Chapter 7: Classroom Practice: Challenging Mathematics Classroom Practices 26 December 2007

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In this chapter we examine classroom practice issues related to teachers providing mathematical challenges in their everyday classrooms. We examine how challenging mathematics can become the essence of mathematics classrooms, how challenging mathematics can be designed for the everyday classroom and how classroom artefacts and practices can be designed for mathematical challenges. Finally, the question of suitable research designs for research into classroom practices associated with the use of challenging mathematics in everyday classrooms is addressed and illustrated.

1. Challenging Mathematics—the essence of mathematics classrooms

Our challenge as educators is not just to make challenging mathematics available in school. It is to enable, invite, and scaffold students to accept and exploit the challenge that richer mathematical understandings can offer them (Mason & Janzen Roth, 2004). Consider the following task used by Mason and Janzen Roth in one of a series of design experiments (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; English, 2003) which will be overviewed later in this chapter.

The Tennis Ball Problem: Consider the following problem for which you are given a set of resources you might use. For each use of a resource, there is a point-cost which reduces the total for your team. Up to 100 points will be awarded for a clear and accurate solution. The resources are:

A tube container just large enough for the 3 tennis balls it holds: 60 points
A tennis ball: 30 points
A cloth tape measure: 20 points
A metre stick: 10 points
A 1-metre string: 5 points

You have 10 minutes to decide on the resouces you might use. You then have a further 10 minutes to answer the question.
Question: By what percent does the height of the tube container for the tennis balls exceed the distance around the tube (or vice versa)?

This task has been used to introduce a unit of mathematics that engages students in looking more deeply into the formulas for the circumference and area of circles. The students who have participated in the design experiment had learnt in previous years that $C = 2\pi r$, $C = \pi d$ and $A = \pi r^2$, but it has been a repeated source of disappointment for the researchers how few students actually consider using the relationships summarised by the formulas to answer the problem and so answer the question without buying any resources.

Why so few? Why have the capable students we teach chosen to develop a purely procedural understanding of the mathematical relationships they are studying? Mason and Janzen Roth posit that this is a consequence of students not perceiving the potential benefits, including the intrinsic satisfaction, available to them if they accept the challenge of developing richer understandings. What is it about classroom practices that fosters this shallow understanding of the power of mathematical relationships?

We could of course, be pessimistic and take Vinner's line that "there are two essential conflicting elements in the human psychology which are active in the domain of teaching and learning mathematics: the need for meaning and the ritual schema" and "there is no chance that one tendency will take over the other. The educators will continue their call for meaningful learning, whereas the masses of students will prefer the ritual (procedural) approach" (2000, p.121). Alternatively, we could put our hands up and say perhaps this lack of depth in student understanding of concepts underpinning formulas, say, is an artefact of classroom practices which result from different teaching styles. The King's College Project (Askew et al., 1996), for example, which studied teaching styles in the UK found three that were prevalent: transmission, discovery learning and connectionist. Teachers adopting a connectionist viewpoint, for example, would use challenging activities such as problem solving in the classroom by building on students' current knowledge and existing connections between mathematical ideas to look flexibly at a problem to make sense of it and generate new connections in innovative ways. This is a far cry from the transmission of procedures that work in familiar contexts but are quickly discarded by students as useless in slightly novel situations (Stillman & Galbraith, 2003, p. 183).

In contrast to the transmission method of instruction which is often seen as the traditional approach in Western mathematics classrooms, a traditional method in Japanese elementary school is to solve a problem through full-class discussion. With a skilful teacher, the children can learn more than the curriculum intends. The Japanese *open approach* (Hino, 2007; Nohda, 1995/2000; Shimada, 1977; Tejima, 1997/2000) "offers opportunities for especially bright students to exercise their creative abilities and devise insightful ways to deal with mathematical topics and problems" (Hashimoto & Becker, 1999, p. 101). When using the *open process* aspect of this approach, the focus is "on different ways of solving a problem when the answer is unique" (Hashimoto & Becker, p. 101). As an example of this practice, suppose the class is given the problem of dividing $\frac{4}{5}$ by $\frac{2}{3}$. One student might observe that 6 is the least common multiple of 2 and 3, and write

$$\frac{4}{5} \div \frac{2}{3} = \frac{4 \times \frac{6}{2}}{5 \times \frac{6}{3}} = \frac{4 \times 3}{5 \times 2} = \frac{12}{10}$$

The children can then come to realise that this method is equivalent to the standard algorithm and can be used with other choices of fractions. "It is important how [the] teacher leads students to make relationships on the basis of different conceptions" (Tejima, 1997/2000, p. 252) of the process, as in this case. However, the different conceptions could also be of the problem formulation or what counts as a solution if it is an open ended problem. From the teacher's point of view this dynamic is unpredictable. Consequently, the teacher requires deep mathematical understanding and sure skills in order to handle the situation. However, when the challenge of such an alternative solution process is taken up rather than dismissed and the approach succeeds, the children deepen their mathematical experience. What is it about the classroom practices in this approach that ensures this “meta-learning” (Nodha, 1995/2000, p. 30) occurs?

1.1 Why do we need challenges in the regular classrooms?

Organising mathematical challenges in overloaded mathematics lessons, whether they be short open or closed problem solving tasks (e.g., see Sriraman & English, 2004), or investigations (Ponte, 2007), mathematical modelling tasks (e.g, Galbraith, Stillman & Brown, in press; Kadijevik, 2006) or projects involving extended challenging tasks particularly those involving real world contexts, can be time-consuming; therefore, there must be valid reasons for doing so. There are, indeed, pay-offs in teachers using challenging mathematical tasks in regular classrooms.

Firstly, when well-planned in accordance with a theory of knowing, such as Gardner’s Theory of Multiple Intelligences (1983, 1999) or Tall’s Theory of Mathematical Growth (2006), challenging mathematics classroom practices enable students to develop a fuller understanding of the many aspects of a mathematical concept, enriching their concept image. A student’s concept image is “the set of all the mental pictures associated in the student’s mind with the concept name, together with all the properties characterizing them” (Vinner & Dreyfus 1989, p. 356). Teaching the same concept using multiple channels and perspectives (i.e., teaching with/through Multiple Intelligences) is one promising way to allow students to learn concepts with deep understanding (Cheung, 2003). Students’ concept image becomes enriched through challenging tasks if, for example, the tasks afford students to see that there are more transparent forms of the same concepts in different situations as happens when The Chickens and Hens Problem, how many legs, is solved using a spreadsheet rather than by-hand. Thus finding a solution to a particular task is of little benefit in itself, rather the enrichment comes when the students are able to look across several situations or several solutions and see the different mainfestations of the concept some of which more easily recognisable and separable from the task context than others.

Secondly, setting the mathematical tasks in real life situations not only makes the challenges more personally relevant to the daily life of students (Kadjevich, 2006), but also affords students opportunities to approach the challenges at different levels of mathematization (Freudenthal, 1973). Julie (2007) is one of the few researchers who have investigated the contexts that learners prefer to learn about in mathematics class or through mathematics. He confirmed that students show “strong interest in issues of direct personal appeal bolstered by a high media visibility” (p. 201). However, in agreement with Skovsmose (1998), he points out that schooling is also about “foregrounding” issues that “learners do not as yet perceive as interesting” (p. 201) so such real contexts for challenges need not be restricted to current interests if a long term perspective is taken. Students will have greater opportunities to encounter similar challenges in their everyday lives as they grow older. Their experiences will be accumulated through these encounters. According to Freudenthal, there are four *levels of mathematization* which could be the basis for real world tasks: (1) the *situation level* where knowledge and strategies specific to the domain in which the task is set are used in the task context; (2) the *referential level* where models and strategies refer to the task context; (3) the *general level* where the focus is on mathematical strategies and models for the task context; and (4) the *formal* *level* involving working with formal mathematical notation and procedures (see Gravemeijer, 1999; Cheung, 2005 for an explanation and illustrative examples). Such tasks need not be at a higher level of mathematization to be challenging, though. In the research of Mason and Janzen Roth (2005), for example, tasks such as whether it is possible to draw a square of area 10 cm^2 have proved to be challenging for Year 9 secondary students.

Thirdly, when students engage in solving challenging mathematical tasks, they are placed at a psychological boundary between their comfort zone and risk-taking. Challenges teach students how to sustain themselves in uncertainty—a skill relevant for lifelong learning—and successes with challenges prepare students for real life, which is a flexible or generic form of readiness for real life. As real life is often messy and may not be easily reduced to simple mathematical forms, challenges help students become aware of the intricate details and the significance of the roles these details may play for solving the challenges. Therefore, it is essential that mathematical challenges should be presented at different forms of mathematization so that students at all grade levels can experience how open-ended real life problems may be approached by mathematicians and their teachers.

1.2 How often should challenges be used and for whom?

It is essential that such experiences are provided regularly (Kadijevic & Marinkovic, 2006; Silver & Stein, 1996). Students need the opportunity to engage with challenging tasks on many occasions. They may not always engage with a particular challenging task but encountering several over time will give multiple opportunities for them to access such tasks and bring them to the realisation that it is an expectation of all to be able to do so. The regular use of challenges in classrooms with appropriate structuring or scaffolding of the task as necessary (Mason, Graham & Johnston-Wilder, 2005; Sheffield, 2003; Stillman, Edwards, & Brown, 2004) indicates to all that solving of mathematical challenges is applicable for all students of various learning abilities (Woodward & Brown, 2006) and experiential backgrounds, and as such it is one viable way to address inclusion and intellectual diversity (Sriraman, 2006). All students should be challenged and should challenge themselves to learn deeper mathematics in our classrooms as challenge is one of the characteristics of academic tasks that motivate learning according to Paris and Turner (1994). Amit, Freid and Abu-Naja see it as a "social obligation" that we ensure "no child be denied the materials, conditions, and kinds of teaching necessary for developing good mathematical thinking and the social and economic benefits deriving from it" (p. 75). Williams (2003a, 2003b, 2006) suggests this is possible through the development of resilience.

