Aesthetics and Creativity:

An exploration of the relationships between the constructs

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Abstract: *In this contribution, we report on an ongoing study that examines the relationship between aesthetics and creativity among working mathematicians. Writings of eminent individuals indicate that aesthetics is an important component of mathematical creativity, however we were interested in researching this relationship among the normal working mathematician. Anecdotally speaking, many working mathematicians often convey a reciprocal relationship between aesthetics and creativity, particularly when mathematical results and proofs are arrived at with considerable strain and stamina. We report on the findings of our ongoing study among working mathematicians in the U.S.A and Germany, and make a case for emphasizing the aesthetic dimension in mathematics education.*

1. Introduction

In the general literature one finds several reports concerning the use of aesthetics as a guide when formulating a scientific theory, or selecting ideas for mathematical proofs.

The first who introduced mathematical beauty as well as simplicity as criteria for a physical theory was Copernicus (Chandrasekhar 1973, p. 30). Since then, these criteria have continued to play an extremely important role in developing scientific theories (Chandrasekhar 1973, p. 30; Chandrasekhar 1979, 1987). This is especially so for truly, creative work that seems to be guided by aesthetic feeling rather than by any explicit intellectual process (Ghiselin 1952, p. 20). Dirac (1977, p. 136), for example, tells about Schrödinger and himself:

> It was a sort of act of faith with us that any questions which describe fundamental laws of nature must have great mathematical beauty in them. It was a very profitable religion to hold and can be considered as the basis of much of our success.

Van der Waerden (1953) reports that Poincaré and Hadamard pointed out the role of aesthetic feeling when choosing fruitful combinations in a mathematical solution process. More precisely, Poincaré asked how the unconscious could find the “right” or fruitful, combinations among the many possible ones. He gave the answer: “by the sense of beauty, we prefer those combinations

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that we like" (van der Waerden 1953, p. 129; see also Poincaré 1956, p. 2047-2048). A similar statement is also given by Weyl (Ebeling, Freund and Schweitzer 1998, p. 209), who points out:

> My work has always tried to unit the true with the beautiful and when I had to choose one or the other I usually chose the beautiful.

Thus theories, that have been described as extremely beautiful, as for example the general theory of relativity, have been compared to a work of art (Chandrasekhar 1987); Feyerabend (1984) even considers science as being a certain form of art.

Mathematics and mathematical thought seem to be directed towards beauty as one profound characteristic. Papert and Poincaré even believe that aesthetics play the most central role in the process of mathematical thinking (see e. g. Dreyfus and Eisenberg 1986, p. 2; Hofstadter 1979). Burton (2004, p. 88) points out that most of the mathematicians involved in a recent study rated highly the importance of aesthetics in their work. Nevertheless, this point of view is in general rarely considered. Davis and Hersh state (1981, p. 169):

> Blindness to the aesthetic element in mathematics is widespread and can account for a feeling that mathematics is dry as dust, as exciting as a telephone book, as remote as the laws of infangthief of fifteenth century Scotland. Contrariwise, appreciation of this element makes the subject live in a wonderful manner and burn as no other creation of the human mind seems to do.

Although aesthetics seem to play a crucial role in creative mathematical processes, there is a limited body of research on the meaning of aesthetics in the domain of mathematics. On the other hand, there is a substantial body of research on the nature of mathematical creativity.

Our question is what is the relationship between the two constructs? Before making this more precise (section 3), we go on with clarifying the concepts we focus on: general and mathematical creativity, and aesthetics.

2. Aesthetics and creativity: meaning of the constructs

The concept of creativity is well-defined in literature. Creativity is a paradoxical construct to study because in many ways it is self defining. In other words, we are able to engage or judge acts of everyday creativity such as improvising on a recipe (Craft, 2002), use a tool in a way it wasn't intended, or intuit emotions and intended meanings from gestures and body language in day-to-day communication (Sriraman, 2009). Children are particularly adept at engaging in creative acts such as imaginary role playing or using toys and other objects in imaginative ways. "Aha!" experiences occur not only in individuals working on scientific problems but also in day-to-day problems such as realizing a person's name or relational identity after having forgotten it. However it is important to distinguish between everyday creativity and domain specific or paradigm shifting creativity. Domain specific creativity or "extraordinary creativity" causes paradigm shifts in a specific body of knowledge and it is generally accepted within that works of "extraordinary creativity" can be judged only by experts within a specific domain of knowledge. Some researchers have described creativity as a natural "survival" or "adaptive" response of humans in an ever-changing environment.

