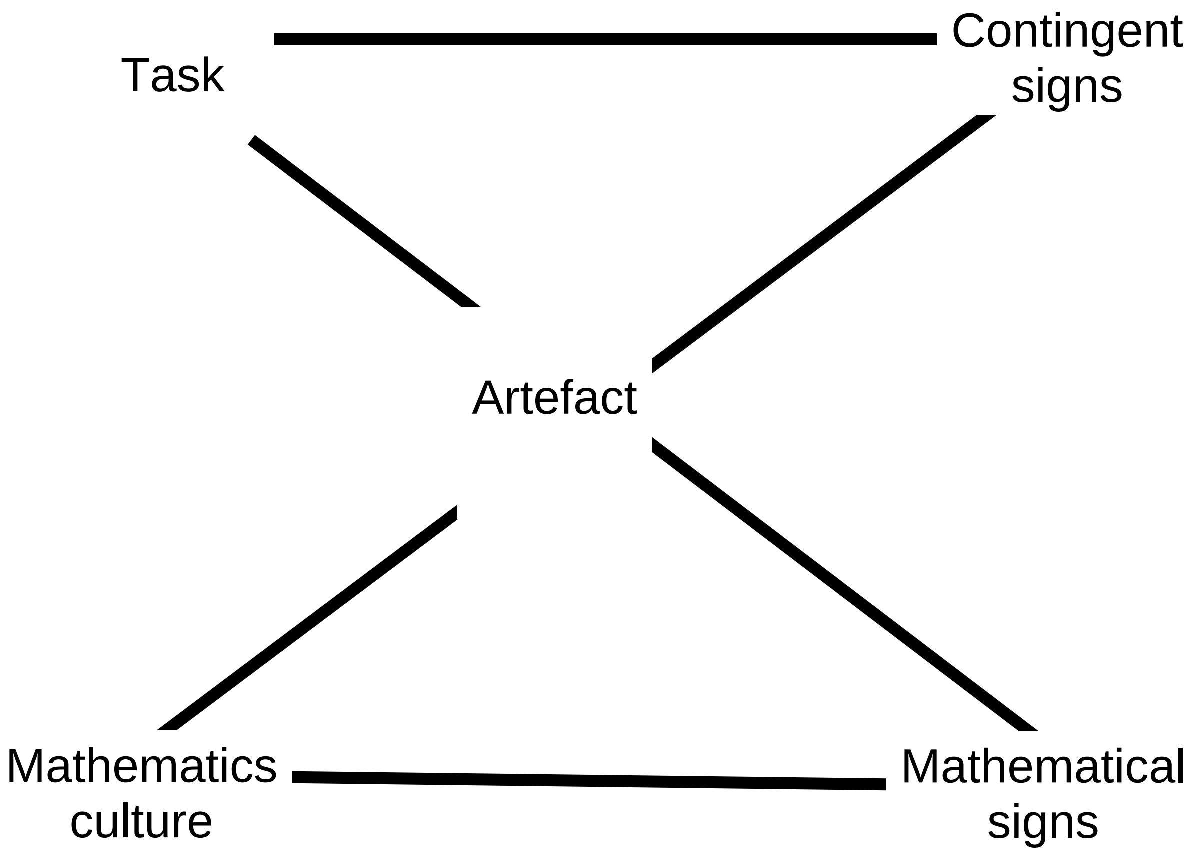


Fig. 32.2 The original scheme that appears in Chapter 28

This scheme may offer a general analytical tool to review also other research studies that deal with semiotic mediation in the mathematics classroom. In fact, the scheme is supposed to contain all the relevant components of the vygotskian approach.

Vygotsky’s triangle is embedded in the scheme of Figure 32.2, which is reproduced below with some slight changes (Fig. 32. 3) in order to show the analogy. The mediated response (R*), for the sake of clarity, is distinguished from the direct response (R), which in the original Vygotskian scheme is represented by the same letter R: it is actually different from it, because it comes after the mediation process. The artefact X (that is, a cultural product of mathematics development) has the very potential to create the “new forms of culturally-based psychological processes.” The “sophisticated forms” mentioned by Vygotsky hint at the long and complex processes through which the artefact exploits its potential (see Fig. 32.3).