"Resilience relates to how a child explains occurrences in their day-to-day encounters with the world" (Williams, 2003b, p. 374). Resilience is "an 'optimistic orientation' to the world characterised by a positive explanatory style where successes are perceived as permanent, pervasive, and personal, and failures as temporary, specific, and external (Seligman, 1995)" (Williams, 2003b, p. 752). According to Williams (2003a), "resilience, and inclination to pursue novel mathematical ideas, appear to be mutually sustaining (overcoming a mathematical challenge conditions an optimistic orientation and an optimistic orientation increases student inclination to pursue the next challenge)" (p. 758). Not all students display resilience but rather can be seen as displaying pessimism or somewhere in between these two. However, Seligman (1995) has found that the resilience of a child can be altered over time. Using Csikszentmihalyi's (1992) concept of flow as a framework, he found "that in overcoming small challenges to gain successes, the child's inclination to undertake future challenges was increased" (Williams, 2003b, p. 378). Williams describes flow and its effects as "an optimal learning condition that may occur when a person works just above their present skill level on a challenge almost out of reach. Individuals or groups in flow become so engrossed with the task at hand that they lose awareness of self, time and the world" (p. 378). These challenges that develop resilience are self-set as individuals or groups spontaneously decide to explore unfamiliar mathematics encounterd in a teacher–set challenging task (Williams, 2006). Thus, what is being advocated here is regular use of challenging problems within which groups or individuals have opportunities to set their own challenges at a level of difficulty that is appropriate for them. Through treachers providing opportunities for mathematical challenges for all students, there is the potential for students to develop resilience, if they are not already displaying it, and an inclination to desire to pursue more mathematical challenges.

2. Designing Challenging Mathematics for Classrooms

2.1 Setting the scene

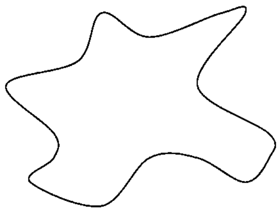
Before we can consider how challenging mathematical activities and tasks might be designed for the everyday classroom there are three issues that must be addressed. Firstly, it is necessary to consider the nature of the mathematical understandings that we expect to be deepened by use of these challenges. Secondly, teachers need to be aware of the nature and extent of the gap between what they are currently doing in mathematics classrooms and what is possible using mathematical challenges. Thirdly, it is necessary to point out that when mathematical challenges are being used, students need to be made aware that the rules of the didactical game (Brousseau, 1997; Mercier, Sensevy, & Schubauer-Leoni, 1999) have altered and so the didactic contract (Brousseau, 1997) needs to be renegotiated.

2.1.1 The nature of the mathematical understandings expected to be deepened. In designing challenging activities and tasks for the regular classroom our purpose is not to restrict and confine students with activites or tasks that students find palatable but the mathematical payoff is limited, rather we want an invitational package that not only draws students towards the activity but also has the mathematical qualities of mathematical inquiry that will sustain it. Thus "we must do more than invite people to our mathematical game—we will have to play host to ensure all the guests feel the pleasure of addressing an intellectual challenge" (Mason, 2000, p. 111) through engaging in meaningful and rewarding mathematical thinking. The processes that are required to solve challenges and the metacognitive knowledge and strategies that can be brought to bear during application of those processes are also considered content when it comes to the understanding necessary to engage with challenges fruitfully in the classroom.

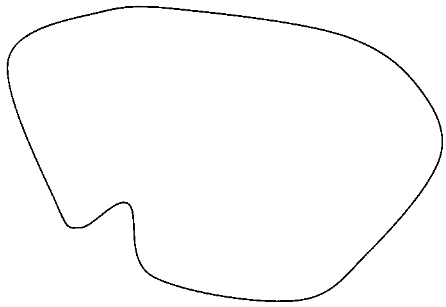
2.1.2 The gap between what is being proposed and present practice. Teachers are often not aware that their own teaching practices contribute to the students in their classes not experiencing or not desiring to be challenged in their mathematics classrooms.

However, the gap between present practice and what is being proposed may not be all that wide as the following vignette from Japan illustrates.

Primary teachers in Japan use the area of a rectangle as a starting point for finding the area of a circle. In Years 4 and 5, 10/11 year olds already know the formula for area of a rectangle (see Takahashi, 2006) and are given many small problems about area for complicated configurations such as those in Figures 1 and 2. Students are asked questions such as: Which has the bigger area 1a or 1b? If the perimeter of a figure is longer is the area bigger?



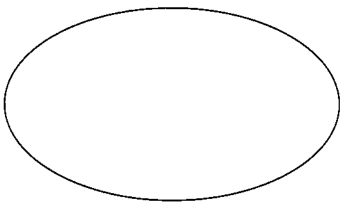
(1 a)



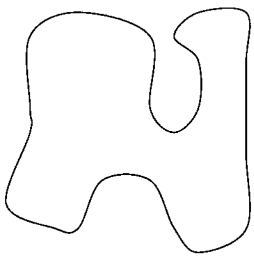
(1 b)

Figure 1. Complicated configurations for comparing area and perimeter for Year 4 & 5 students.

How can you find the area for a complicated configuration such as 2 a or 2 b?



(2--a)



(2 b)

Figure 2. Complicated configurations for finding area for Year 4 & 5 students.

Students begin by cutting out rectangles then progress to drawing rectangles to find answers to these questions. Then students are asked to compare areas of the following figures (Figure 3).

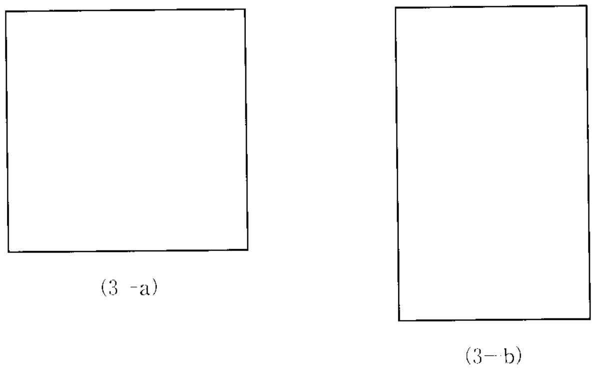


Figure 3. Comparing area of rectangles task for Year 4 & 5 students.

One student might propose to compare these figures by regarding them as tthe bases of boxes and then filling them with small balls represented by circles as in Figure 4. So the area of Figure 3b is bigger than that of Figure 3a.

Figure 4. Using balls to compare areas.

Another student raises a doubt about the applicability of the proposed method pointing out that circles will never fill up the rectangles. At this point there are several ways for the teacher to handle the challenge to the first solution method. One possibility is to use circles of different radii progressively making the radii smaller and smaller. In this case students will see the essence of the idea that a certain infinite process is necessary. Then the teacher has the opportunity to explain, or students can be given the opportunity, to find for themselves that to find the area of configurations such as Figures 2 a and 2 b, even if rectangles are used, an infinite process is still needed. This would be a good prelude to consider the area of a circle. Another possibility is to ask what happens if we use dice to cover the bases of the boxes instead of balls.

However, if the teacher thinks these questions will merely make the students puzzled, they might take the point of view that it would be better just to skip the question (and so skip the challenge) saying that it would result in the same answer by using rectangles or circles anyway. In this instance, the teacher just wants to make things simple and ignores the challenge of using circles, which is much more difficult – an opportunity lost.

After these preparations, students are able to find the area of a triangle themselves by using rectangles and known formulae for these (Figure 5).

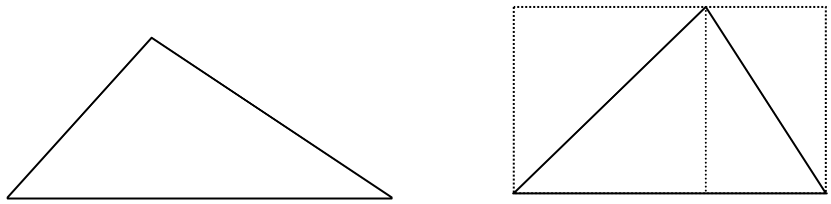


Figure 5. Finding area of triangle from rectangles solution by Year 4 & 5 students.

Students then find areas for a parallelogram, trapezium, a pentagon and finally a circle (Figure 6). However, unless the challenge is taken up in the teaching moment, rather than avoided, the link to the deeper notion of infinite processes (Figure 7) is lost.

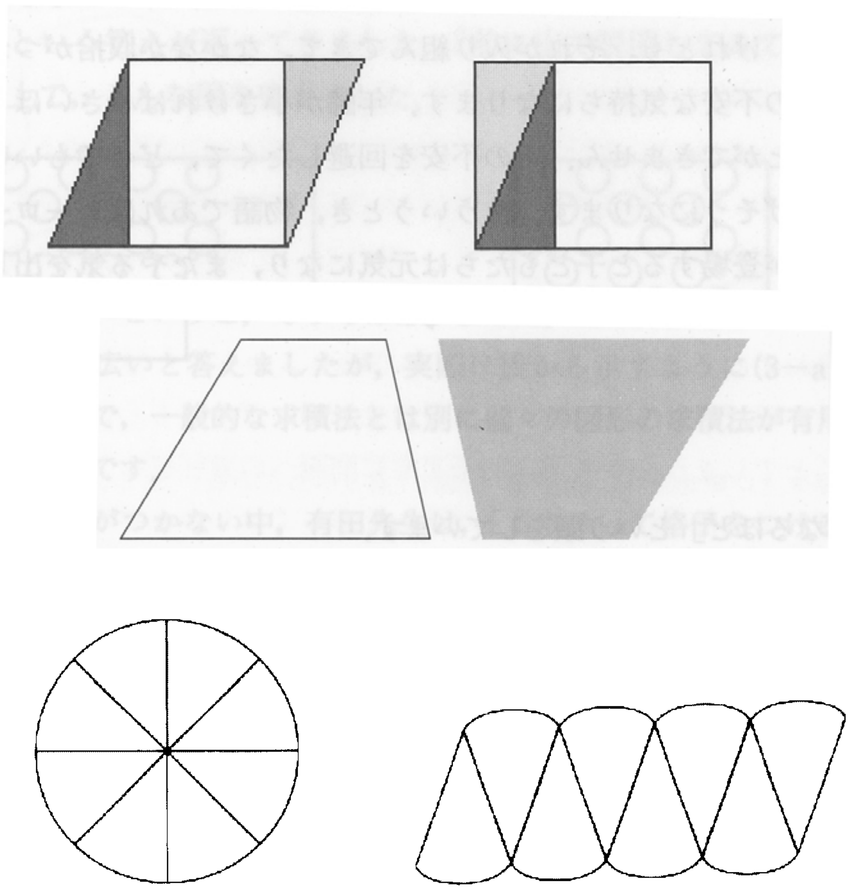


Figure 6. Finding areas of standard figures by Year 4 & 5 students.