The Handbook of Creativity (Sternberg, 2000) which contains a comprehensive review of all research available in the field of creativity suggests that most the approaches used in the study of creativity can be subsumed under six categories: mystical, pragmatic, psychodynamic, psychometric, social-personality and cognitive. The mystical approach to studying creativity suggests that creativity is the result of divine inspiration, or is a spiritual process. In the history of mathematics, Blaise Pascal claimed that many of his mathematical insights came directly from God. This is somewhat analogous to the ancient Greeks belief in muses as a source of inspiration for artistic works. The pragmatic approach is focused on developing creativity. For instance, George Polya's emphasis on the use of a variety of heuristics for solving mathematical problems of varying complexity is an example of a pragmatic approach. The psychodynamic approach to studying creativity is based on the gestaltist idea that creativity arises from the tension between conscious reality and unconscious drives as popularized by Jacques Hadamard who constructed case studies of eminent creators such as Albert Einstein. The psychometric approach to studying creativity entails quantifying the notion of creativity with the aid of paper and pencil tasks such as the Torrance Tests of Creative Thinking developed by Paul Torrance. These tests are used by many gifted programs in middle and high schools, to identify students that are gifted/creative and show traits of divergent thinking. The test is scored for fluency, flexibility, and the statistical rarity of a response. Some researchers also call for use of more significant productions such as writing samples, drawings, etc to be subjectively evaluated by a panel of experts instead of simply relying on a numerical measure. The social-personality approach to studying creativity focuses on personality and motivational variables as well as the socio-cultural environment as sources of creativity. Finally the cognitive approach to the study of creativity focuses on understanding the mental processes that generate new and novel ideas. Most of the contemporary literature on creativity suggests that creativity is the result of confluence of factors from the six aforementioned categories. Two of the most commonly cited confluence approaches to the study of creativity are the "systems approach" of Mihaly Csikszentmihalyi; and "the case study as evolving systems approach" of Doris Wallace and Howard Gruber (see Csikszentmihalyi, 1992; Wallace & Gruber, 1992)

The systems approach takes into account the social and cultural dimensions of creativity, instead of simply viewing creativity as an individualistic psychological process and studies the interaction between the individual, domain and field. The field consists of people who have influence over a domain. For example, editors of research journals would have influence on any given domain. The domain is defined a cultural organism that preserves and transmits creative products to other individuals in the field. Thus creativity occurs when an individual makes a change in a given domain, and this change is transmitted through time. The personal background of individuals and their position in a domain naturally influence the likelihood of their contribution. It is no coincidence that in the history of science, there are significant contributions from clergymen such as Pascal, Copernicus and Mendel, to name a few, because they had the means and the leisure to "think". Csikszentmihalyi (1992) argues that novel ideas that result in significant changes are unlikely to be adopted unless they are sanctioned by a group of experts that decide what gets included in the domain. In contrast to Csikszentmihalyi's argument that calls for focus on communities in which creativity manifests, "the case study as evolving systems approach" treats each individual as a unique, evolving system of creativity and ideas, where each individual's creative work is studied on its own (Wallace & Gruber, 1992). The case study as an evolving system has the following components to it. First, it views creative work as multi-faceted. So, in constructing a case study of a creative work, one has to distill out the facets that are relevant and construct the case study based on the chosen facets. These facets are: uniqueness of the work, epitome (a narrative of what the creator achieved), systems of belief (an account of the creator's belief s system), modality (whether the work is a result of visual, auditory or kinesthetic processes), multiple time-scales (construct the time-scales involved in the production of the creative work), dynamic features of the work (documenting other problems that were worked on simultaneously by the creator), problem-solving, contextual frame (family, schooling, teachers influences), and values (the creator's value system).

Cultural and social aspects play a significant role in what the community, in general, and the school system, in particular, considers as "creativity" and how they deal with it. Numerous studies indicate that the behavioral traits of creative individuals very often go against the grain of acceptable behavior in the institutionalized school setting. For instance, negative behavioral traits such as indifference to class rules, display of boredom, cynicism or hyperactivity usually result in disciplinary measures as opposed to appropriate affective interventions. In the case of gifted students who 'conform' to the norm these students are often prone to hide their intellectual capacity for social reasons, and identify their academic talent as being a source of envy. History is peppered with numerous examples of creative individuals described as "deviants' by the status quo. Even at the secondary and tertiary levels there have been criticisms about the excessive amount of structure imposed on disciplines by academics as well as Euro-centric attitudes and male epistemology centered attitudes towards knowledge generation. Such a criticism particularly resonates in the world of science and mathematics, especially during elementary and secondary schooling experiences level, where minority, ethic minorities, first nation and female gifted/creative students are marginalized by practices that are alien to their own cultures (Sriraman, 2009).

Based on extensive classroom based research and informed by findings from the field of psychology and the history of science, five pedagogical principles to maximize general creativity in the classroom have been posited by Sriraman & Dahl (2009). The five principles are: (a) the Gestalt principle, (b) the Aesthetic principle, (c) the free market principle, (d) the scholarly principle, and (e) the uncertainty principle.