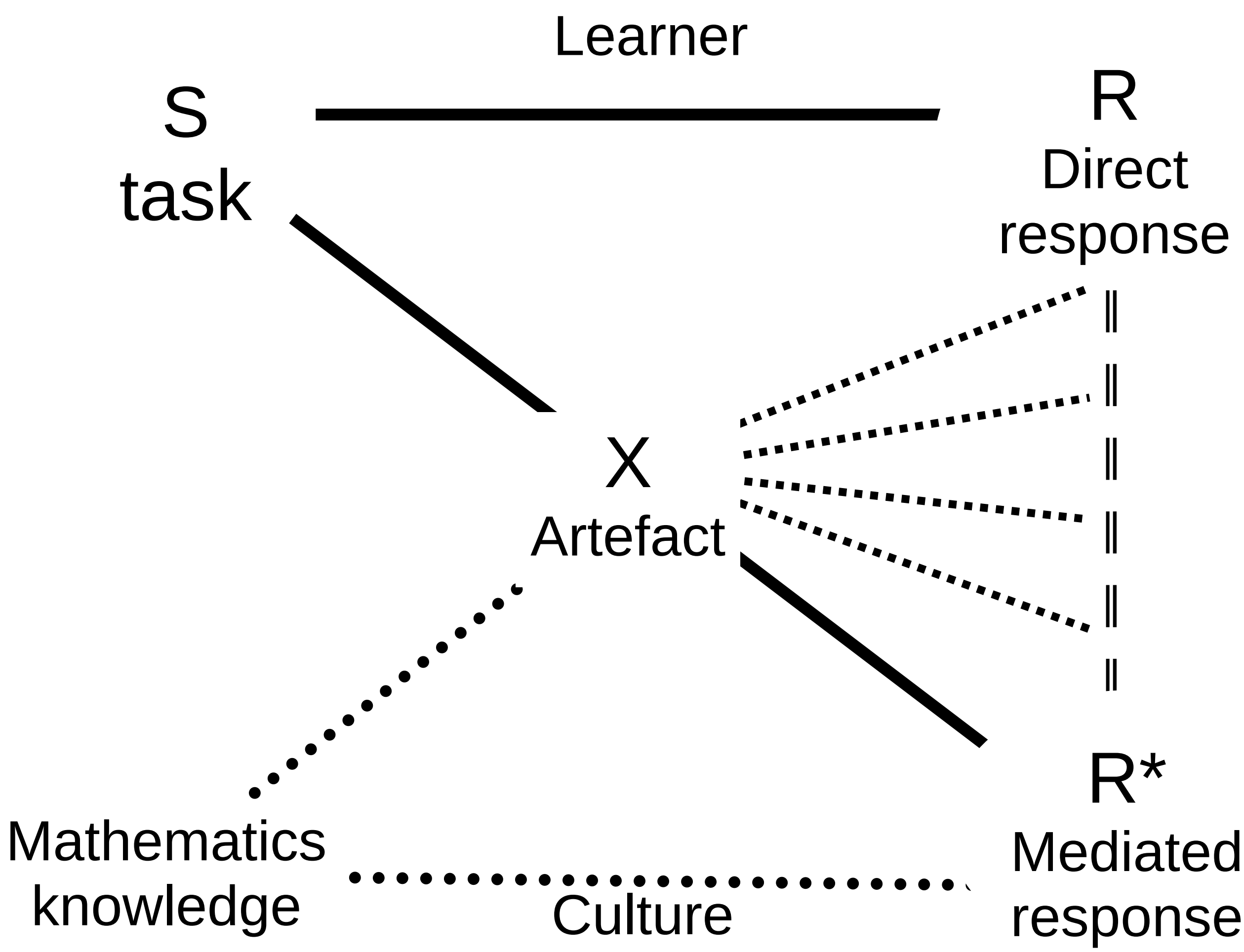


Fig. 32.3 The revised scheme

At the beginning of a didactical process the learner is asked to produce an answer to a given task. It is not obvious that this answer is a “mathematical’ one. As Mariotti (personal communication) says, with intriguing expression, “mathematics might be only in the eye of the observer.” In the schemes of Figures 32.2 and 32.3, the plane of mathematical meanings is the plane of culture. The planes of culture and the planes of direct solutions given by the learner (producing in the process some personal meanings), at the beginning, might be separated: the responsibility of creating a bridge between them rests with the teacher, who may use, for this purpose, suitable artefacts. In Bartolini Bussi and Mariotti’s chapter (28) particular kinds of artefacts are addressed (manipulatives and ICT tools), but, in principle, the artefact might be a chapter of a textbook, the text of an historical source, and so on. The responsibility of “drawing into the operation” the artefact and the related knowledge lies firstly with the teacher, when he/she acts within the *zone of proximal development* of the student, and later this responsibility is with the student, when the process of *internalization* has been realized (Vygotsky, 1978).

In this scheme, the artefact is assumed as a sign, that is, “the auxiliary stimulus that possesses the specific function of the reverse action.” This is consistent with those essemiotic approaches to mathematics, which assume a Peircean perspective, as expressed by Otte (2001) in the following quote:

> All our cognitive access to reality is relative and mediated by signs, rather than being direct and absolute. But signs must be incorporated and thus depend on objects in order to possibly come into reality and to function as signs […] Signs have meanings and refer to objects. Meanings and objects of signs may both be signs themselves.[…]

> Reality or the world in which we live essentially consists of two types of entities, signs, which have meaning, and objects, which represent pure actual existence. Existents can react with other existents, but they do not mean anything. Meanings, in contrast, are possibles, that is their objectivity lies in the future. The meaning of a natural law or of a mathematical concept, for example, is to be seen in its potential future applications. The meanings of a sign are not to be confused with the sign itself. A sign may have different types of meanings, depending on code or on context (Otte, 2001, p.1).

As argued by Bartolini Bussi and Mariotti (chapter 28), this approach is also consistent with Hasan's (2002), which refers to the linguistic field. All the relevant participants of Hasan's list find their position in this scheme:

- The mediator is the teacher, who is in charge of starting the process of semiotic mediation;
- What is mediated is a mathematical meaning;
- The mediatee is the learner;
- The circumstances of mediation include the artefact, the task, the spatio-temporal location (the mathematics classroom and the ways of organizing interactions) and so on.

In the same text, Hasan (2002) expressed concern that, at least in studies addressing the field of linguistics, "because these tedious details have not been the object of reflection, the full contribution of semiotic mediation to human life have remained hidden, and at the same time, its problematic nature has failed to be recognised." (p.4).

May this comment be applied to the field of mathematics education? As a starting exercise, one can analyse the chapters of this Handbook, where semiotic mediation is put in the foreground with reference to classroom experiments, searching first, for consistency with this position and second, for a proactive attitude to develop further research.

Arzarello and Robutti's chapter (27) is consistent with this approach and adds further elements for deepening the analysis of small group and classroom processes. The authors consider the teacher's role in nurturing the construction of meanings in small group interaction with the teacher. The analysis, which is rich, complex and very detailed, concerns mainly short-term processes, whilst long-term processes are addressed by Bartolini Bussi and Mariotti in chapter 28. It is likely that, in the near future, a stricter and deeper coordination of the theoretical constructs elaborated by both teams is realised. It is not an accident that both teams are part of the same coordinated Italian national project.

Also Boero, Douek and Ferrari's chapter (12) is consistent with this approach. The authors consider the prominent function of natural language (the most important semiotic tool) in the classroom processes where the teacher is supposed to play an important cultural role. One example in particular may be used to illustrate the consistency of this approach with that of Bartolini Bussi and Mariotti: Measuring the Height of Plants in a Pot With a Ruler.
The idea of enlarging the kind of signs useful in mathematical activity is shared by Radford (this volume, chapter 24)), who, however in this handbook, does not apply his theoretical framework to the analysis of classroom experiments. In many published papers concerning empirical research in the mathematics classroom, Radford has focused attention on the learner's level, studying in particular the evolution of learning during peer interaction in small group work (Radford 2000, 2003b). In other papers by the same author (Radford 2003a) the cultural level is addressed. Although Radford does not represent the process with the same scheme of Figure 32.3, the teleology of classroom interaction, controlled by the teacher, is especially evident in some of his papers. For instance, in one of the most recent ones (Radford 2006), concerning the construction of the concept of rate of change and derivatives by means of a concrete artefact (a

conical container with coloured water, which flows out from the bottom of the container at a constant rate) the author wrote:

> Certainly, the students were actively engaged in what has been termed a “negotiation of meaning”. But this term can be terribly misleading in hat it may lead us to believe that the attainment of the concept is a mere consensual question of classroom interaction. As I mentioned previously, in addition to its social dimension, meaning also has a cultural-historical dimension which pulls the interaction up in a certain direction – more precisely, in the direction of the cultural conceptual object. It is in fact this cultural object that shapes and explains the teacher’s intervention in the previous excerpt. Through the design of the lesson and the teacher’s continuous interpretation of the students’ learning, classroom interaction and the students’ subjective meaning are pushed towards specific directions of conceptual development. Cultural conceptual objects are like lighthouses that orient navigators’ sailing boats. They impress classroom interaction with a specific teleology. (p. 58)

The scheme may be usefully applied to ICT activity in the mathematics classroom, as shown in Bartolini Bussi and Mariotti’s chapter, with reference to Cabri. ICT constitues now a vast field of didactical research, where the construct of (semiotic) mediation is often in the foreground (e.g., Tabach, Hershkowitz & Arcavi, chapter 29; Bottino & Chiappini, chapter 31). It is left to the reader to examine the quoted chapters by means of the analytical tool proposed here.

One must not, forget, however that manipulatives also exist in the mathematics classrooms, in both developed and less developed countries and that in some parts of the world only simple traditional manipulatives are available in the mathematics classroom (in

Africa, for instance, it is not uncommon to have lessons under a tree[1].) In North America too, manipulatives are widespread in mathematics classrooms (together with the new virtual forms of ‘manipulatives’ that are realized on computer screens). Recently a monograph of the *Montana Mathematics Enthusiast* has been devoted to Zoltan Paul Dienes (Sriraman, 2007). Is it possible and meaningful to analyse the semiotic potential of the Dienes materials, by means of the above scheme?