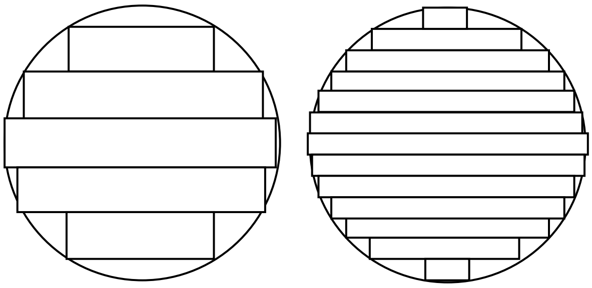


Figure 7. Finding the area of a circle by the infinite process of using rectangles of decreasing width.

Thus, teachers are the key to effecting change in the classroom. However, they need to be convinced themselves that it is necessary to go in a new direction (Pekhonen, 2007) and to engender students with confidence that challenges are essential in the classroom. Continually avoiding the addressing of challenges, robs students of opportunities to learn from engaging in challenges. It is imperative, therefore that the teacher suspend judgement for a while when these serendipitous challenges arise and see what prevails.

2.1.3 Clarifying changes in expectations

According to Mercier, Sensevy and Schubauer-Leoni (1999), "Brousseau defines the didactical contract as a system of reciprocal expectations between teacher and pupils, concerning knowledge, which contract is setting both pupil's and teacher's acts and can explain them thereby" (p. 343). Furthermore, "Modelling a teaching situation consists of producing a game specific to the target knowledge among different subsystems: the educational system, the student system, the milieu, etc." (Brousseau, 1999, p. 47). But what happens when the implicit rules for this didactical game have to be changed? If students are use to doing several short quick tasks in every lesson where the mathematisation of such tasks has been fully laid out before them, introducing a task that now requires, say, that they work out the mathematisation themselves and that they are expected to persist for several lessons working on the same task is a contravention of the usual didactical game. Students need to be alerted to the changes in the teacher's intentions and expectations in the new didactical situation. Thus, when introducing mathematical challenges into the classroom, whether they be short or long tasks, abstract or set in a real world context, attention must be given to this issue of clarifing expectations. Failure to do so can result in little outcomes related to the mathematical purpose of the lesson or organisational difficulties and frustrations for both teachers and students (see Cheung, 2006, for an example).

2.2 Task design

Kadijevick and Marinkovic (2006) in making a plea that regular curriculum content be the source of challenging tasks in the regular classroom, suggest that mathematical quizzes involving questions which are solvable in 10 to 30 seconds be used. These quizz items should "require a prompt and meticulous [form of] thinking, contributing to the development of mathematical reasoning" (p. 34). Sample items are:

A mouse's body is 12 cm in length and this is one third of the length of its tail. How long is the tail? (15 seconds are allowed for primary students to answer)
Write down an expression for 100 by using a) 5 six times; b) 3 seven times. (30 seconds are allowed for Years 7-9 students to answer)

The development and use in regular classrooms of sets of challenging tasks which are isomorphic with respect to a) the mathematical content or b) solution method are also advocated by Kadijevick and Marinkovic (2006). Examples of the former types of tasks are:

Marking Pens: Two marking pens pens cost more than three pencils. Do 5 marking pens cost more than 7 pencils (no discount offered)?

Parking Stations: Two parking stations in a town are competing for customers. At present more cars can be parked in one parking station which has two levels than in another one with three levels. Because of this, the parking station with three levels will be extended to five levels. Will an extension of the first mentioned station from 2 levels to 3 levels enable it to continue to have more parking space than the extended second station?

Examples of tasks which are isomorphic with respect to method would be non-routine area tasks which are all solvable using translation of geometric figures to gain an insight into a simple method of solution. Exemplar tasks can be found in Kadijevick and Marinkovic (2006). Design features that should be borne in mind in developing such tasks are: “1) there is a good mathematical idea behind the task; (2) the task is not routine; (3) the task is interesting with respect to formulation and content; (4) the task has a nice and perhaps unexpected solution(s); (5) the task requires its solver to stretch his/her mind; and (6) the solution of the task is usually short and not complicated, enabling the solver to use knowledge and skills traditionally learned in the classroom” (p. 35).

Another source of challenging tasks for the regular classroom is real-world tasks. Designing extended real-world investigative tasks that present manageable and engaging challenges for lower secondary students is not without challenges as was shown in the RITEMATHS project (http://extranet.edfac.unimelb.edu.au/DSME/RITEMATHS/). Despite thoughtful considerations by the teacher in both designing the tasks (e.g., Figure 8) used in one part of the project and providing timely task scaffolding at points during task implementation when students were expected to be challenged by the cognitive demand of tasks, there are always differences between the expected student moves and challenges and what transpires. Some students take up the challenges as expected but for others these same challenges do not eventuate as the significance of particular requirements of the tasks is missed, or the mathematical implications of results produced during the task which should generate challenge are not realised. At other times unforeseen challenges arise for individual students as they discover different complexities in their unanticipated interpretation of tasks (Stillman, 2006). As an example of the former, in the task, *Shot on Goal*, one pair of students were challenged by their interpretation of the task when they tried to mathematise the run line. The teacher expected the students to consider a player approaching the goal on a run line parallel to the side line. Instead of advancing down their specified run line in 1 m intervals which would have kept the line of the path parallel to the sideline, this pair of students took a stepped trajectory towards the goal considerably increasing the challenge beyond what was expected of them.

Many ball games have the task of putting a ball between goal posts. The shot on the goal has only a narrow angle in which to travel if it is to score a goal. In field hockey or soccer when a player is running along a particular line (a run line parallel to the side line) the angle appears to change with the distance from the goal line. At what point on the run line, has the attacking player opened up the goal to maximise the possibility of scoring the goal?
Assume you are not running in the GOAL-to-GOAL corridor. Find the position for the maximum goal opening if the run line is a given distance from the side line As the run line moves closer or further from the side line, how does the location of the position for the widest view of the goal change?

Figure 8 Major elements of *Shot on Goal* Task from RITEMATHS research project.

2.2.1 Rephrasing as a means of tweaking a task for different grade levels.

Another illustrative example of a challenging mathematical task set in an everyday context applicable to Forms 1-6 (grades 7-12) is presented below for the ensuing discussion. This task will be further modified in order to demonstrate how mathematical challenges might be introduced into regular classrooms at all grade levels through a process of rephrasing. This task has been used by Cheung (2006) in teaching experiments in Macao but it is a common task in secondary schools in many contexts worldwide although not all versions or implemtations are challenges (see, French, 2002; Harvey, Waits & Demana, 1995; Pierce & Stacey, 2006). As will be discussed later in this chapter, whether or nor a task is challenging for a particular student is very much dependent on the actions of the teacher and other students and the student's own abilities and experiences.

Open Box Problem (Version 1): Use a sheet of 12 cm x 12 cm cardboard, cut away four corners, fold and glue to form an open box to hold as many things or as much as possible.

When this problem is introduced for student project work, individually or as a group, it is advisable for the teacher to phrase it in a manner that establishes a better connection with everyday life and appears more personally relevant to the students. For example, the problem may be stated as:

Open Box Problem (Rephrased in an Everyday Context): Using the cardboard (12 *cm* x 12 *cm*) provided, form any kind of open box to contain as many paper clips as possible (or contain as much rice as possible).

This problem after rephrasing provides students opportunities to change a realistic practical problem into a mathematical problem for solution. If this happens, the student is encouraged to approach the problem at the lowest level of mathematization, that is, the situation level (Freudenthal, 1973; Gravemeijer, 1999). Before assigning the problem to the students, the teacher needs to consider carefully what background knowledge and skills are available for the students in order to arrive at a solution. At the situation level, students engage in both hands-on and minds-on activities. They need to make an open box and then find out the one box that contains the largest number of clips. Teaching experiments conducted in Macao with this problem (Cheung, 2006) revealed that some younger students do not know how to make an open box. They simply treat the task as a paper folding exercise to come up with a container and then see how many clips it can contain without spilling. Their attempts to make such containers are by no means systematic. They simply treat the task as a mathematical game where the container holding the most wins not realising they might not have stumbled upon the one that contains the most possible. They do not see the mathematical purpose is to find the relationship between volume and height of the open box. They fail to see the relationship between the height of the box and the length of the side of the square cut from the corners of th cardboard.

Another interesting observation is that some students are reluctant to cut away corners from the cardboard when the open box is made. Interviews conducted after the teaching experiments revealed that there is a misconception preventing them from doing so, namely, students possess an intuition that the more that is cut from the cardboard the less the volume of the open box enclosed from the remaining cardboard will be. This illustrates that teachers should always be ready to learn from their students, and teachers should treat their not knowing an unexpected phenomenon such as this as an asset, not a short coming. This is because these moments give direction for future teaching which can be based on these misconceptions or alternative conceptions.

In this task, as in all mathematical challenges, teachers need to know what the gap is between what is proposed for them to accomplish mathematically and what the students may do.