The Gestalt principle: Although psychologists have criticized the Gestalt model of creativity because it attributes a large "unknown" part of creativity to unconscious drives during incubation, numerous studies with scientists and mathematicians have consistently validated this model. In all these studies after one has worked on a problem for a considerable time (preparation) without making a breakthrough, one puts the problem aside and other interests occupy the mind. Jacques Hadamard put forth two hypotheses regarding the incubation phase: (1) The 'rest-hypothesis' holds that a fresh brain in a new state of mind makes illumination possible. (2) The 'forgetting-hypothesis' states that the incubation phase gets rid of false leads and makes it possible to approach the problem with an open mind. The Soviet psychologist Krutetskii explained that the experienced of sudden inspiration is the result of previous protracted thinking, of previously acquired experience, skills, and knowledge the person amassed earlier. This period of incubation eventually leads to an insight on the problem, to the "Eureka" or the "Aha!" moment of illumination. Most of us have experienced this magical moment. Yet the value of this archaic Gestalt construct is ignored in the classroom. This implies that it is important that teachers encourage the gifted to engage in suitably challenging problems over a protracted time period thereby creating the opportunities for the discovery of an insight and to experience the euphoria of the “Aha!” moment.

The Aesthetic principle: Many eminent creators have often reported the aesthetic appeal of creating a “beautiful” idea that ties together seemingly disparate ideas, combines ideas from different areas of knowlege or utilizes an atypical artistic technique. In mathematics, Georg Cantor’s argument about the uncountability of the set of real numbers is an often quoted example of a brilliant and atypical counting technique

The Free market principle: Scientists in an academic setting take a huge risk when they announce a new theory or medical break through or proof to a long standing unsolved problem. The implication for the classroom is that teachers should encourage students to take risks. In particular they should encourage the gifted/creative students to pursue and present their solutions to contest or open problems at appropriate regional and state math student meetings, allowing them to gain experience at defending their ideas upon scrutiny from their peers.

The scholarly principle: One should embrace the idea of “creative deviance” as contributing to the body of knowledge, and they should be flexible and open to alternative student approaches to problems. In addition, they should nurture a classroom environment in which students are encouraged to debate and question the validity of both the teachers’ as well as other students’ approaches to problems. Gifted students should also be encouraged to generalize the problem and/or the solution as well as pose a class of analogous problems in other contexts. Allowing students problem posing opportunities and understanding of problem design helps them to differentiate good problems from poor, and solvable from non-solvable problems. In addition, independent thinking can be cultivated by offering students the opportunity to explore problem situations without any explicit instruction.

The Uncertainty Principle: Real world problems are full of uncertainty and ambiguity as indicated in our analysis so far. Creating, as opposed to learning, requires that students be exposed to the uncertainty as well as the difficulty of creating original ideas in science, mathematics, and other disciplines. This ability requires the teacher to provide affective support to students who experience frustration over being unable to solve a difficult problem. Students should periodically be exposed to ideas from the history of mathematics and science that evolved over centuries and took the efforts of generations of artists, scientists and mathematicians to finally solve. At the secondary school levels, one normally does not expect works of extraordinary creativity, however the literature indicates that it is certainly feasible for students to offer new insights into a existing/current scientific problems or a new interpretation or commentary to a literary, artistic or historical work.

While creativity, particularly mathematical creativity can be somewhat described in an objective way, feelings of aesthetics are very subjective. The subjectively experienced feelings of mathematical beauty are not so easily to be described. For this we have to bring out the characteristics of mathematical aesthetics. What does it mean, for example, that a theorem, a proof, a problem, a solution of a problem (the process leading up to a solution, as well as the finished solution), a geometric figure, or a geometric construction is beautiful?

Although assessments about beauty are very personal, there is a far-reaching agreement among scholars as to what arguments are beautiful (Dirac 1977). Thus it makes a sense to search for factors contributing to aesthetic appeal. Before starting on this journey, Hofstadter (1979, p. 555) sounds a note of warning when suggesting, that it is impossible to define the aesthetics of a mathematical argument or structure in an inclusive or exclusive way:

> There exists no set of rules which delineates what it is that makes a peace beautiful, nor could there ever exist such a set of rules.

However we can find in the literature several indications of criteria determining the aesthetic rating (see also Brinkmann 2000, 2004a, 2004b, 2006).

The Pythagoreans took the view that beauty grows out of the mathematical *structure*, found in the mathematical *relationships* that bring together what are initially quite independent parts in such a way to form a unitary whole (Heisenberg 1985). Chandrasekhar (1979) names as aesthetic criteria for theories their display of "*a proper conformity of the parts to one another and to the whole*" while still showing "some *strangeness* in their proportion". Weyl (1952, p. 11) states that beauty is closely connected with *symmetry,* and Stewart (1998, p. 91) points out that *imperfect symmetry* is often even more beautiful than exact mathematical symmetry, as our mind loves surprise. Davis and Hersh (1981, p. 172) take the view that:

> A sense of strong personal aesthetic delight derives from the phenomenon that can be termed *order out of chaos*.

And they add:

> To some extent the whole object of mathematics is to create order where previously chaos seemed to reign, to extract structure and invariance from the midst of disarray and turmoil.

Beutelspacher (2003, p. 86) points out, that mathematical beauty is first *simplicity*; a mathematical description should be *brief* and *concise*. Whitcombe (1988) lists as aesthetic elements a number of vague concepts as: *structure, form, relations, visualisation, economy, simplicity, elegance*, *order*. Dreyfus and Eisenberg (1986) state, according to a study they carried out, that *simplicity, conciseness* and *clarity* of an argument are the principle factors that contribute to the aesthetic value of mathematical thought. Further relevant aspects they name are: *structure, power, cleverness* and *surprise*.