There is another lesson that can be learnt from Hasan’s paper (2002). She commented on her list as follows:

> We can still maintain that the mediator has the initiative and active power to impart the semiotic/semantic energy, but here the user/mediator has far less control on what happens to this mediated energy: the mediator may impart semiotic energy, but the mediatee may or may not respond to its force, or respond to it in a way not intended by the user. At the heart of semiotic mediation there is this element of uncertainty. Notably, this is not a fact that to my knowledge has ever been brought to attention in the Vygotskian literature: semiotic mediation in the Vygotskian literature appears to always act felicitously. (p.4)

One may paraphrase this quotation stating that, when the teacher has carefully designed tasks to mediate some meaning and has presented these tasks to the pupils (the “mediates,” in Hasan’s words), it is not certain that they will respond. In every didactical project on semiotic mediation, there is an element of uncertainty. Yet in most cases, the literature shows only examples of felicitous processes.

Concluding Points

[1] http://homepage.mac.com/globalhealing/SouthernSudanEd./PhotoAlbum31.html

In this concluding chapter, we have addressed what we consider to be some of the key 21st century issues in the advancement of mathematics education and mathematics education research. We have explored some of the multidisciplinary debates on powerful mathematical ideas for this century, including the impact of cultural, social, and political factors on the nature of mathematics, its future growth, and its place in the curriculum. The changing nature and role of mathematics in the work place and other everyday settings also have implications for the mathematics we include in the curriculum. To this end, we listed a number of key mathematical understandings and processes that employers generally consider essential to productive outcomes. At the same time, we noted some issues that need attention in addressing the links between the mathematics required in the work place and the formal mathematics taught in schools. Nevertheless, the need to develop our students' abilities to work creatively with mathematical ideas as they engage in collaborative, knowledge-building communities in the classroom remains a significant and challenging goal for mathematics educators.

We have also stressed the need for research to support more equitable mathematics curriculum and learning access for all students, with a focus on their exposure to the powerful mathematical ideas of the new millennium. We have considered ways in which research can facilitate the creation of learning environments that can increase this access, as well as ways in which teacher education and development can help achieve our goal of powerful mathematics for all. Increasing the dialogue between research and practice in mathematics teacher education remains a major goal. The need for research to inform the important and challenging issues of assessment -- whether it be assessment of students, teachers, programs, tools, or researchers -- has also been emphasized in this chapter.

A brief overview on the nature and role of semiotic mediation in mathematics education, which has become a popular theoretical construct in recent years, has also been provided in this chapter. It was pointed out that, while we use a range of artefacts to convey (mediate) intended mathematical meaning to students, we cannot be certain that the artefacts will actually achieve this goal. The majority of research projects addressing semiotic mediation do not highlight this uncertainty, however; future research needs to place greater emphasis on why some chosen artefacts do not adequately convey the desired mathematical meanings.

We have also devoted considerable discussion in this chapter to the proliferation of research designs and methodologies in the past decade; these have provided us with unprecedented opportunities for investigating (and ultimately improving) the mathematical learning of students and other members of our societies. With their varied emphases on cognitive, social, and cultural perspectives, these newer research designs place us in a unique position to look at mathematical learning from multiple perspectives. The design approach, for example, creates optimal opportunities for researchers and teachers to cooperate to produce meaningful changes in classroom practice (e.g., Richardson & Placier, 2001). This means that goals and design constraints are drawn from the local context, suggesting a design strategy that deliberately creates opportunities for the stakeholders to influence the design process and focus on adaptation to already existing practices. In this way design research becomes a basis for the development of warranted practices with which the teachers may experiment in their classroom. Both preservice and inservice teachers should encounter situations where they have access to knowledge about innovations in mathematics teaching and, in partnership with researchers, use, share, and develop this knowledge in design projects (Michelsen & Sriraman, 2007).

Our research endeavors in meeting the foregoing challenges need to be subjected to greater scrutiny, especially in terms of the way in which we deal with our “subjects” and the types of “products” we try to generate. Our subjects in mathematics education are complex systems: dynamic, interactive, self-monitoring, and continually adapting. We need to deal with such complex systems in a more disciplined way (English, 2007b; Lesh, 2006). At the same time, we need to broaden the nature of the products that we seek to generate from our research. More than ever before, our research needs to produce knowledge that can be shared and re-used. Our products are many and varied, including tools for creating, observing, classifying, or measuring our subjects, and models or conceptual systems for constructing, describing, or explaining the complex systems with which we are dealing.

It is our hope that future studies will draw on the best aspects of the research designs we have addressed in this chapter to create rich, multidisciplinary approaches to mathematics education research. In so doing, we can generate products, including new theories, which provide more integrated and culturally sensitive frameworks for improving learners' access to powerful mathematical ideas.

References

Adler, J., Ball, D., Krainer, K., Lin, F., & Novotna, J. (2005). Reflections on an emerging field: Researching mathematics teacher education. *Educational Studies in Mathematics*, *60*, 359–381.

Afamasaga-Fuata'i, K. (2002). A Samoan perspective on Pacific mathematics education. In B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.), *Mathematics education in the South Pacific*, Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia (pp. 1-13). Auckland: MERGA.

Anderson, J. (1999). Being mathematically educated in the 21st century: What should it mean? In C. Hoyles, C. Morgan, & G. Woodhouse (Eds.), *Rethinking the mathematics curriculum* (pp. 8-21). London: Falmer Press.

Ball, D. L. (2000). Working on the inside: using one's own practice as a site for studying teaching and learning. . In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 365-402). Mahwah, NJ: Lawrence Erlbaum.

Bartolini-Bussi, M. (1990). Mathematics knowledge as a collective enterprise. *Proceedings of the 4th SCTP conference* (pp. 121-151). Bielefeld, West Germany: Institut fur Didaktik der Mathematik.

Bartolini-Bussi, M. G., & Bazzini, L. (2003). Research, practice, and theory in didactics of mathematics. *Educational Studies in Mathematics, 54*(2-3), 202-223.

Bartolini-Bussi, M. G. & Boni, M (2003). Instruments for semiotic mediation in primary school classrooms, *For the Learning of Mathematics*, 23 (2), 12-19.

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843-908). Charlotte, NC: Information Age Publishing.

Bauersfeld, H. (1980). Hidden dimensions in the so-called reality of the mathematics classroom. *Educational Studies in Mathematics, 11*, 23-41.

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York: Macmillan.

Bell. A., & Lins, R. (Eds.). (2001). *Perspectives on school algebra*. Dordrecht, The Netherlands: Kluwer.

Ben-Zvi, D., & Garfield, J. (Eds.). (2004). *The challenge of developing statistical literacy, reasoning, and thinking*. Dordrecht, The Netherlands: Kluwer.

Bereiter, C., & Scardamalia, M. (2006). Education for the knowledge age: Design-centered models of teaching and instruction. In P. A. Alexander, & P. H. Winne (Eds.), *Handbook of educational psychology* (pp. 695-714). Mahwah, NJ: Lawrence Erlbaum.

Berry, R. Q., III. (2004). The equity principle through the voices of African American males. *Mathematics Teaching in the Middle School, 10*, 100-103.