Students in Macao learn how to conceptualize and calculate the volume of a cuboid before Primary 6. Cuboid and capacity are two mathematical terms that may prompt students to adopt a mathematical approach for a solution. The problem could be rephrased accordingly.

Open Box Problem (Rephrased in an Everyday Context with mathematical hints): Using the cardboard (12 *cm* x 12 *cm*) provided, form an open box in the form of a cuboid with an internal capacity as large as possible (Paper clips, rice, ruler, calculator etc. may be provided if students opt to solve the problem using an experimental approach, or a combination of experimental and mathematical approaches).

Students need to form a net in the form of a cross in order to have it folded and glued into an open box. Four identical square corners with equal lengths need to be cut from the cardboard in order to form the cross. Students may experiment with how much rice or how many paper clips the open box is able to contain, and/or go straight to use of the volume formula of the cuboid to calculate the internal capacity of the open box. For those students who adopt an experimental approach, they need to continue making more boxes of different sizes and make the necessary size comparisons accordingly. For those students who adopt a mathematical approach, they need not make more boxes once they can make sense of the calculations done to the first box. Instead, they can try different heights of the cuboids and calculate the corresponding internal capacity to come up with an optimal solution. Because the mathematical approach adopted makes references to a concrete case in order to help reasoning, this solution process is at the referential level of mathematization (Freudenthal, 1973; Gravemeijer, 1999). In order to reach the optimal solution, students may make tabulations to find out relationships between the height of the cuboid and the internal capacity of the open box. However, those students who insist on adopting an experimental approach will not know whether their solutions are optimal or not. What they are targetting are better and better solutions by making more and more boxes. The level of mathematization still remains at the situation level of mathematization. However, realizing that mathematical challenges are essential to develop reasoning, the teacher should suspend judgment for a while and should not usher students to adopt a higher level of mathematization right away "as it is the extent to which the locus of knowledge generation is with the learners which makes the difference" (Watson, 2004, p. 366).

This task can be assigned as a group project so that students not only learn *with* each other but also learn *from* each other. When students of a heterogeneous group engage in collaborative exchanges they can contribute their intellectual strengths and at the same time have their weaknesses scaffolded by their peers. The effectiveness of peer scaffolding is, however, mediated by the appropriateness of task-related questions framed by their peers "and the extent to which one learner attends to the questions (and other contributions) of the other" (Clarke, 2001, p. 310).

For a project, the teacher could, for example, ask the group to guess which of the following five cases, that is, those with 1 *cm*, 2 *cm*, 3 *cm*, 4 *cm*, and 5 *cm* square corners, should be cut to form the required net. Research done in Macau classrooms (Cheung, 2006) shows that the 1 *cm*, 2 *cm* and 4 *cm* cases are students' favorite choices. Some students choose the 1 *cm* case because it entails the least amount of paper cut from the cardboard and therefore they think this results in an open box of the the largest internal capacity. Some students choose the 4 *cm* case because they believe that if the cuboid resembles a cube then the volume of this cube should be the largest. However, they forget that the net is to be folded into an open, not closed, box. Students choose the 2 *cm* case because they perceive the open box to have a broad base area and yet have considerable height to produce a large enclosed volume. In this task, the mathematical challenge becomes one of seeking to find the relationship between the height of the open box and the internal capacity of the box formed. Even senior primary (Cheung, 2006) and junior secondary students without knowledge of advanced mathematics, for example, inequality and Calculus, can generate some solutions using a variety of approaches. At the referential level of mathematization, one is to be aware that it is still very difficult for the teacher to explain to students why the 2 *cm* case produces an open box with maximum capacity. The teacher can only point out that it is the largest amongst the five cases under consideration. In this regard, reformulating the mathematical challenge to higher levels of mathematization, such as the general or formal levels (Freudenthal, 1973; Gravemeijer, 1999), is required for a convincing explanation. This case will be considered in the context of answering our next question.

2.2.2. Does the nature of the task change with increasing grade level?

With increasing grade level the sophistication and the breadth of mathematics that students can bring to solving a challenge increases. Returning to the example of the Open Box Problem, the problem can easily be rephrased to afford higher levels of mathematization as shown.

Open Box Problem (Rephrased in Mathematical Context Affording Higher levels of Mathematization): Given a sheet of cardboard of dimensions *a cm* x *a cm*, what is the maximum capacity of the open box that can be formed from it?

This statement of the problem may be regarded as a typical textbook problem for senior secondary students. At the general level of mathematization, no open box needs to be made by the students and the length of the sides of the cardboard needs not be specified numerically. Students can simply make sketches of the net of an open box and use the sketch to formulate a non-linear equation relating capacity of the open box (i.e., $V\ cm^3$) with length of the side of the corner cut (i.e., x *cm*). Using the non-linear equation, students find the maximum value of V without recourse to the original problem situation.

$V = f(x) = x\ .\ (a - 2x)^2$, a being the length of the side of a piece of square paper, is therefore a model used for relating variables having non-linear relationships of third degree. Students may solve this problem if they possess knowledge of "inequality". At the formal level, using $V = f(x)$, students solve the maximization problem using established formal mathematical methods within the mathematics discipline, for example, differentiation. One difficulty faced by students at the general and formal level of mathematization is that their attention is often exhausted on the correct formulation of $V = f(x)$, and success in tackling the task depends on whether they can make use of the mathematical knowledge and skills taught to them during formal lessons. If they cannot formulate an equation and if they are unwilling to investigate the problem at a lower level of mathematization, they cannot proceed further. In this sense, such textbook problems may not be considered as mathematical challenges at all. In contrast, relaxing problem conditions and constraints can make problems more challenging than can requiring the use of sophisticated formulae and techniques.

Open Box Problem (Rephrased with a Relaxation of Problem Conditions and Constraints): Given a 12 *cm* x 12 *cm* square piece of cardboard, construct an open box in the form of a cuboid with an internal capacity as large as possible You are not allowed to waste any of the cardboard. Any cardboard that is cut away should be taped back to the net.

This problem is open-ended and can be assigned to both junior and senior secondary students. Students generally need to start from the situation level of mathematization to come up with one tentative solution first. After that, they need to attempt alternative methods and find out if other solutions exist. Since it is often time-consuming to experiment with more cases by hand, students may simulate the problem situations using a computer or graphing calculator. If a maximal solution is being sought, then the problem needs to be solved at the general or formal levels using sophisticated mathematical methods, for example, partial differentiation, or alternatively use graphical or symbolic manipulation techniques assisted by technology. The ultimate solution relies on the application of the Lagrange Multiplier Method. The constraint is that the area of the net equals 144 cm^2 because no cardboard paper is to be wasted. After derivation, V can be shown approximately equal to 166.28 cm^3.

2.2.3. Do we need new topics for challenges or can we find them within the existing curriculum?

Although the topics in existing curricula have not been exhausted as sources, there are some areas that could prove fertile ground for a source of challenges. Sriraman and English (2004) make a case for combinatorics as the topic is "accessible to students starting at the elementary levels because it builds from simple enumerative techniques" (p.183). Sriraman (2006) suggests that in addition the area of number theory is a good source of challenges. Sriraman and Strzelecki (2004) offer suggestions for challenges in number theory. In section 4, Sriraman's use of challenges from both these areas in secondary classrooms will be reviewed. In Sriraman and English (2004) examples are given of the use of combinatorics problems as challenges in primary school.

2.2.4 How can technology be incorporated into task design to facilitate the use of mathematical challenges?

Use of electronic technologies such as calculators and image digitisers can reduce the cognitive demand of tasks through "supplementation" and/or "reorganisation" of human thought (Borba & Villarreal, 2005) by carrying out routine arithmetic calculations, algebraic manipulations, or graph sketching; acting as an external store of interim results; or overlaying visual images within an interactive coordinate system to facilitate analysis. However, these technologies also have potential to influence the complexity of what students do as they transform classroom activity and allow new forms of activity to occur. Regulation of this complexity is a further opportunity for teachers to mediate cognitive demand, and therefore the challenge, of tasks through careful crafting of tasks and management during implementation. In particular, use of multiple representations, easily accessible with graphing calculators and tasks amenable to electronic technology use, harness opportunities for students to use technology to stimulate higher order thinking in investigating real-world situations. Within tasks diagrammatic, numerical, symbolic, graphical, and algebraic representations can be intentionally employed to support bridge making from one representation to another and to provide opportunities for interpretation across representations as well as from each representation back to the situation being investigated.

As Dede (2004) points out, several projects implementing well-formulated technology-based designs have demonstrated that "typical middle years students [are capable of] mastering science and mathematics previously thought appropriate to teach only" to students at higher schooling levels (p. 111). However, two challenges middle years students face when engaging in extended investigations for the first time (Loh et al., 2001), are inability to recognise when to keep records and failure to plan and monitor progress effectively. It is thus prudent for teachers designing extended tasks for the lower secondary years, initially at least, to provide timely instructions throughout task statements supporting recording of key information, a planned solution, checking and verification of results. As student task expertise and familiarity with technology grow, some "fading" of this scaffolding (Guzdial, 1994) should occur, particularly that related to task structuring and technological tool selection and instructions. This is not to say mathematical analysis tools need be withdrawn. On the contrary, "learning to 'work smart'" in a technology-rich learning environment may involve "learning to establish one's own scaffolds for performance, and fading these may be beside the point" (Pea, 2004, p. 443).

3. Designing Classrooms for Mathematics Challenges

3.1. How do we teach students strategically to address a challenge?

To help students not only address the mathematical challenges with which they have been presented but also to deepen their mathematical reasoning and develop their mathematical creativity, students can be encouraged to question the answers to the challenges, not just answer the questions. The use of a student-centred inquiry approach that encourages students to think like mathematicians, asking questions that enable them to make sense of mathematics, is one of the critical aspects of the Project M^3: Mentoring Mathematical Minds (Gavin, Chapin, Daly & Sheffield, 2006) program, a curriculum and research study in the United States designed to nurture mathematical talent and creativity in elementary students by creating challenging, creative and motivational curriculum units for students. The deeper mathematical reasoning occurs when students begin to create and solve their own challenges, realising that each solution is just the beginning of a new investigation.