Cuoco, Goldenberg and Mark (1995, p. 183) take the view that the beauty of mathematics lies largely in the *interrelatedness of its ideas*. Ebeling, Freund and Schweitzer (1998, p. 230) point out, that the beautiful is as a rule connected with *complexity*; complexity is necessary, even though not sufficient, for aesthetics. In this context, the degree of complexity plays a crucial role, as e. g. a study with students carried out by Brinkmann (2004a) indicates: the permissible degree of complexity for a beautiful problem depends on the mathematical ability of each individual.

Complexity and simplicity are both named as principal factors for aesthetics: how do these notions fit together? If simplicity is named, it is mainly the simplicity of a solution of a complex problem, the simplicity of a proof to a theorem describing complex relationships, or the simplicity of representations of complex structures. It looks as if simplicity has to be combined in this way with complexity, in order to bring out aesthetic feelings (Brinkmann 2000).

We have to consider that all the quoted criteria for aesthetics are given by qualitative characteristics[3], and hence by their nature they are fuzzy quantities. Thus aesthetic considerations will depend on individual judgements.

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[3] Birkhoff (1956) made an attempt to quantifying aesthetics in a general way, but his proposal seems not to be very convincing.

3. Research questions

The focus of our interest was to explore the relationship of aesthetics and creativity in the field of mathematics. In other words, we first asked:

I. Do creative mathematicians necessarily also experience the feeling of mathematical beauty?

II. Is creativity a necessary condition for the aesthetic appeal?

III. Is an aesthetic appeal necessary for creative work?

As for question III it is clear, that the aesthetic appeal is not the only reason why researchers do mathematics.

Based on experience, we could definitively answer the second question by “no”. Two examples may illustrate this.

Example 1: Learning mathematics in school is not necessarily done by creative processes. Students could, for example, just be informed about some mathematical contents and feel that they are beautiful. In a school lesson, e. g., there was shown a film on fractals to the students. There was no creativity demanded, the students just looked at the film and were fascinated!

Example 2: If mathematicians read an article in a mathematics journal they might experience some kind of mathematical beauty, without being involved in a creative process.

Thus we extended the second question by asking:

II.’ In which situations may mathematical beauty be experienced?

Further we asked, in more detail:

IV. In which way success/failure in a creative problem solving process influences the former aesthetic appeal of the problem?

Inspired by (mostly historical) literature (see section 1), we wanted also to know,

V. if contemporary creative mathematicians deal the point of view, that beauty serves as an orientation in the work of mathematicians, and

VI. if the aesthetic appeal for mathematics can be compared with that one for arts and music, and if arts inspired some creative endeavours in mathematics.

Additionally, we were interested

VII. if there is a hierarchy to the aesthetic appeal of mathematics and if so, what the relationship between this and the degree of corresponding creative work is.

4. Methodology

The aesthetic appeal of a mathematical object is not a characteristic possessed by this object but dependent on personal feelings. Thus we had to find out the relationship of aesthetics and creativity as it is personal experienced by contemporary mathematicians. For this, interviews with creative academics are a suitable means.

For reasons of practicability we decided to develop a questionnaire that should be sent and responded by e-mail. We addressed mathematicians in USA and Germany, known for notable works in different areas of mathematics. It was important for us to explore the statements of mathematicians working in the very different fields of mathematics in order not to get a one-sided view.

The questionnaire we used is shown in figure 1. In Germany, mathematicians involved in the study received the questionnaire both in English and in German language and they wrote their answers in German. Thus semantic misunderstandings due to language problems were excluded to the extent possible.

Questionnaire

1. What do you feel is extremely beautiful / beautiful? Please give some examples. Can you order these examples according to their aesthetic appeal on you?
2. Did you experience some mathematical objects as beautiful? Which ones? Can you compare those feelings of mathematical beauty with the aesthetic appeal of the examples you gave above?
3. In which situations did you experience mathematical beauty? Were you involved in a creative process?
4. Do you see any link between the degree of creativity in your work and the degree of aesthetic appeal of the mathematics you dealt with? If so/If not please explain.
5. Can you compare/contrast your aesthetic appeal for the subject as you transitioned from a learner to a researcher of mathematics?
6. Can you describe a learning or a teaching moment when you experienced a sense of beauty for the subject matter? If so did it result in pursuit of research (in other words in a creative process)?
7. Was every creative work you did in mathematics connected with aesthetic feelings? Is(Was) an aesthetic appeal of the mathematics you deal(t) with necessary for your creative work?
8. Does/Did the aesthetic appeal of mathematics influence the choice of research questions?
9. Was your creative work lead by aesthetic feelings? Did aesthetic feelings play a role in your “choice”/route of finding a solution? When multiple solutions/solution paths were available did aesthetic feelings play a role for your choice of one solution over the others?
10. Dirac wrote about Schrödinger and himself: "It was a sort of act of faith with us that any questions which describe fundamental laws of nature must have great mathematical beauty in them." Can you agree with the similar statement: "It was a sort of act of faith in my work that any fundamental mathematical truth must have great beauty in it."
11. Does success/failure in a creative problem solving process influence the former aesthetic appeal of the problem? In other words, if the pursuit of a problem for an

extended period of time does/did not result in an insight, does/did it diminish the aesthetic appeal in some way? On the other hand if the pursuit resulted in fruition...does/did it enhance or deepen the aesthetic appeal? Does success enhance the appeal of the problem that way, that it leads you to other related problems?