Berry, R. Q., III. (2005). Introduction: Building an infrastructure for equity in mathematics education. *The High School Journal, 88*(4), 1-5.

Biehler, R., Scholz, R., Strässer, R. & Winkelmann, B. (Eds) (1994). *Didactics of mathematics as a scientific discipline*. MA: Kluwer, Norwell.

Bishop, A. J. (Ed) (2003). Second International Handbook on Mathematics Education. Dordrecht: Kluwer.

Boaler, J. (1997). *Experiencing school mathematics: Teaching styles, sex, and setting*. Philadelphia: Open University Press.

Boaler, J. (1998). Open and closed mathematics: Student experiences and understandings. *Journal for Research in Mathematics Education*, *29*, 41-62.

Boaler, J. (1999). Participation, knowledge, and beliefs: A community perspective on mathematics learning. *Educational Studies in Mathematics*, *40*, 259-281.

Boaler, J. (2002). Learning from teaching: Exploring the relationship between reform curriculum and equity. *Journal for Research in Mathematics Education*, *33*, 239-258.

Boaler, J. (2003). When learning no longer matters: Standardized testing and the creation of inequality. *Phi Delta Kappan, 84,* 502-507.

Boaler, J. (2006a). Promoting respectful learning. *Educational Leadership, 63*(5).

Boaler, J. (2006b). Urban Success: A multidimensional mathematics approach with equitable outcomes. *Phi Delta Kappan, 87,* 364-369.

Boaler, J., & Greeno, J. G. (2000). Identity, agency, and knowing in mathematical worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning*. Westport, CT: Ablex Publishers.

Bol, L., & Berry, R. Q. (2005). Secondary mathematics teachers' perceptions of the achievement gap. *The High School Journal, 88*(4), 32-45.

Borko, H., Stecher, B. M., Alonzo, A., Moncure, S., & McClam, S. (2005). Artifact packages for characterizing classroom practice: A pilot study. *Educational Assessment, 10*(2), 73-104.

Brousseau, G. (1984). The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics. In H. G. Steiner (Ed.), *Theory of*

Mathematics Education (Occasional paper 54, pp. 110-119). Bielefeld, Germany: Institut fur Didaktik der Mathematik.

Brousseau, G. (1992). Didactique: What it can do for the teacher. In R. Douady, & A. Mercier (Eds.), *Reseach in didactique of mathematics* (pp. 7-40). Grenoble, France: La Pensee Sauvage.

Brown, A. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *Journal of the Learning Sciences 2*(2), 141-178.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher, 18* (1), 32-42.

Buxton, G. A. (2005). Creating a culture of academic success in an urban science and math magnet high school. *Science Education, 89*(3), 392-417.

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades 1-3. *Journal for Research in Mathematics Education, 15*, 179-202.

Carraher, D. (2007). Early algebra. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 669-706). Charlotte, NC: Information Age Publishing.

Civil, M., & Andrade, R. (2002). Transitions between home and school mathematics: Rays of hope amidst the passing cloud. In G. de Abreu, A. J. Bishop, & N. C. Presmeg (Eds.), *Transitions between contexts of mathematical practices* (pp. 149-169). Dordrecht, The Netherlands: Kluwer.

Clarke, D. J. (2001). *Perspectives on practice and meaning in mathematics and science classrooms*. Dordrecht, The Netherlands: Kluwer.

Clayton, M. (1999). What skills does mathematics education need to provide? In C. Hoyles, C. Morgan, & G. Woodhouse (Eds.), *Rethinking the mathematics curriculum* (pp. 22-28). London: Falmer Press.

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420-464). New York: Macmillan.

Clements, D. H., & Sarama, J. (2007). Early childhood mathematics learning. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 461-555). Charlotte, NC: Information Age Publishing.

Cobb. P. (2000). Conducting teaching experiments in collaboration with teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 307-334). Mahwah, NJ: Lawrence Erlbaum.

Cobb, P., & Hodge, L. L. (2002). A relational perspective on issues of cultural diversity and equity as they play out in the mathematics classroom. *Mathematical Thinking and Learning, 4* (2/3), 249-284.

Cobb, P., Stephan, M., McClain, K. & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *The Journal of the Learning Sciences, 10* (1/2), 113-163.

Cohen, J. H. (2003). Supplemental education: Six essential components. *Principal, 82*(5), 34-37.

Confrey, J. & Lachance, A. (2000). Transformative teaching experiments through conjecture-driven research design check chapter title. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 231-266). Mahwah, NJ: Lawrence Erlbaum.

Cooney, T. D., & Grouws, A. D., & Jones, D. (1988). An agenda for research on teaching mathematics. In A. D. Grouws, T. D. Cooney, & D. Jones (Eds.), *Perspectives on research on effective mathematics teaching* (pp. 253-261). Reston, VA: National Council of Teachers of Mathematics.

Craig, J. R., & Kacer, B. A. (2000, November). *Using an innovative configuration component map to assess the relationship between student achievement and the degree of implementation of extended school services in a sample of Kentucky middle schools.* Paper presented at the annual meeting of the Mid-South Educational Research Association, Bowling Green, KY.

D'Ambrosio, U. (1999). Literacy, Matheracy, and Technoracy: A trivium for today. *Mathematical Thinking and Learning, 1*(2), 131-154.

D'Ambrosio, U. (2007). Peace, social justice and ethnomathematics. In B. Sriraman (Ed.), *International perspectives on social justice in mathematics education* (pp. 25-34). Monograph of The Montana Enthusiast. Montana: The University of Montana and The Montana Council of Teachers of Mathematics.

Davis, P. J., & Hersh, R. (1986). *Decartes' dream: The world according to mathematics* . Brighton, England: Harvester.

Dick, T. (2007). Keeping the faith: Fidelity in technological tools for mathematics education. In G. W. Blume & M. K. Heid (Eds.), *Research on technology and the teaching and learning of mathematics: Synthesis, cases, and perspectives. Vol. 2: Cases and perspectives* (pp. 333-339). Greenwich, CT: Information Age.

DiSessa, A. A., & Abelson, H. (1986). Boxer: A reconstructible computational medium. *Communications of the ACM, 19,* 859-868.

Diversity in Mathematics Education Center for Learning and Teaching. (2007). Culture, race, power, and mathematics education. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 405-434). Charlotte, NC: Information Age Publishing.

Doerfler, W. (1993). Computer use and views of the mind. In C. Keitel & K. Ruthven (Eds.), *Learning from computers: Mathematics education and technology* (pp. 159-186). Berlin, Germany: Springer.

Doerr, H. M., & Lesh, R. A. (2003). A modeling perspective on teacher development. In R. A. Lesh, & H. M. Doerr, (Eds.), *Beyond constructivism: A models and modelling perspective on teaching, learning, and problem solving in mathematics education* (pp. 125-140). Mahwah, NJ: Lawrence Erlbaum.

Doerr, H. M., & English, L. D. (2003). A modeling perspective on students' mathematical reasoning about data. *Journal for Research in Mathematics Education, 34*(2), 110-137.

Dunkin, J., & Biddle, B. (1974). *The study of teaching*. New York: Holt, Rinehart, & Winston.