Just as students answer questions of "who, what, when, where, why, and how" in writing an article for the school newspaper, they can learn to ask and answer these same questions as they investigate, create and extend mathematical challenges. All stdents can and should challenge themselves to deepen and extend their mathematical reasoning and abilities.

Some suggested questions for students as they create and solve mathematical challenges are:

- **Who?**
  Who has a new or different idea? Who is right?

- **What or what if?**
  What sense can I make of this problem? What is the answer? What are the essential elements of this problem? What is the important mathematics? What patterns do I see in this data? What generalizations might I make from the patterns? What proof do I have? What if I change one or more parts of the problem?

- **When?**
  When does this work? When does this not work?

- **Where?**
  Where did that come from? Where should I start? Where might I go next? Where might I find additional information?
- **Why or why not?**
  Why does that work? If it does not work, why not?

- **How?**
How is this like other mathematical problems or patterns that I have seen? How does it differ? How does this relate to "real-life" situations or models? How many solutions are possible? How do you know you have found all the possible solutions? How many ways might I use to represent, simulate, model, or visualize these ideas? How many ways might I sort, organize, and present this information?

Students of all ages can learn to deepen their mathematical reasoning and enjoyment by asking themselves these questions. The questions might be asked in any order as they fit the problem under consideration.The use of these questions is developed further in *Extending the Challenge in Mathematics: Developing Mathematical Promise in K - 8 Students* by Sheffield (2003) where a variety of challenges with samples of student work are presented.
Students use the heuristic shown in Figure 9 as they work on problems.

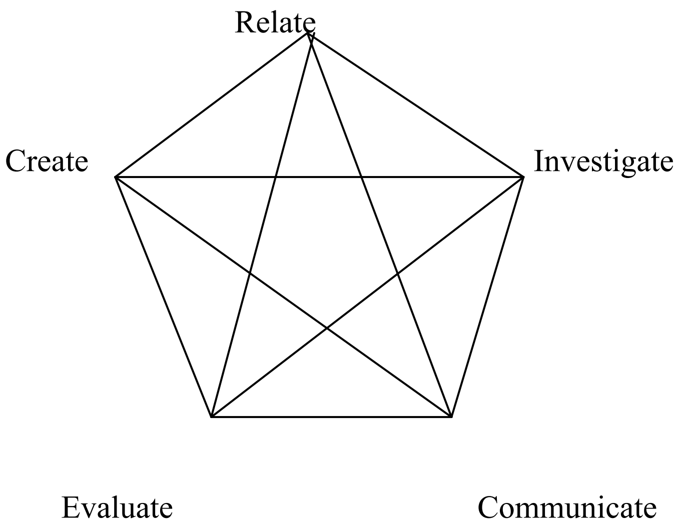


Figure 9. Heuristic used whislst solving challenges.

Students may start at any point on the diagram and proceed in any order that makes sense to them. They might do the following:

- Relate the problem to other problems that they have solved.
- Investigate the problem. Think deeply and ask questions.
- Evaluate their findings.
- Communicate their results.
- Create new questions to explore.

As they begin to learn to think like mathematicians, students might change the order of the steps and of the questions that they ask once they begin the exploration of the problem. Throughout the problem solving, students should be evaluating their work,

making connections, asking questions, communicating results, and creating new problems to investigate.
For example, in one problem in *Extending the Challenge in Mathematics*, students begin by investigating which of the following numbers can be the sum of two consecutive numbers: 15, 18, 57, 58, 229 and 228. They relate this to earlier problems that they have done with odd and even numbers. They then investigate numbers that can be the sum of three consecutive numbers and relate this to work with multiples of three and finding the mean of a set of consecutive numbers. Students are then encouraged to ask related, sometimes divergent questions. Some students extend their investigations to sums of 4 or more consecutive numbers and investigate why numbers that are a sum of 4 consecutive numbers are not multiplies of 4. Some students do more work with mean and median while others investigate other patterns with sequences and series. One very interesting investigation involves trying to find all numbers that cannot be written as the sum of any number of consecutive counting numbers. Finding these numbers and proving that they can never be the sum of consecutive numbers challenges even the most outstanding mathematics students.

In this way, the challenges for the students are differentiated according to their background and interests, and all students develop a deeper understanding of the topic under consideration. The teacher observes and supervises, challenges students who are ready to move to a higher level, gives hints to students who might be frustrated and ready to give up on a difficult question, and decides when to bring students together as pairs, small groups, or as a whole class to discuss their findings and probe possible misconceptions.

3.2 How can we make sense of the pedagagogic challenge of having students appreciate challenge in mathematics?

Perhaps the giants of twentieth century pedagaogy offer valid starting points. Piaget (1950) offers us a way to think about students' changing orientations to their subject matter. He suggests that as learners we are all more comfortable adding on to our current knowledge and understandings, a process he calls *assimilation*. Yet it is not by assimilation that students come to change their conceptual orientations: students are reluctant to change how they perceive themselves in relation to their environment (*accommodation*), and will do so only in the face of relatively persistent discomfort with the fit between their orientation and some aspect of their environment. It is not difficult for teachers to offer discomfort (cognitive dissonance, Atherton, 2003; Festinger, 1957) although some teachers may be reluctant to do so (Pierce, & Stacey, 2006). The challenge lies in having students accept the discomfort as an invitation to change or grow, rather than rejecting the discomfort as a source of frustration. Here, it is Vygotsky's idea of the zone of proximal development (1978) that helps us organise our pedagogy when we offer students opportunities to grow beyond what they are already capable of. We must fit what we offer to students into this region beyond what they can already do independently, and what they can do with the scaffolding we can offer them in their relationship with us and the classroom in which they are learning. Scaffolding, a term first introduced by Wood, Bruner and Ross (1976), is the interpersonal and strategic supports that teachers and classmates can offer learners to enable them to learn beyond what they can do on their own. “Each task has a relative cognitive value for an individual. Tasks that are too easy or too hard have limited cognitive value” (Diezmann & Watters, 2002, p. 78). However, if students engage in tasks of high relative cognitive value for them, potentially they can explore the cognitive challenge of engaging with mathematically challenging tasks and enhance their learning (Diezmann & Watters, 2000, 2002). This suggests that we can offer significant invitations to grow or change (dissonance) (Neighbour, 1992) only when we also offer significant scaffolding that enables students to see their engagement beyond their comfort zone as likely to generate success for them.

It must be borne in mind, however, that these are starting points, not ending points. If this is truly scaffolding in the sense of Wood, Bruner and Ross (1976), it needs to be accompanied by cycles of diagnosis of the student’s level of performance and the need for scaffolding which then results in an adjustment of the level of scaffolding needed (Pea, 2004; Stone, 1993). Thus, the level of scaffolding fades (Collins, Brown, & Newman, 1989; Guzdial, 1994) over time. Pervasive forms of support where the scaffolding is not dismantled enable only what Pea (2004, p. 431) calls “distributed intelligence” with the conquering of the challenges, in this case, being “accomplished” not achieved. Scaffolding also needs to be targetted to the needs of individuals and not be “provided to the whole class on the pretext that all students will benefit” (Diezmann & Watters, 2002, p. 78). As Diezmann and Watters add, “the gifted students are likely to be most adversely affected by unnecessary scaffolding” (p. 78).

3.3 The role of textbooks

The role of textbooks in classroom practice issues related to teachers providing mathematical challenges in their everyday classrooms should not be underestimated. This can have an impact on what happens in classrooms regarding the use of challenges due to the nature of the content of the textbooks and also how the textbooks are used in the classroom.

Textbooks could be written so that challenging activities is the philosophy and leading idea behind them, and not merely fragmented parts of the content of the book. Sadly, this is rarely the case in practice. There is research evidence that many mathematics textbooks in various countries contain few challenging tasks and often the level of challenge is not as high as expected. Haggarty and Pepin (2001, 2002), for example, in a study conducted in 15 lower secondary schools in England, France and Germany, found that French textbooks provided students with more challenging tasks than did their English and German counterparts. Furthermore, in Sweden where mathematics textbooks tasks are grouped into strands according to difficulty to assist the teacher in differentiation of learning to suit learners’ differing abilities, Brändström found when she examined Year 7 mathematics textbooks “the level of challenge is low in almost all strand, even those intended to be higher” (2005, p. 4). However, there is some recent evidence that the situation is improving in some countries (e.g., Germany) where there has been “a shift in mathematics textbooks for all grades from rather algorithmically oriented tasks to more demanding problems” (Reiss & Törner, 2007, p. 440). It would appear, though, that in many countries teachers may need to look for specialist publications rather than the chosen textbook for sources of challenges.

Even when textbooks have a more than adequate supply of challenging tasks, how these tasks are implemented in the classroom by teachers can affect the cognitive demand placed on students as they engage with the task. Henningsen and Stein (1997) and Stein, Glover and Henningsen (1996) have found that the high level of cognitive demand of textbook tasks can be reduced by teachers who remove the challenging aspects of such tasks. Teachers often diffuse challenge in textbook tasks when students start to struggle whereas struggle is what challenges are all about. This practice is fueled by a belief that "all but the most able pupils [need] routine and relatively low level demands made of them" (Haggarty & Pepin, 2001, p. 124). On other occasions teachers pre-empt difficulties that students might have with challenging tasks and intervene unnecessarily (Diezmann & Watters, 2002; Stillman, Brown & Galbraith, 2007). One of the practices teachers must foster when using challenging tasks is patience to allow struggle and not to intervene too early.

3.4 Managing the challenge

One of the reasons that is often given for a reluctance by teachers to use challenges in the classroom is a lack of knowledge of effective ways to manage all the divergent processes arising in the classroom when challenging projects or tasks are used. Several strategies are offered that teachers who use challenges in their classrrom employ to cope with this diversity.