12. Can you compare your aesthetic appeal for the arts and music to your aesthetic appeal for mathematics? Have the arts inspired you in your creative endeavors in mathematics?

Figure 1: Questionnaire used in the study

The questions 1, 2, 4 and 5 in the questionnaire refer to question VII in section 3. We expected that it might be difficult to classify aesthetics according to degree of appeal. The idea was, to use metaphors in the sense of comparing the aesthetical appeal of a mathematical object with that one of a nonmathematical object were the characteristic of beauty is more familiar ("This mathematical object is as beautiful as …"). Of course, the aesthetic rating of nonmathematical objects is also individually different. The first questions of the questionnaire is explained in order to explore individual levels of feelings of beauty (in every day life), and thus to get a sort of scale for aesthetic feelings. Question 2 in the questionnaire intends to integrate the aesthetic appeal of (certain) mathematics in the greater beauty hierarchy of the respective individual. Question 4 and 5 in the questionnaire examine a possible link between the degree of aesthetics and the degree of creativity; question 5 is based on the assumption that a researcher is more involved in creative working processes than a learner.

Question 3 in the questionnaire corresponds to question II' in section 3; the questions 6 and 7 refer to the questions I and III in section 3. The research question IV in section 3 is implemented in the questionnaire under 11, research question V under 8, 9 and 10, and research question VI under 12.

The answers given by mathematicians should be analysed with regard to common or differing arguments.

5. Results

The mathematicians involved in our study are working in very different fields of mathematics: algebra, discrete mathematics, geometry, non-classical and mathematical logic, history of mathematics. We have received the answers to our questions from eight mathematics researchers, these are about half of the researchers to whom we asked to answer our questionnaire. With one exception, those who refused to answer our questions, argued this with time problems. One mathematician answered with no further comments: "that's not much good to me"; a comment hardly to be interpreted.

In seven of the eight responses we received, the aesthetical component is regarded as an important and influencing factor in the work of mathematicians; we will simultaneously present the given answers to our single research questions in detail.

However, one mathematician does not agree with a considerable function of aesthetics in the *daily work* of mathematicians. He states, that pure mathematics involves theories that are often very transparent and conclusive and therefore beautiful; but that in every-day work there is no place for questions about the aesthetic appeal of a working subject. He argues, that mathematicians cannot allow themselves the luxury of working on problems they like. In the work of a mathematician, creativity is generally demanded in problem solving processes for realistic applications. Contexts in which mathematics is honestly applicable to something, often involve “dirty” calculations and tiresome considerations. Thus, reality given, he sees other sources than the aesthetical appeal for mathematical creativity: e. g. curiosity or challenge by the problem. While working on a problem he tries to draft simple models, but the results often still involve ugliness. He does not see any link between the degree of creativity in the every-day work of mathematicians and the degree of aesthetic appeal of the mathematics they deal with.

We will now present the answers of the other seven mathematicians, named here M1 to M7.

First, some of them gave a sort of *definition* for their personal feelings *of mathematical beauty*, respectively some characteristics for mathematical aesthetics:

M1: I define beauty, roughly speaking, as the experience of realization (and surprise) that an extremely complicated phenomenon is actually just a special case of a general principal. This general principal might be a radically different perspective. This sort of experience occurs all the time in mathematics and physics.

M2: I feel what is beautiful is that which gives pleasure to the mind, the sight, the hearing or the heart, or a combination of these.

M4: I feel the simple and unexpected as particularly beautiful. More precisely, how you can get far with only very few assumptions. Usually, beauty is to equate with simplicity in my research work.

M6: A concrete mathematical model can hold an aesthetic appeal, as well as a virtual one. A particularly clear and concise new proof or just a particularly smart example can cause this feeling. Also the application of elegant mathematical methods for a practical problem can be joined by the feeling of beauty. Certainly, a particularly intensive appeal comes from the suddenly (and sometimes unexpected) pure discovery, the clear understanding of a mathematical phenomenon, often from a completely new perspective, a new harmonic interplay of different fields, first appearing not to be related to each other.

M7: Beauty within mathematics manifests itself on the one hand by typical mathematical-logical arguments, especially if these arguments show unexpected and important connections, in an (at first) surprising manner and then mostly also in a surprising simple manner. On the other hand, mathematical beauty shows itself in the structures used by mathematicians, for me especially in algebraic structures. To be precise, first especially then, if these structures fit extremely good to an important problem class and are at the same time so abstract that the treatment of the relevant (non-trivial) problems becomes “simple”. Second, within the treatments of such structures them themselves, especially then, if the conceptual grasp of these structures is as advanced, that an “elegant” solution of structure internal problems is possible.