English, L. (2002). (Ed.). *Handbook of research in mathematics education.* Mahwah, NJ: Lawrence Erlbaum

English, L. D. (2007a). Interdisciplinary modelling in the primary mathematics curriculum. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice* (pp.275-284). Hobart: Mathematics Education Research Group of Australasia.

English, L. D. (2007b). Cognitive psychology and mathematics education: Reflections on the past and the future. In B. Sriraman (Ed). Zoltan Paul Dienes and the dynamics of mathematical learning. *The Montana Mathematics Enthusiast, Monograph No. 2* (pp. 119-126).

Missoula: The University of Montana ands the Montana Council of Teachers of Mathematics.

Ernest, P. (1998). The epistemological basis of qualitative research in mathematics education. In A. R. Teppo (Ed.), *Qualitative research methods in mathematics education* (pp. 22-39). Reston, VA: National Council of Teachers of Mathematics.

Er-sheng, D. (1999). Mathematics curriculum reform facing the new century in China. In Z. Usiskin (Ed.), *Developments in school mathematics education around the world* (pp. 58-70). Reston, VA: National Council of Teachers of Mathematics.

Even, R., & Ball, D. (2005). The International Commission on Mathematics Instruction (ICMI) – The 15th ICMI Study: The professional education and development of teachers of mathematics, *Journal of Mathematics Teacher Education*, *7*, 279-293.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. G. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, *27*, 403-434.

Fey, J. T. (2003). *Computer algebra systems in secondary school mathematics education*. Reston, VA: National Council of Teachers of Mathematics.

Filloy, E., & Sutherland, R. (1996). Designing curricula for teaching and learning algebra. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.). *International handbook of mathematics education* (pp. 139-160). Dordrecht, The Netherlands: Kluwer.

Fischbein, E. (1975). *The intuitive sources of probabilistic thinking in children.* Dordrecht, The Netherlands: Reidel.

Franke, M. L., Kazemi, E., & Battey, D. (2007). Understanding teaching and classroom practice in mathematics. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 225-256). Charlotte, NC: Information Age Publishing.

Fredericksen, J. R., & Collins, A. (1989). A systems approach to educational testing. *Educational Researcher, 18*(9), 27-32.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.) *Handbook of research on mathematics teaching and learning* (pp. 243-275). New York: Macmillan.

Gainsburg, J. (2006). The mathematical modeling of structural engineers. *Mathematical Thinking and Learning, 8*(1), 3-36.

Gravemeijer, K. (1998). Developmental research as a research method. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identify*, (Book 2, pp. 277-296).Dordrecht, The Netherlands, Kluwer.

Greeno, J. G., & Gresalfi, M. (in press). Opportunities to learn in practice and identity. In P. A. Moss, D. Pullin, E. H. Haertel, J. P. Gee, & L. J. Young, (Eds.), *Opportunity to learn.* New York: Cambridge University Press.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276-295). New York: Macmillan.

Greer, B., & Mukhopadhyay, S. (2003). What is mathematics education for? *Mathematics Educator, 13*(2), 2-6.

Grouws, D. A. (Ed.). (1992*). Handbook of research on mathematics teaching and learning.* New York: Macmillan.

Gutierrez, K. D., & Rogoff, B. (2003). Cultural ways of learning: Individual traits or repertoires of practice. *Educational Researcher, 32*(5), 19-25.

Gutstein, E. (2003). Teaching and learning mathematics for social justice in an urban, Latino school. *Journal for Research in Mathematics Education, 34*, 37-73.

Gutstein, E. (2006). The real world as we have seen it: Latino/a parents' voices is this correct "a parent's voices" on teaching mathematics for social justice. *Mathematical Thinking and Learning, 8*(3), 331-358.

Gutstein, E. (2007). Connecting community, critical, and classical knowledge in teaching mathematics for social justice. In B. Sriraman (Ed.), *International perspectives on social justice in mathematics education* (pp. 109-118). Monograph of The Montana Enthusiast. Montana: The University of Montana and The Montana Council of Teachers of Mathematics.

Gutstein, E. (in press). So one question leads to another: Using mathematics to develop a pedagogy of questioning. In N. S. Nasir & P. Cobb (Eds.), *Improving access to mathematics: Diversity and equity in the classroom*. New York: Teachers College Press.

Gutstein, E., Fey, J. T., Heid, M. K., De-Loach- Johnson, I., Middleton, J. A., & Larson, M. (2005). Equity in school mathematics education: How can research contribute? *Journal for Research in Mathematics Education, 36,* 92-100.

Hand, V. (2003). *Reframing participation: Meaningful mathematical activity in diverse classrooms.* Unpublished doctoral dissertation, Stanford University, CA.

Hasan, R. (2002). Semiotic mediation, language and society: three exotropic theories – Vygotsky, Halliday and Bernstein Language, *Society and Consciousness: The Collected Works of Ruqaiya Hasan* Vol 1 (Edited by: Jonathan Webster, London:Equinox. Available on line at http://www.education.miami.edu/blantonw/mainsite/Componentsfromclmer/Component13/Mediation/SemioticMediation.html

Hawkins, J. (1997). *The national design experiments consortium: Final report*. New York: Center for Children and Technology, Educational Development Center.

Hersh, R. (1979). Some proposals for reviving the philosophy of mathematics. *Advances in Mathematics, 31*, 31-50.

Hiebert, J. S., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371-404). Charlotte, NC: Information Age Publishing.

Hoyles, C., Bakker, A., Kent, P., & Noss, R. (In press). Attributing meanings to representations of data: The case of statistical process control. *Mathematical Thinking and Learning, 9*(4).

Hoyles, C., Noss, R., & Pozzi, S. (2001). Proportional reasoning in nursing practice. *Journal for Research in Mathematics Education, 32*(1), 4-27.

Hoyles, C., Wolf, A., Molyneux-Hodgson, S., & Kent, P. (2002). *Mathematical skills in the workplace*. London: Science, Technology and Mathematics Council. Retrieved May 24, 2006, from http://www.ioe.ac.uk/tlrp/technomaths/skills2002/

Ireson, J., Hallam, S., & Hurley, C. (2005). What are the effects of ability grouping on GCSE attainment? *British Educational Research Journal, 31*, 443-458.

Jaworski, B., & Gellert, U. (2003). Educating new mathematics teachers: Integrating theory and practice and the roles of practicing teachers. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick & F. K. S. Leung (Eds.), *Second international handbook of mathematics education* (823–675). Dordrecht: The Netherlands: Kluwer.

Jenkins, H., Clinton, K., Purushotma, R., Robinson, A. J., & Weigel, M. (2006). *Confronting the challenges of participatory culture: Media education for the 21st century*. Chicago: IL: MacArthur Foundation.

Johnson, D. (2001). The cyber children have arrived. *The Futurist, 35*(5), 14-15.

Jones, G. A. (Ed.). (2005). *Exploring probability in school: Challenges for teaching and learning*. New York: Springer.

Jones, G. A., Langrall, C. W., & Mooney, E. S. (2007). Research in probability: Responding to classroom realities. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 909-956). Charlotte, NC: Information Age Publishing.