The use of open questions by teachers (Sullivan & Clarke, 1991) is often advocated in mathematics classrooms as a means of facilitating deeper thinking as would be required when using mathematical challenges in the classroom. However, as Herbel-Eisenmann and Breyfogle (2005) point out "merely using open questions is not sufficient" (p. 484) to ensure this occurs. Often teachers use a questioning technique called "funnelling" (for examples see Herbel-Eisenmann & Breyfogle, 2005; Goos, Stillman & Vale, 2007, pp. 51-54) where questioning is used in such a manner that group or "classroom discussion [converges] to the thinking pattern of the teacher" rather than that of the students (Goos, Stillman & Vale, p. 54). This can be desirable when students are early in their experiencing of challenging tasks in the classroom, particularly extended challenging tasks, as the teacher's intention may be to scaffold students along a particular solution pathway so that they all experience the processes involved in solving such a task and have some insight into the complexity of their management of strategic resources in this process. If funnelling questions are being used for this purpose, it is necessary that the teacher brings to the foreground the "metacognitive purpose of the questions" being used "and explicitly encourages students to start asking themselves these same questions. As students take responsibility for doing this, the teacher then fades the scaffolding" (Goos, Stillman & Vale, p. 54). The levels of scaffolding provided and how long these are sustained are dependent on the level of schooling of the students undertaking the activity, the abilities of individuals (Diezmann & Watters, 2002) and the previous experience of the students with challenging tasks.

Another questioning technique called "focussing" (for examples see Herbel-Eisenmann & Breyfogle, 2005; Goos, Stillman & Vale, 2007, pp. 54-58) allows students to "articulate their thinking" (Goos, Stillman & Vale, p. 58) about the challenge. During classroom discussion of a challenge "the teacher asks *clarifying questions* and *restates aspects of the solution* to keep attention focused on the *discriminating aspects* of the particular student's solution. However, for this to be used effectively, the teacher must be able to see the essence of a mathematical task and, on a moment-by-moment basis, the essence of a task solution proferred by a student" (Goos, Stillman & Vale, p. 58). The guiding basis for group or class discussion is the lines of thought of the students, not the teacher (Doerr & English, 2006). Use of this technique effectively requires that teachers have both well developed PCK and SCK (specifically Mathematical Content Knowledge [MCK]) with respect to the use of challenges (see Chapter 6 and Leikin, 2006, for an explanation of these terms).

It is not unusual that students set challenges in the classroom, no matter what form they take, will "seek to reduce the task complexity by seeking specific input from the teacher" (Doerr & English, 2006, p. 9). Often, however, teachers using extended challenges find themselves under sustained pressure as many students simultaneously seek their help with different parts of the task. In this situation a technique observed in the RITEMATHS research project (see section 4.1.1 this chapter) has proved useful. One or two students who the teacher knows has expertise in the part of the task in question are announced to be "experts" for the other students to consult for a limited period of time. The student experts are only to be consulted in the same way as a student would ask for assistance from the teacher with their own solution. They are not meant to tell or impose their solution on the student asking for assistance. For example, a student might be using a spreadsheet for a numerical solution to a task and decide to graph the data but does not know how to do this. An announced spreadsheet expert can then be consulted instead of the teacher.

3.5 How can teachers introduce mathematical challenges into the regular classroom?

When challenging tasks are used with middle school students, teachers use a variety of methods to introduce the tasks to students. In section 4.2.1, Olga Medvedeva's approach when introducing abstract mathematical challenges in the regular classroom is outlined. Her approach is based on the ideas of Davydov (1972/1990) and is basically a guided deep analysis of an abstract mathematical task beginning with the heuristic, solve a simplier problem, in order to facilitate students' identification of the essential relationships and underlying structure of the problem to enable them to generalise to structurally similar problems.

When teachers use extended real world challenging tasks in lower secondary school (Years 8-10), the cognitive demand required for task formulation by middle school students is high potentially leading to a blockage in this early phase of the solution if students find the level of challenge too high for them to engage with the task. To help overcome this, teachers use a variety of methods to ensure students do not have difficulty interpreting the situation (Stillman & Brown, 2007). These include physical demonstrations often involving concrete props in which students participate or observe, writing activites such as stating the aim of the task or the goal they have to reach, debating, dynamic computer simulations, and scale drawing or other forms of diagram drawing by both students and teacher. These activities serve to bridge the enactive and iconic worlds as well as are a means of introducing some structure that reduces some of the cognitive load when there are no cues as to how to deal with the information in the situation presented.

4. Designing Research for Challenging Mathematics Classroom Practices

4.1 Fruitful Research Designs for Examining Challenges

In the following we will suggest and illustrate three types of classroom based research designs that we believe are fruitful for exploring and researching classroom practices related to the role of challenging tasks in everyday mathematics classrooms. These are design-based research, Japanese lesson study and teaching experiments conducted by teacher researchers. Each will now be elaborated and illustrated.

4.2 Design-based Research

Design-based research (Collins, Joseph, & Bielaczyc, 2004) where iterative cycles of design, implementation, evaluation, and refinement are used to improve educational practice has potential for researching classroom practices related to the use of challenges and is already being used for this purpose (e.g., Mason & Janzen Roth 2004, 2005, 2006; Stillman, 2006). The purpose of design experiment research as a form of educational research is to explore the qualities of student understandings and their development of further understanding as the development of instructional resources progresses through these cycles (Lobato, 2003). Researchers and teachers work collaboratively to test theories in everyday classroom settings. Both theory and practice inform the design phases and are informed by what transpires during each teaching experiment (Palinscar, 2005) as shown in Figure 10. This example is from an Australian research project where the final implementation cycle for a set of extended tasks developed by teachers and researchers in the RITEMATHS project (http://extranet.edfac.unimelb.edu.au/DSME/RITEMATHS/) is shown to be informed by a theory about the mediation of cognitive demand of the tasks by teacher actions as well as wisdom of practice documents prepared by teachers about previous experiences with the tasks and conditions for success identified by classroom observation in the previous two cycles. Also, during this final implementation it is indicated that data will be collected about student perspectives on practice. The project will be described more fully in the next section.

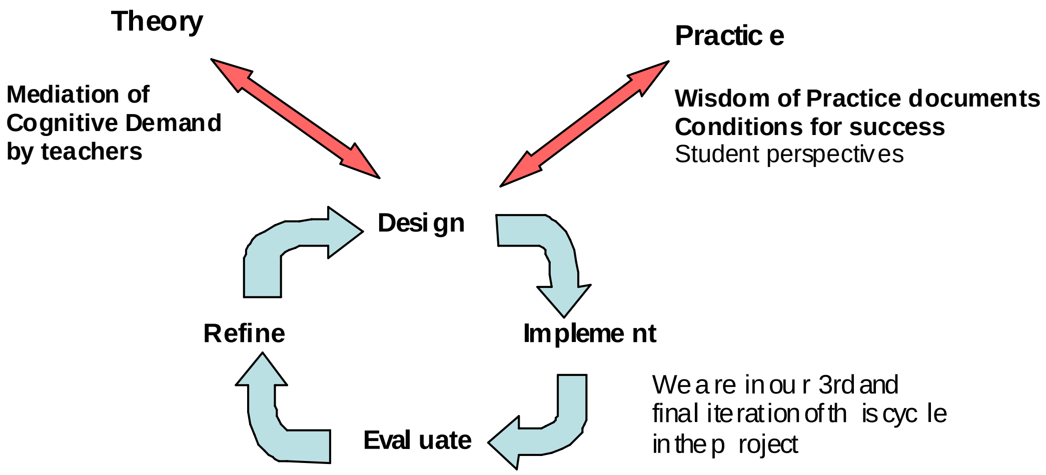


Figure 10. Design research cycle used for teacher/researcher meeting in the RITEMATHS Resaerch Project.

4.2.1 An Australian Example

As part of an Australian research project [1] how project teachers engineer learning environments in their classrooms to manage increased cognitive demand of lessons where task contexts involve real-world applications and how students negotiate this challenge are being investigated. The part of this project of relevance to this ICMI study is the study of classroom practices at Year 9 level associated with the use of extended challenging tasks by teachers.

Teachers at this level of schooling make use of challenges of this form in order to facilitate students' development of an integrated view of necessary competencies to approach challenging tasks, deepen knowledge of mathematical concepts and procedures they have already encountered in the classroom through application in novel and complex situations and the use of a wide variety of representations, sometimes simultaneoulsy in the same phase of the solution and in different phases of the solution. According to Kadijevick (2007), "the degree to which mathematical learning is successfully attained depends on the degree to which learner[s] can successfully cope with the coordination of different mathematical entities (competencies, activity, knowledge types, representations, etc)" (p. 7). It was not the intention of the teachers who were designing and using extended tasks in this project to focus on just one of these aspects at a time (a common failing of mathematics teachers according to Kadijevik, 2007).

The design and sequencing of extended investigative tasks so the cognitive demand matches students' needs at a particular stage in the development of their mathematical, technological, and investigative procedure knowledge were issues of interest to teachers in the project. At the beginning of the project it was hypothesised that management of

[1] RITEMATHS is a collaborative research project, funded by the Australian Research Council Linkage Scheme, involving the Universities of Melbourne and Ballarat, six schools and Texas Instruments as industry partners.

cognitive demand of teaching tasks in technology-rich teaching and learning environments is mediated through careful tuning by the teacher of the interplay between (a) task scaffolding, (b) task complexity, and (c) complexity of technology use (Stillman, Edwards, & Brown, 2004).