As for the answer to *question 1* there were listed non-mathematical examples for things or situations felt as extremely beautiful, e. g.:

M2: seeing Bryce Canyon and hiking it in the cool of the day; watching a swan on a lake; observing a cat at play or play with it; listening to a symphony or playing a musical instrument well; absorbing words fitly spoken; experiencing a well-composed and delivered lecture or sermon; tracing the lines of a child's face or the lines in the hands of an elderly person; experiencing fireworks on the 4th of July; seeing the flag of the USA flying in the breeze; watching a sunset or a jet stream float across the sky,

M3: Prince William Sound, Quadratic Reciprocity, Bach's Toccata and Fugue in D minor, Hamlet, Golden Retrievers,

M5: nearly everything in the field of e-music: Gregorianik, Monteverdi, Bach, Mozart, Beethoven, Schubert, Schumann, Brahms; literature: Fontane (everything), H.M. Enzensberger, a lot of poems; painting; beautiful, quiet landscapes; beautiful, intelligent women;

M7: well played classical music; close to nature mountainous landscapes; a warm summer evening on the terrace of the own house, together with my wife and a good bottle of wine; intelligent conversations with good friends.

But, no one ordered his examples; a ranking seems to be very difficult or quite impossible in this field. This applies accordingly for the answers given to *question 2*. Thus, we did not succeed with our initial idea to classify aesthetics according to degree of appeal by using metaphors in the sense of comparing the aesthetical appeal of a mathematical object with that one of a nonmathematical object (see section 4). Nevertheless we experienced examples to what affects mathematicians, both, in non-mathematical areas (see above) and in the field of mathematics, as follows:

M1: Prime examples are Newton's theory of gravitation usurping the theory of epicycles. An example in mathematics is the notion that number theory and the classical theory of algebraic geometry are two aspects of the theory of schemes.
I would not say that I experienced actual objects as beautiful. Instead, I find that certain mathematical objects naturally facilitate a new, more general perspective, which allows one to answer old questions in a new way. […] such examples abound in mathematics.
One example is the Lebesque theory of integration. The theory provides a different perspective of integration than the Riemann theory, and makes certain questions related to the compatibility of the integral and the sum much easier to deal with.
Another example is the application of the theory of cohomology to the theory of schemes. Using a complicated construction called Etale cohomology, Grothendieck's school was able to answer questions of Weil regarding objects whose definitions seemed to have nothing to do with cohomology.
In both examples … there is an experience of great surprise that such a novel new theory can be used to answer old questions whose statements have nothing to do with the new theory.

M2: Reading or writing or explaining an elegant proof of a theorem, or an elementary proof of a great theorem, or proofs which involve the interaction or algebra and geometry or other branches of mathematics coming together, or...

M3: Quadratic Reciprocity, Godel's theorem, Fundamental theorem of algebra, Church-Turing thesis, Riemann zeta function.

M4: Robinsons Q; the proof of the prime number theorem by Cornaros and Dimitracopulos within a week number theory; Erdoes' proof of the Tschebyscheff's theorem, that there always exists a prime number between n and 2n, and Paolo D'Aquino's proof, that this may be implemented in a week number theory; my definition for two perpendicular circles, using only “there exists”, “and”, “unequal” and the “point-circle incidence”; the whole numbers and their simplest properties, about which we hardly know something - they are probably the most beautiful objects of mathematics; in addition, simple statements, simple axioms, unexpected relations.

M5: Within mathematics I feel relations between objects as beautiful. In the field of mathematics a particular relation often proves to be something that “fits”. We feel this as being beautiful. This reaches from simple examples as: “the equation is solvable” over particular ways of solution, but also to particularly comprehensive structures (pentagon, soccer ball …).

M6: Actually, mathematics as a whole seems to me to be characterized by particular aesthetic, and one could give a nearly unlimited list of mathematical phenomena of great beauty.

One mathematician (M5) pointed out, that on the one hand one can compare the aesthetic appeal of mathematical objects with that one of non-mathematical things, but on the other hand, within mathematics, there appears in the moment of the awareness of the fitting also a particular verity, that cannot be taken away again. This, he does not experience as much in arts.

Another mathematician (M6) stated that “the magnificent sight from a mountain peak on the mountain scenery of the Alps is, after a strenuous hiking tour, even more overwhelming” than mathematics.

When asked if there is seen a link between the degree of creativity in one selves work and the degree of aesthetic appeal of the mathematics dealt with (*question 4*), most of the interviewed mathematicians answered that they did so:

M1: I do see such a link. My feeling is that, the more esoteric and general the new perspective, the more beautiful it is. If I (or someone else) must develop extremely abstract (and seemingly irrelevant) machinery in order to answer classical questions which seem to have nothing to do with that machinery, than it surely takes a very new perspective to see the link between the old and the new. Thus, the paradigm shift is much more extreme, and the experience of realizing that the old phenomenon is a very special case of the new theory is that much more surprising.