Kaput, J., & Roschelle, J. (1999). The mathematics of change and variation from a millennium perspective: New content, new context. In C. Hoyles, C. Morgan, & G. Woodhouse (Eds.), *Rethinking the mathematics curriculum* (pp. 155-170). London: Falmer Press.

Kaput, J. J. (1989). Linking representations in the symbolic systems of algebra. In S. Wagner & C. Kieren, (Eds.) *Research Issues in the learning and teaching of algebra* (pp. 167-194). Reston, VA: National Council of Teachers of Mathematics; Hillsdale, NJ: Lawrence Erlbaum.

Katz, M., & Jury, M. (March, 1997). Literacies in a changing workplace: A look at the uses of literacy in a multi-ethnic, high-electronics factory. Paper presented at the annual meeting to the *American Educational Research Association*, Chicago.

Kelly, A. E., & Lesh, R. A. (2000). Trends and shifts in research methods. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp.35-44). Mahwah, NJ: Lawrence Erlbaum.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan.

Koch, L. (1997). The growing pains of change: A case study of a third grade teacher. In J. Ferrini-Mundy & T. Schram (Eds.), Recognizing and recording reform in mathematics education project: Insights, issues and implications. *Journal for Research in Mathematics Education Monograph 8,* (pp. 87-109). Reston, VA: National Council of Teachers of Mathematics.

Kuhs, T. M., & Ball, D. L. (1986). *Approaches to teaching mathematics: Mapping the domains of knowledge, skills, and dispositions.* East Lansing: Michigan State University, Center on Teacher Education.

Ladson-Billings, G. (1997). It doesn't add up: African American students' mathematics achievement. *Journal for Research in Mathematics Education, 28,* 697-708.

Ladson-Billings, G. (2006, April). *From the achievement gap to the education debt: Understanding achievement in U.S. schools*. Paper presented at the annual meeting of the American Educational Research Association, San Francisco.

Lakoff, G., & Núñez, R. (1997). The metaphorical structure of mathematics: Sketching out cognitive foundations for a mind-based mathematics. In L. English (Ed.), *Mathematical Reasoning: Analogies, Metaphors, and Images* (pp. 21-89). Mahwah, NJ: Lawrence Erlbaum.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* New York: Basic Books.

Lamon, S. J. (2006). *Teaching fractions and ratios for understanding: essential content knowledge and instructional strategies for teachers*. Mahwah: Lawrence Erlbaum.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Towards a theoretical framework for research. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629-668). Charlotte, NC: Information Age Publishing.

Lampert, M. (1998). Studying teaching as a thinking practice. In J. Greeno & S. G. Goldman (Eds.), *Thinking practices* (pp. 53-78). Mahwah, NJ: Lawrence Erlbaum Associates.

Lankshear, C., & Knobel, M. (2003). *New literacies: Changing knowledge and classroom learning*. Maidenhead, UK: Open University Press.

Lappan, G. (1999). Preparing students for college and the workplace: Can we do both? In Z. Usiskin (Ed.), *Developments in school mathematics education around the world* (pp. 150-161). Reston, VA: National Council of Teachers of Mathematics.

Lave, J. (1988). *Cognition in practice: Mind, mathematics, and culture in everyday life*. Cambridge, England: Cambridge University Press.

Lave, J. (1997). The culture of acquisition and the practice of understanding. In D. Kirschner & J. A. Whitson (Eds.), *Situated cognition: Social, semiotic, and psychological perspectives*. (pp. 17-35). Mahwah, NJ: Lawrence Erlbaum.

Lave, J., & Wenger, E. (1991). *Situated cognition and legitimate peripheral participation*. Cambridge, England: Cambridge University Press.

Lee, J., & Wong, K. K. (2004). The impact of accountability on racial and socioeconomic equity: Considering both school resources and achievement outcomes. *American Educational Research Journal, 41*, 797-832.

Lehrer, R., & Chazan, D. (Eds.) (1998). *Designing learning environments for understanding of geometry and space*. Mawhah, NJ: Lawrence Erlbaum.

Lesh, R. (2006). Modeling students modeling abilities: The teaching and learning of complex systems in education. *The Journal of the Learning Sciences, 15* (1), 45-52.

Lesh, R., & Clarke, D. (2000). Formulating Operational Definitions of Desired Outcomes of Instruction. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 113-150). Mawhah, NJ: Lawrence Erlbaum.

Lesh, R., & Doerr, H. M. (Eds.). (2003). *Beyond constructivism: Models and modeling perspectives on mathematic problem solving, learning and teaching*. Mahwah, NJ: Lawrence Erlbaum.

Lesh, R., & English, L. (2005). Trends in the evolution of the models-modelling perspective on mathematical learning and problem solving. *International Reviews on Mathematical Education (Zentralblatt für Didaktik der Mathematik)*, vol.37, no.6, 487-489.

Lesh, R., Hole, B., Hoover, M., Kelly, A., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 591-646). Mahwah, NJ: Erlbaum.

Lesh, R., Kaput, J., & Hamilton, E. (Eds.) (2007) *Foundations for the Future: The Need for New Mathematical Understandings & Abilities in the 21st Century*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Lesh, R., & Lamon, S. (Eds.) (1993). *Assessments of authentic performance in elementary mathematics.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Lesh, R., Lovitts, B., & Kelly, A. E. (2000). Purposes and assumptions of this book. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 17-34). Mahwah, NJ: Lawrence Erlbaum.

Lesh, R. & Sriraman, B. (2005a). Mathematics education as a design science. *International Reviews on Mathematical Education (Zentralblatt für Didaktik der Mathematik)*, 37(6), 490-505.

Lesh, R. & Sriraman, B. (2005b). John Dewey Revisited- Pragmatism and the models-modeling perspective on mathematical learning. In A. Beckmann, C. Michelsen & B. Sriraman [Eds.], *Proceedings of the 1st International Symposium on Mathematics and its Connections to the Arts and Sciences*. May 18-21, 2005, University of Schwaebisch Gmuend: Germany.Franzbecker Verlag, pp. 32-51.

Lesh, R. & Yoon, C. (2006). What is distinctive in (our views about) models and modeling perspectives on mathematics problem solving, learning and teaching? In B. Werner, P. Galbraith, H. Henn, and M. Niss (Eds), *Modeling and applications in mathematics education: The 14th ICMI Study ICMI* (pp. 161-170). New York: Springer.

Lesh, R., & Zawojewski, J. (2007). Problem solving and modelling. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 763-804). Charlotte, NC: Information Age Publishing.

Lesh, R. A., Hamilton, E., & Kaput, J. (2007). Directions for future research investigating foundations for the future. In R. Lesh, E. Hamilton, & J. Kaput (Eds.). *Foundations for the future in mathematics education*. New York: Routledge.

Lesh, R. A., & Kelly, A. E. (2000). Multitiered teaching experiments. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 197-230). Mahwah, NJ: Lawrence Erlbaum.

Lester, F. K. Jr. (1994). Musings about mathematical problem-solving research. *Journal for Research in Mathematics Education, 25*, 660-675.

Lester, F. K. Jr. (2005). On the theoretical, conceptual, and philosophical foundations for research in mathematics education*. International Reviews on Mathematical Education (Zentralblatt für Didaktik der Mathematik)*, 37(6), 457-467.