Task scaffolding is the degree of cognitive processing support provided by the task setter enabling task solvers to solve complex tasks beyond their capabilities if they depended on their cognitive resources alone. Task structure (e.g., carefully sequenced steps or a bald task statement), type of technology chosen (e.g., a real world interface tool such as a data logger or a mathematical analysis tool such as a calculator), and whether technological assistance rather than by-hand calculation is privileged, all contribute to task scaffolding. Whose choice it is to decide all of these also contributes to the level of task scaffolding. The *complexity* of a real world task can be characterised by identifying and assessing the level of those attributes of the task that contribute to its overall complexity. These are potentially numerous contributing via the mathematical, linguistic, intellectual, representational, conceptual, or contextual complexities of the task (Stillman & Galbraith, 2003). Overall task complexity also varies along a continuum from simple to complex with the latter presenting a challenge for many students. For a particular task, students focus on only a subset of attributes when assessing overall task complexity (Stillman & Galbraith, 2003) but these indicative cues contribute to their sense of challenge with the task.

One project school developed a lower secondary mathematics curriculum (Years 8–10) providing opportunities for engagement in extended investigation and problem solving tasks set in real-world contexts considered meaningful for students by the teachers. A major focus was in Year 9 (14-15 year olds). During the Year 9 program, in keeping with local curriculum requirements, students were introduced to a mathematical model being used to describe the relationship between variables in a real situation and then being used to predict an outcome in terms of a response variable when a control variable is altered. A series of extended real-world tasks designed by one teacher, and the implementation and refinement of these tasks was studied in depth over the life time of the project.

Adoption and implemenation by other classroom teachers with different motivations for the use of real-world tasks and/or electronic technologies in the lower secondary years so as to retain the integrity of the task is not guaranteed even if the design can be shown to be worthwhile. Some of the tasks from this first school were modified by members of the research team and teachers at other project schools where they were then implemented to fit the different conditions existing at that school.

One research question investigated during the project was: How can tasks be implemented in different contexts (e.g., shorter time frame and teachers and students used to more highly structured investigations) but the level of challenge and engagement retained?
As Blum and Leiß (2007) point out, there is a lack of research as well as knowledge amongst teachers "of appropriate ways for teachers to act when diagnosing students' solution processes and when intervening in cases of students' difficulties" (p. 223). In particular, they highlight a lack of knowledge of strategies for "independence-supporting"

interventions in demanding mathematical tasks (p. 230). The difficulty for teachers is to decide when it is necessary to intervene and the nature of that intervention. In several implementations of the tasks in the project it was observed that blockages to students' progress differed in type and cognitive demand (Stillman, Brown, & Galbraith, 2007). When blockages were induced by a lack of reflection on interim results or incorrect or incomplete knowledge, students were observed overcoming these blockages without teacher intervention when allowed to continue to struggle and resolve the situation themselves. Students appeared to do this by genuinely reflecting on their mental image of the problem and their approach. This reflection, sometimes stimulated by reflective questions built into the task booklet, included the potential to recognise and hence rectify the application of incorrect or incomplete knowledge. However, there were also instances of blockages observed where students engaged in cognitive dissonance which prevented them from activating procedures to unblock their progress. These students persisted in attempting to assimilate, rather than accommodate (Piaget, 1950) new contradictory information into their chosen structure for the task. In this instance successful teacher intervention that supported independent progression on the task involved the promotion of reflective learning where the teacher firstly tried to alter the students' current mental model through reflection and then the actions of the student. Thus, for example, rather than say a group's model was wrong, the teacher used the group's model to produce an incongruity that the students themselves were able to perceive before focussing attention on what actions might be employed to rectify the situation. Being able to recognise when students are facing just lower intensity blockages which they should be able to resolve themslves if they engage in genuine reflection (self-initiated or orchestrated by the teacher or task sheet), is a critical teaching competency when using challenging tasks in the classroom. Students do learn from resolving these situations themselves so by allowing them to persist rather than pre-empting when intervention is necessary seems a pre-requisite for task implementation that does not reduce the challenge intended in the task. Likewise, being able to recognise and intervene in a manner that promotes reflective learning when students are experiencing cognitive dissonance beyond that which they can resolve themselves will also ensure the level of challenge and engagement with such tasks is retained.

4.2.2 A Canadian Example—Can Students Think Like Archimedes?

Mason and Janzen Roth (e.g., 2004, 2005, 2006, 2007) have used three cycles of design experimentation using instructional resources and strategies that deliberately and aggressively attempt to reorient students' approaches to learning mathematics toward the challenge of developing conceptual understandings rather than the stockpiling of additional procedures. This teaching program uses challenging mathematical activities (e.g., The Tennis Ball Problem, Section 1, this chapter) drawn from an academic study of the history of mathematics to accomplish its educational goals.

The curriculum design for these teaching experiments began with a study of the history of mathematicians' inquiries into circle relationships, especially those of Archimedes (Cuomo, 2000; Eves,1960) and early Chinese mathematicians (Lui, 2003). History of mathematics provides a context for presenting a narrative of mathematics as an ongoing process developing through thoughtful effort our communal mathematical understandings (Arons, 1988; Mason, 1999). Historical framings of mathematics provide a way to put a human face on the mathematics that students encounter, offering context and story-lines to enliven the content and role/process models for students to view as examples (positive and negative) for their own mathematical efforts (Mason, 2003; Stinner & Williams, 1998). Although individuals' understandings do not necessarily follow the historical order in which mathematics developed, the historical structure of the discipline offers a framework within which educators can think about the educational sequence and structure of topics (Mason, 2001; Rudge & Howe, 2004). The history of mathematics gives us the mathematical version of the inquiry processes behind the content. Design experiment research may give us the mechanism, over time, to develop instructional versions of those processes which preserve the spirit, challenge, and intrinsic rewards of the mathematician's original inquiries.

The research has now completed three full cycles of design, implementation and redesign. The first cycle of curriculum development incorporated the responses of mathematicians and educators to instructional activities attempting to represent the cognition of Archimedes that is summarised by the pi-based circle formulae all students learn to use. It is difficult to reconstruct the cognition of ancient mathematicians in producing their results in mathematics as often just the result with justification, not the thinking that produced it, is all that is recorded in surviving treatises. However, in the case of Archimedes, a copy of a letter written by himself entitled, *The Method*, and preserved on the surface of a palimpsest was found in 1906. This discovery and technological advances since its restoration to the scrutiny of academics in 2001 provide us with an insight into his thinking about the inquiry processes he carried out in investigating the relationships among the measures of circles, not just the final result (Hoffman, 1988; Netz & Noel, 2007). "Specifically, in the case of Archimedes' work with circles and with pi, *The Method* shows his thinking to be: a) Geometric: Inscribing, circumscribing; b) Empirical: Specific examples, specific quantities; c) Algebraic: General quantities, relationships; and d) Conceptual: extending sequences to infinity (early calculus)" (Martin & Janzen Roth, 2005, p. 2). For the students who were to be the audience for the curriculum activities designed by Martin and Janzen Roth, the qualities of the thinking of Archimedes that were desired were: "a) Tangible – practical, tactile, visible; b) Exploratory – questing, noticing, connecting; and c) Thoughtful – abstracting, wondering, generalizing" (Martin & Janzen Roth, 2005, p. 2). In the second cycle (see Mason & Janzen Roth, 2005), academic grade 9 students interacted with a prototype unit of instruction, including a sequence of 6 guided instruction student inquiries built around historical vignettes. In the final cycle (see Mason & Janzen Roth, 2006, 2007), the unit was adapted to challenge the understandings of the nature of mathematics held by a group of academic Year 12 students.

Each cycle has provided opportunities to better understand how to engage students in the challenge of deep understanding of algebraic formulae. First and foremost, students hold a wide range of beliefs and values about the nature of academic mathematics and its rightful place in school. Some held an *instrumentalist view of mathematics* (Ernest, 1989; cf traditional view, Dionne, 1984) as a set of ideas and formulae to be learned for use in applications and in further study. Others held a *conceptual view of mathematics* as a collection of ideas with internal and interconnected coherence (cf Platonist view, Ernest, 1989; formalist view, Dionne, 1984). Some saw mathematics as a *field of present-tense inquiry* in which they could participate through problem-solving and inquiry in the kinds of thinking that are part of our culture and history (cf problem solving view, Ernest, 1989; constructivist view, Dionne, 1984); others saw mathematics as a *collection of artefacts from past inquiries* to be apprehended and remembered. When probed, students portrayed their beliefs as deeply embedded in their lived histories as learners of mathematics, products of their personal experiences and their social environments.

Students' initial understandings of the functions and relationships that the circle formulae summarise (for mathematicians) were disappointingly shallow. Yet, the shallowness is clearly remediable, through engaging students in challenging mathematical inquiries related to those relationships (see Mason & Janzen Roth, 2005, 2006 for details).

It is known that "beliefs have a powerful impact on our thinking and action, and they work for the rationality of our decisions. Thus to know student's beliefs is vitally important" for those of us attempting to bring change through curriculum development (Furinghetti & Pehkonen, 2000, p. 23). It is thus crucially important to find that students of all orientations towards mathematics, including instrumentally orientated students, accepted the challenge presented to them and were open to perceiving mathematics as Mason and Janzen-Roth's historically grounded curriculum presents it, as a complex human enterprise available to their abilities.

4.2 Japanese Lesson Study

Japanese Lesson Study (Fernandez & Yoshida, 2004; Isoda, Stephens, Ohara & Miyakawa, 2007) "refers to a process in which teachers progressively strive to improve their teaching methods by working with other teachers to examine and critique one another's teaching techniques. ...[It] functions as a means of enabling teachers to develop and study their own teaching practices" (Baba, 2007, p. 2). It appears to be an ideal method to ensure classroom practices related to using challenges in mathematics classrooms can be improved in the long-term by generating, accumulating, sharing "practitioner knowledge ... within a system [that ensures the transformation of] such knowledge into a professional knowledge base" (Hiebert, Gallimore, & Stigler, 2002, p. 10). Indeed, Isoda (2007) in sketching a brief history of lesson study in mathematics education in Japan, points out that it was the vehicle "for the emergence of teaching methods that focus on problem-solving, which today are globally recognised as models of constructivist teaching" (pp. 13-14). This has led to "the problem-solving approach [becoming] well known as a major way of teaching mathematics in Japan" (p. 14). Although lesson study methodology for research and professional development has spread to other countries, what seems to be particularly characteristic of Japanese lesson study which would be critical if this methodology is to be used for further study and dissemination of classroom practices related to the use of challenges in mathematics classrooms, is the pivotal role played by university researchers and supervisors whose research intersts, in this instance, would be such classroom practices. These "university researchers are expected to have accumulated deep knowledge of teaching practice … so that they can provide constructive and well informed comments on lessons observed and the ensuing discussions" (Stephens & Isoda, 2007, p. xx).