M4: Yes, there exists a very significant link. If I would not have found an aesthetically fascinating part of mathematics, I would have probably stopped a long time ago to do research, or even would have left mathematics totally. Only to prove something because others are interested in it and maybe one could make a career for oneself – and could keep one’s head above water – is not enough motivation for me, to wake me out of my natural laziness.

M5: In my research work I have been, nearly always, only interested in those new findings, which were aesthetically satisfying. Findings that are “only true”, with which we do not “feel” why they are true, are boring.

M6: The intensity of the emotional experience when suddenly all pieces of the puzzle of a new mathematical discovery fit together, is surely related to the intensity of the creative study of the respective topic.

Nevertheless, we have got also different answers. M2 responded question 4 by “no” and M7 replied: “I never thought about this: afterwards, an answer, on the truth of which you could rely, is not possible for me”.

The comparison of the aesthetic appeal for the subject as a learner with that one as a (more creative working) researcher, demanded by *question 5*, emphasizes in some cases the answers given to question 4:

M1: As a learner of the subject, one experiences new perspectives on a weekly basis, so one sees the general “explain” the more specific relatively often. However, as a researcher, one develops a much deeper faith in the fact that a novel perspective can have enormous power. … I had to have a great deal of faith that my more general theory would work, since I spent many months developing it before knowing whether it would answer the question I wanted it to answer. I am still amazed that the theory answers the question I set out to answer, and this deepens my faith in the power of beauty as I have defined it.

M5: The experience to rediscover a connection that was already drawn by a great scientist is an almost equivalent experience [with respect to aesthetic feelings]. It is “weaker”, as it is not one’s own, but as a rule it is much more “stronger”, as it concerns something much more important or may concern much deeper relations.

M6: The feeling of beauty … develops further while working intensively.

The answers given to the *questions 3, 6 and 7* show situations in which mathematical beauty has been experienced during a creative process. But, they show also that creative processes are neither necessary nor sufficient for the experience of mathematical beauty: on the one hand, there exist creative processes, which are not accompanied by aesthetic feelings, on the other hand aesthetic feelings may arise in situations in which one is not involved in a creative process (as already marked in section 3). The following extract of the given answers give more insight:

M1: I have experienced mathematical beauty many times, in particular, whenever I learn new mathematical theories that answer old questions, or answer questions unexpectedly. I have also experienced mathematical beauty in the process of solving problems. For example, in trying to solve a problem about a type of algebraic surface, I attempted again and again to use old tools and techniques. The old ways failed me. In desperation, I started reinventing some of these old tools in a more general context. To my amazement (after several months of developing this new theory), my invention turned out to be exactly the right tool to answer the questions I wanted to answer. This is an example of a new perspective making old questions answerable. This sort of thing does not always happen when tackling a mathematics problem. Much of the time, all that is required is the clever and agile use of old tools.

M2: I have experienced mathematical beauty in reading, in hearing lectures, in talking with students/colleagues, in working on research,... Often I was not involved in a creative process.

M3: [I have experienced mathematical beauty in] studying, teaching, doing research.

It has been a cumulative process - no particular moment. Both teaching and research have contributed to my appreciation of the beauty of mathematics.

M4: Not every creative work was connected with aesthetic feelings. Most of these works were accompanied by nail-biting and unpleasant feelings, sleeplessness and things like that. Only after having proved the theorem, one could speak about the aesthetic effect of the result.

M6: Even if a difficult proof for a statement is finally produced, one is not satisfied until its inherent aesthetic really reveals the context to be expressed. Also in mathematics creativity does not necessarily lead to aesthetics, but the aesthetic claim is a great motivation for one's creative work.

In *question 7* we have asked also if an aesthetic appeal of the mathematics one deals or dealt with is/was necessary for creative work. As to this point, the answers differed – some were agreements, others negations:

M1: Yes, since the experience of creativity is tied to the experience of changing ones perspective. Aesthetic appeal is necessary to me for my creative work since the process of changing perspective is quite painful to me. I have learned, however, that the payoff (i. e. the aesthetic appeal I experience) is more than ample payment for the labor of altering my perspective.

M2: I would not say aesthetic appeal was necessary for my creative work.

M3: No, some research was driven by practical considerations, e.g. my PhD thesis, tenure, but later work by intellectual interest which I equate with aesthetic feelings. The work of which I am most proud was driven by intellectual interest.

M4: My entire research work aims exclusively the hidden beauty of geometrical theories.

M6: There exists a plenty of such experiences [of mathematical beauty]; they are the researcher's actual drive.

M7: No.

The answers of the interviewed mathematicians to *question 8*, referring to aesthetic appeal as a possibly guiding principle when choosing a research question, all pointed in the same direction, whereas *question 9*, referring to aesthetics as possibly having a leading function when choosing a route of finding a solution, or one solution over other possible ones, provided diverse answers.