Lester, F. K. Jr. (Ed.). (2007*). Second handbook of research on mathematics teaching and learning.* Charlotte, NC: Information Age Publishing.

Lester, F. K. Jr., & Kehle, P. (2003). From problem solving to modeling: The evolution of thinking on research about complex activity. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 501-518). Mahwah, NJ: Lawrence Erlbaum.

Lombardi, J., & Lombardi, M. M. (2007). *Croquet learning space and collaborative scalability*. Paper presented at DLAC-II Symposium, June 11-14, Singapore.

Matthews, L. E. (2005). Towards design of clarifying equity messages in mathematics reform. *The High School Journal, 88*(4), 46-58.

Michelsen, C., & Sriraman, B. (2007). Design science perspectives in mathematics and science education classroom research. *Proceedings of the 9th International History, Philosophy and Science Teaching Group (IHPST)* 2007, Calgary, in press.

Moreno, L.A. & Sriraman, B. (2005). The articulation of symbol and mediation in mathematics education. *International Reviews on Mathematical Education (Zentralblatt für Didaktik der Mathematik)*, vol.37, no.6, 476-486.

Moses, R. P., & Cobb, C. E. (2001). *Radical equations: Civil rights from Mississippi to the Algebra Project*. Boston: Beacon Press.

Moses-Snipes, P. R., & Snipes, V. T. (2005). The call: The importance of research on African American issues in mathematics and science education. Negro Educational Review, 56(2/3), 103-105.

Moyer, P. S., Niezgoda, D., & Stanley, J. (2005). Young children's use of virtual manipulatives and other forms of mathematical representations. In W. J. Masalski & P. C. Elliott (Eds.), *Technology-supported mathematics learning environments: Sixty-seventh yearbook* (pp. 17-34). National Council of Teachers of Mathematics: Reston, VA.

Mukhopadhyay, S. & Greer, B. (2007). How many deaths? Education for statistical empathy. In B. Sriraman (Ed.), *International perspectives on social justice in mathematics education* (pp. 119-136). Monograph of The Montana Enthusiast. Montana: The University of Montana and The Montana Council of Teachers of Mathematics.

Mullis, I. V. S., Martin, M. O., Gonzales, E. J., & Chrostowski, S. J. (2004). *TIMSS 2003 international mathematics report: Findings from IEA's trends in international mathematics and science study at fourth and eighth grades*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Lynch School of Education, Boston College.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

Nohda, N. (2000). Teaching by open-approach method in Japanese mathematics classroom. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 39-54). Hiroshima, Japan: Hiroshima University.

Noss, R., & Hoyles, C. (1996). Tools and technology. In R. Noss & C. Hoyles (Eds.), *Windows on mathematical meanings: Learning cultures and computers* (pp. 52-69). Dordecht, The Netherlands: Kluwer Academic Publishers.

Nunes, T., Schliemann, A. D., & Carraher, D. W. (1993). *Street mathematics and school mathematics*. Cambridge, UK: Cambridge University Press.

Núñez, R. (2007). The cognitive science of mathematics: Why is it relevant for mathematics education? In R. Lesh, E. Hamilton, & J. Kaput (Eds.). *Foundations for the future* (pp.127-154). Mahwah, NJ: Lawrence Erlbaum.

Otte, M. (2001). Mathematical epistemology from a semiotic point of view, Paper presented at PME 2001 in Utrecht. Available on line at http://www.math.uncc.edu/~sae/dg3/otte-new-PME25.pdf

Pappert, S. (1980). *Mindstorms: Children, computers, and powerful ideas.* New York: Basic Books.

Parker, D. L. (2003). Taking care of Mother Earth: Technology and the environment. *Teaching Children Mathematics, 9*, 414-419.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second handbook of research on mathematics learning and teaching* (pp. 257-517-317). Charlotte, NJ: National Council of Teachers of Mathematics.

Polaki, M. V. (2000). *Using instruction to trace Basotho students' growth in probabilistic thinking*. Unpublished doctoral dissertation, Illinois State University, Normal. Should the word "normal" appear?

Pollak, H. (1997). Solving problems in the real world. In L. A. Steen (Ed.), *Why numbers count: Quantitative literacy for tomorrow's America* (pp. 91-105). New York: College Entrance Examination Board.

Power, R. J. D. & Dal Martello, M. F. (1990). *The Dictation of Italian Numerals, Language and Cognitive Processes*, 5 (3), 237-254.

Pratt, D. (2005). How do teachers foster students' understanding of probability? In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 171-190). New York: Springer.

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis, *Educational Studies in Mathematics* 42(3), 237-268.

Radford, L. (2003a). On the epistemological limits of language. Mathematical knowledge and social practice during the Renaissance, *Educational Studies in Mathematics* 52(2), 123-150.

Radford, L. (2003b). Gestures, speech and the sprouting of signs, *Mathematical Thinking and Learning* 5(1), 37-70.

Radford, L. (2006). The Anthropology of Meaning. *Educational Studies in Mathematics*, 61(1-2), 39-65.

Remillard, J. T., & Jackson, K. (2006). Old math, new math: Parents' experiences with standards-based reform. *Mathematical Thinking and Learning, 8*(3), 231-260.

Richardson, V., & Placier, M. (2001). Teacher Change. In V. Richardson (Ed.), *Handbook of Research on Teaching*. Washington D.C.: American Educational Research Association. 905- 947.

Roschelle, J., Kaput, J., & Stroup, W. (2000). SimCalc: Accelerating students' engagement with the mathematics of change. In M. J. Jacobson & R. B. Kozma (Eds.), *Innovations in science and mathematics education* (pp. 47-76). Mahwah, NJ: Lawrence Erlbaum.

Roschelle, J., Tatar, D., Shechtman, N., Hegedua, S., Hopkins, B., Knudsen, J., et al. (2007). *Scaling up SimCalc project: Can a technology enhanced curriculum improve student learning of important mathematics?* Technical Report 01. SRI International.

Ross, J. A., McDougall, D., Hogaboam-Gray, A., & LeSage, A. (2003). A survey measuring elementary teachers' implementation of standards-based mathematics teaching. *Journal for Research in Mathematics Education, 34*, 344-363.

Roth, W-M. (2005). Mathematical inscriptions and the reflexive elaboration of understanding: An ethnography of graphing and numeracy in a fish hatchery. *Mathematical Thinking and Learning, 7*, 75-110.

Rousseau, C. K., & Powell, A. (2005). Understanding the significance of context. A framework to examine equity and reform in secondary mathematics. *The High School Journal, 88*(4), 19-31.

Rowan, B., Correnti, R., & Miller, R. J. (2002). *What large-scale survey research tells us about teacher effects on student achievement: Insights from the "Prospects" study of elementary schools* (CPRE Research Report Series PR-051). Philadelphia: University of Pennsylvania, Consortium for Policy Research in Education.

Rowan, B., Harrison, D., & Hayes, A. (2004). Using instructional logs to study mathematics curriculum and teaching in the elementary grades. *Elementary School Journal, 105*, 103-127.

Rubel, L., & Meyer, M. R. (2005). The pursuit of mathematics for all! *Teaching in the Middle School, 10*, 479-483.