4.3 Teaching experiments or Teaching-Research

Teaching experiments where the researcher is also the teacher and the primary purpose is to improve the teaching in particular classrooms where the research is being conducted have proved a valuable source of insight into practices related to the use of mathematical challenges in the classroom. This type of teaching-research has its roots in action research and some of the promises and challenges of the design are discussed in Czarnocha and Prabhu (2004).

4.3.1 A North American Example. Sriraman conducted several teaching experiments (Sriraman, 2002, 2003a, 2003b, 2004a, 2004b, 2006) when he was the class teacher at a rural mid-western high school in the United States of America with a heterogeneous group of Year 9 students enrolled in a beginning algebra course. The goal of these teaching experiments was to offer mathematical challenges not provided by the regular curriculum and to study how students abstract and generalise. In these teaching experiments students were given series of combinatorics problems which they were to work on independently in their journals over an extended period of time.

In the first teaching experiment (Sriraman, 2004b; Sriraman & English, 2004) students worked on problems over a four month period that included four Steiner-triple arrangement problems. The problems were framed in the context of recreational arrangement problems such as inviting people over for dinner, schoolchildren on a walk and prisoners chained in triplets (see Gardner, 1997 for examples). A Steiner triple system is an arrangement of *n* objects in triplets such that every pair of objects appears in a triplet exactly once. The students worked on the problems independent of other students and explicit instruction from the teacher. More than 50% of the students were able to devise strategies that required a high level of abstraction and systematisation to count all possible arragements.

In the second teaching experiment (Sriraman, 2002, 2003a, 2004a, 2004c; Sriraman & English, 2004) students worked on a series of five problems of increasing complexity assigned every second week over a three month period. The problems were all based on the Pigeonhole Principle which states that: If *m* pigeons are put into *m* pigeonholes, there is an empty hole if, and only if, there is a hole with more than one pigeon. The principle is believed to have first been made by Dirichlet in 1834 under the name Schubfachprinzip (i.e., "drawer principle" or "shelf principle") (see Chapter 6 for the Dirichlet Principle). Almost 50% of students were able to use the Pigenhole principle intuitively, by focusing "on understanding the structure of a given problem, in addition to engaging in reflective abstration" (Sriraman & English, 2004, p. 184; Sriraman, 2004c). However, all students engaged in thinking mathematically, that is, constructing mathematical representations, reasoning, abstraction and generalisation through the use of these problems.

A third teaching experiment (Sriraman, 2003b, 2006) focussed on Diophantine *n*-tuples. Students were introduced to elementary Diophantine equations as recreational journal problems. The problem chosen for investigation was the classic *n*-tuple Diophantine problem supposedly posed by Diophantus himself for solutions in the rationals. A Diophantine *n*-tuple is a set of *n* positive integers such that the product of any two is one less than a square integer. It was hoped that a very elementary version of the problem would kindle student interest and eventually result in an attempt to tackle the as yet unsolved 5-tuple problem in integers: Does there exist a diophantine 5-tuple? The author initiated this problem by simply mentioning in class the 3-tuple problem, if one considers the integers 1, 3, and 8, then it is always the case that the product of any two is always one less than a perfect square. Indeed $1\times3 = 2\times2 - 1$; $1\times8 = 3\times3 - 1$; and $3 \times 8 = 5\times5 - 1$. This remark led students to wonder if other such 3-tuples existed? This problem was then assigned as a recreational journal problem. Students in the class found different 3-tuples, which led to the following questions naturally: What is the pattern for 3-tuples? Are there 4-tuples? These questions were the catalyst for an investigation of the unsolved 5-tuple problem over the course of the school year. In this experiment students started working on problems independently but once they had received written feedback in their journals about their solution they were allowed to work with others. In one lesson each week there was a time for presenting solutions and defending strategies. Many students were surprised at the difficulty of solving these seemingly easy problems as they began writing algorithms and computer programs to check for integer solutions. The progress of the problem depended completely on the "will" of the students. Mathematical notation was created as a class only after every student had expressed the particular idea in their own words. It was crucial that students initiated the process of conjecture, proof and refutation out of their need to resolve the unimagined difficulties that arose from a seemingly easy problem. All students in this class willingly engaged in trying to solve one of the unresolved conjectures of our time over a seven month time period through the process of conjecture, proof and refutation. The mathematics created by the students in trying to solve the classic 5-tuple Diophantine problem clearly indicates that students are capable of original thought that goes beyond mimicry and application of procedures taught in the classroom. The students' proof did not resolve the 5-tuple problem by any means but thesescfourteen-year old students persisted over an extended time period in trying to solve this challenging problem.

Sriraman (2006) concluded that the use of journals to nurture the process of conjecture-proof-refutation was invaluable to the teacher as it was in the other teaching experiments. It allows for constant communication between the individual student and the teacher, and allows room for reluctant students to express themselves. Journals also allow the teacher insight into the affective drives of the students, as well as their capacity for originality and creativity. Journal problems also allow for extended investigations that are student driven and convey that mathematics is an evolving process of discovery leading to generalities that are either proved or disproved. In Sriraman's experience, "journal problems of varying levels of difficulty, which are characterized by an overarching mathematical generality, is a novel and non-intrusive way of differentiating the curriculum for all students, and not simply for the able students" (2006, p. 7).

4.3.2 A Russian Example. The work of Olga Medvedeva in researching her own classroom is described in Sriraman and English (2004). Her teaching approach when using challenging problems in the classroom is based on an implementation of Davydov's ideas (1972/1990) about different types of generalisation in instruction. The approach is illustrated using a combinatorial problem, The Walking Problem.

The Walking Problem (Medvedeva, 2002, cited in Sriraman and English, 2004): Consider the problem of walking in a 6 x 4 "rectangular city". In how many possible ways can a person move from a point A to a point X travelling only up and right along the edge of each grid?

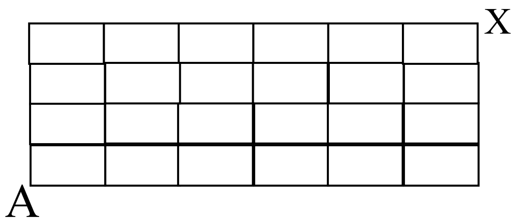


Figure 11. 6 x 4 "rectangular city".

Firstly, the students would be asked to work on a simplier problem such as the finding of a path in a smaller "rectangular city", say 2 x 3. Secondly, in order to facilitate abstraction, several students are then asked to read aloud the directions for their path using the words "right" and "up". These paths are then represented using the leters R and U and students write several such strings for different paths they discover. The idea is that the teacher will help students associate path length with string length. Thirdly, the problem of finding all possible paths in the smaller rectangular city can be restated in a generalised form as a combinatorial problem such as determining all possible five-letter strings with 3 R's and 2 U's that represent a valid path from A to X. Fourthly, students then predict and enumerate paths for other dimensions of the "rectangular city". Finally, students are encouraged to conjecture and test a formula for an n x m city through specialising (i.e., using specific cases). According to Davydov, "scientific knowledge ... requires the cultivation of particular means of abstracting, a particular analysis, and generalisation, which permits the internal connections of things, their essence, and particular ways of idealising the objects of cognition to be established" (1972/1990, p. 86). "The essence of a thing is none other than the basis (included in itself) for all of the changes that occur with it in interaction with other things" (Rubinstein, cited by Davydov, 1972/1990, p. 194). Sriraman and English (2004) add that a further step in keeping with this notion of establishing the essence of the task could be that students be asked to pose problems that extend or are of a similar structure to the given problem. Thus, what Medvedeva is suggesting for teachers who follow this type of classroom practice with students is the use of challenging problems to foster "theoretical abstraction" in everyday mathematics classrooms (Mitchelmore & White, 2007).

5. Conclusion

Classroom practice issues related to teachers providing mathematical challenges in their regular classrooms have been addressed in this chapter. The regular use of challenges for all students in the everyday classroom is advocated. Design features of challenging tasks and how these might be varied for different levels of schooling and degrees of contextualisation are addressed. Whilst it is pointed out that the intended curricula in most countries have not been exhausted as sources of challenges, combinatorics and number theory are suggested as fertile topics to explore for more challenges. Technology is suggested as a means of mediating the cognitive demand of some challenging tasks.

Questioning which students ask of themseleves during the solution of a challenging task and the questioning engaged in by teachers in interacting with individuals, groups or the whole class during problem solution are promoted as the means by which students and teachers effectively manage the demand of mathematical challenges. It appears that the ideal of textbooks being written from the perspective of challenging activities being the motivation rather than an add-in is still a long way from being realised. However, it is pointed out that using challenging tasks in the classroom does not mean the experience is necessarily challenging for students as it depends on a number of factors including how they are implemented in the classroom as such tasks are by no means "teacher proof". In the final section, it is suggested that design based research, Japanese lesson study and teaching experiments conducted by teacher researchers on their own classrooms are potentilally fruitful research designs for the study of classroom practice issues related to the use of challenging tasks in the mathematics classroom. A number of case studies using these research designs have been included showing how these research designs has proved fruitful in practice investigating such issues as the type of teacher interventions to be used when students are blocked in their progress so as not to remove the challenge, and how topics outside the normal curriclulm can be used to promote student abstration and generalisation through challenges worked mainly independently.

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