Rumberger, R. W., & Palardy, G. J. (2005). Does segregation still matter? The impact of student composition on academic achievement in high school. *Teachers College Record, 107*, 1999-2045.

Saxe, G. (1991). *Culture and cognitive development: Studies in mathematical understanding*. Hillsdale, NJ: Lawrence Erlbaum.

Saxe, G. B. (1996). Studying cognitive development in sociocultural contexts: The development of a practice-based approach. In R. Jessor, A. Colby, & R. Shweder (Eds.), *Ethnography and human development* (pp. 275-303). Chicago: The University of Chicago Press.

Schoenfeld, A. (1999). Looking toward the 21st Century: Challenges of educational theory and practice. *Educational Researcher 28*(7), 4-14.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York, Macmillan.

Seaton, D. M., & Carr, D. (2005). The impact of participation in an ancillary science and mathematics program (SEMAA) on engagement rates of middle school students in regular mathematics classrooms. *School Science and Mathematics, 105*, 423-432.

Secada, W. (1995). Social and critical dimensions for equity in mathematics education. In W. Secada, E. Fennema, & L. Byrd Adajian (Eds.), *New directions for equity in mathematics education* (pp. 147-164). Cambridge: Cambridge University Press.

Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher, 34*(4), 14-22.

Shaughnessy, J. M. (1992). Research in probability and statistics: Reflections and directions. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 465-494). New York: Macmillan.

Shaughnessy, J. M. (2007). Research on statistics learning and reasoning. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 957-1009). Charlotte, NC: Information Age Publishing.

Shaughnessy, J. M., Garfield, J., & Greer, B. (1996). Data handling. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.). *International handbook of mathematics education* (pp. 205-238). Dordrecht, The Netherlands: Kluwer.

Sierpinska, A., & Kilpatrick, J. (1998).(Eds.). *Mathematics Education as a research domain: A search for identity*. Dordrecht, the Netherlands: Kluwer.

Silver, E. A., Smith, M. S., & Nelson, B. S. (1995). The QUASAR Project: Equity concerns meets mathematics education reform in the middle school. In W. G. Secada, E. Fennema, & L. B. Adajian (Eds) *New directions for equity in mathematics education* (pp. 9-56). Cambridge, MA: Cambridge University Press.

Simon, M. (2000). Research on the development of mathematics teachers: The teacher development experiment. . In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 335-359). Mahwah, NJ: Lawrence Erlbaum.

Sowder, J. (1992). Estimation and number sense. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 371-389). New York: Macmillan.

Sowder, J. T. (2007). The mathematical education and development of teachers. In F. K. Lester (Ed.), *Second handbook of research on mathematics learning and teaching* (pp. 157-223). Charlotte, NJ: National Council of Teachers of Mathematics.

Sriraman, B. (2007) (Ed). Zoltan Paul Dienes and the Dynamics of Mathematical Learning, *The Montana Mathematics Enthusiast*, Monograph 2, September 2007, ISSN 1551-3440.

Stecher, B., & Borko, H. (2002). Integrating findings from surveys and case studies: Examples from a study of standards-based educational reform. *Journal of Education Policy, 17,* 547-569.

Steffe, L. P., & Olive, J. (2002). Design and use of computer tools for interactive mathematical activity (TIMA). *Journal of Educational Computing Research, 27,* 55-76.

Steffe, L. P., Thompson, P. W., & von Glasersfeld, E. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 267-306). Mahwah, NJ: Lawrence Erlbaum.

Stevens, R. (1999). *Disciplined perception: Comparing the development of embodied mathematical practices in school and at work*. Unpublished doctoral dissertation. University of California, Berkeley.

Stigler, J. W., Gonzales, P., Kawanaka, T., Knoll, S. & Serrano, A. (1999) *The TIMSS Videotape Classroom Study: Methods and Findings from an Exploratory Research Project on Eighth-Grade Mathematics Instruction in Germany, Japan, and the United States*. Washington, DC: U.S. Department of Education.

Tabach, M., Hershkowitz, R. & Arcavi, A. (this volume), Computerised Environments in Mathematics Classrooms: A Research-Design View.

The Learning Federation. (2007). *About TLF Content*. Retrieved March 16, 2007 from http://www.thelearningfederation.edu.au/tlf2/showMe.asp?md=p&nodelID=2)

Tirosh, D., & Graeber, O. A. (2003). Challenging and changing mathematics teaching classroom practices. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick & F. K .S. Leung (Eds.), *Second international handbook of mathematics education* (643–687). Dordrecht: The Netherlands: Kluwer.

Verschaffel, L., Greer, B., & DeCorte, E. (2007). Whole number concepts and operations. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 557-628). Charlotte, NC: Information Age Publishing.

Vianna, E. & Stetsenko, A. (2006). Embracing history through transforming it: contrasting Piagetian versus Vygotskian (activity) theories of learning and development to expand constructivism within a dialectical view of history, *Theory and psychology*, 16(1), 81-108.

Vygotsky, L. S. (1978). *Mind in Society: The Development of Higher Psychological Processes*, edited by M. Cole, V. John-Steiner, S. Scribner & E. Souberman. Cambridge: Harvard University Press.

Vygotsky, L. (1981) The instrumental method in psychology. In J. Wertsch (Ed.), *The concept of activity in Soviet psychology* (pp. 189-240). Armonk, NY: Sharpe.

Watson, J. M. (2006). *Statistical literacy in school: Growth and goals*. Mahwah, NJ: Lawrence Erlbaum.

Wenger, E. (1998). Communities of practice: Learning, meaning, and identity. Cambridge, MA: Cambridge University Press.

Wilder, R. (1986). The evolution of mathematical practice. In T. Tymoczko (Ed.). *New directions in the philosophy of mathematics* (pp. 185-199). Boston: Birkhauser.

Woodward, J., & Brown, C. (2006). Meeting the curricular needs of academically low-achieving students in middle grade mathematics. *The Journal of Special Education*, *40*, 151-159.

Zbiek, R. M., Heid, M. K., Blume, G., & Dick, T. P (2007). Research on technology in mathematics education: The perspective of constructs. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1169-1207). Charlotte, NC: Information Age Publishing.

Zevenbergen, R. (2001). Is streaming an equitable practice? Students' perceptions of streaming in the middle years of schooling. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.), *Numeracy and beyond*, Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia (pp. 563-570). Sydney: MERGA.

Mechanistic Perspectives	**Systemic Perspectives**
Testing Creating an ordeal (barrier, or filter) for accepting or rejecting (but not helping).	Examining Inspecting closely and generating high-fidelity descriptions.
Measuring Partitioning (fragmenting) into distinguishable Pieces.	Documenting Gathering tangible evidence which is credible to decision-makers.
Evaluating Assigning a value without specifying conditions or purposes.	Assessing Taking stock, orienting with respect to known landmarks and goals.
Summative Information Focuses on decision-making issues of Administrators.	Formative Information Focuses on decision-making issues of students and teachers.
Improving Tests The goal is to make credible evaluations better aligned with instructional goals.	Improving Teachers' Assessments The goal is to make authentic assessments (such as those based on teachers' judgments) more credible and reliable.

Table 32.1 Mechanistic vs. Systemic Perspectives on Assessment