The mathematicians affirmed the influence of the aesthetic appeal on their choice of research questions, some everywhere, some partly. M6 for example stated: "The already recognizable or through new findings expected aesthetic is a great drive for the choice of the research subject." Two mathematicians named some restrictions:

M1: It does, but not completely. The reason is that even modest questions which do not seem to have a great deal of aesthetic appeal lead to beautiful and unexpected insights. This has happened time and again in the history of mathematics. I don't find any intrinsic appeal in the statement of Fermat's last theorem. However, the enormous depth of the mathematics it has inspired is dense with beauty.

M2: I would say more of my work is motivated by just wanting to know the truth.

Regarding the question if creative work (the choice of solutions / solution paths) was lead by aesthetic feelings, the following answers were given:

M1: It has been lead by aesthetic feelings, and in my choice of finding a solution, much to my detriment. Let me explain: I have had some success in picking aesthetic approaches to solving problems. Furthermore, in one important case …, aesthetics was my only guide to finding a solution. The idea for the solution was very simple, but my faith had to be very great indeed in order to build a theory around this idea to make sure it actually worked. I had never before experienced such a powerful affirmation of faith. During a more recent project, my experience led me to believe that beauty was my primary guide for discovering mathematical truths. This led me astray for many months.

M2: No. Not especially.

M3: No, selecting technique was more a practical matter of that which with I was familiar or able to study and learn. Ideas are tried and discarded mostly based on how well they seem to fit.

M4: Actually, no. Finding a solution for any, however aesthetic result is matter of trench war. One stands deep in the mud and does not come out until the supposition is proved.

M5: Clearly. Aesthetics means in mathematics at first „simplicity“. We try to get complex situations under control by trying to strive for the point from that everything becomes easy. Insofar every substantial mathematical work has to assume this aesthetic principle.

M6: If alternatives for a proof are visible, one would generally decide for the way that appears to be more elegant.

M7: Of course: if you see several ways of solution for a problem, you take that one, which appears as more „elegant“.

Four mathematicians agreed with the statement "It was a sort of act of faith in my work that any fundamental mathematical truth must have great beauty in it." (*question 10*), while three mathematicians negated it:

M1: By my definition of beauty, this is true. Since I have defined beauty to be the realization that a new (general) perspective explains older mathematics, and since one can replace "fundamental" by "very general", the statement in the question is "It was a sort of act of faith in my work that any very general mathematical perspective must have great beauty in it." This is almost a tautology.

M2: Not especially. I would not regard DeBranges' proof of the Bieberbach conjecture as particularly beautiful, but it IS true.

M3: No. There remain fundamental results - e.g. the four color solution; the solution of Fermat's conjecture - that are fundamental but (at this time) lack beauty.

M4: Yes.

M5: Absolutely. Hardy said (cited from memory): Beauty is the ultimate goal. … There is no permanent place for ugly mathematics.

M6: If someone feels the clarity of structures as aesthetic value, the search for underlying structures („life, universe and everything“) is not separable from the search for inherent aesthetic.

M7: No.

By *question 11* we wanted to know, if failure respectively success in a creative problem solving process influences the former aesthetic appeal of the problem. Here we list the very different answers:

M1: Fruition often leads to new insights which put the original question in a larger context. This can lead to generalizations, which are inherently more beautiful.
Certainly, success enhances ones understanding of the problem, and hence naturally suggests related, tractable problems. And tractability plays an important role in what mathematics gets done.

M2: I don't think success or failure changes the aesthetic appeal.

M3: It seems to me, failure makes the problem more attractive - e.g. Riemann hypothesis, twin prime conjecture, 3n+1 problem, P/NP problem. (I've worked on the latter two.)

M4: Of course, failure influences the direction of future projects, but this is not the case for the aesthetic estimation of that what had to be proved but could not be proved.
Success always leads to a deepening of the subject and to similar questions, as one hopes to stand a better chance with such questions.

M5: This question does not hit the mark. The feeling of beauty arises with the discovery of a new, simple, unexpected, … connection. Insofar: Only a solution brings the flow.

M6: Of course, the disappointment about the fact that a supposed intrinsic aesthetic did not reveal, in spite of intensive efforts, has an effect, and generates doubts, if the expected special beauty of the context really exists. The felt aesthetic of a scientific finding reinforces positively the own activities.

M7: Problems never have had an aesthetic appeal to me: one meets problems or is confronted with problems. If they do not arise directly out of practical interests, they may appear more or less “natural”, but they withdraw (for me) from classification according aesthetical points of view.

Thus, we can state no changes of aesthetic appeal in some cases, but also enhanced attractiveness as a result of success, and even enhanced attractiveness as a result of failure in other cases. However, to respond question 11 turned out to be problematic for two mathematicians, because of their personal definition/feeling of aesthetics.

Regarding *question 12*, the mathematicians M1 to M5 gave very similar statements. They pointed out analogies between their aesthetic appeal for the arts and music and that one for mathematics, but stated - like M6 and M7 too - that the arts have not inspired them in their creative endeavors in mathematics. Here are some excerpts of the answers:

M1: In thinking about mathematics, one often loses self-consciousness and forgets about the everyday minutiae of life. The state of concentration which is often required for learning about or discovering mathematics is akin to a meditative state: ones ordinary senses are ignored. This is a similarity between the aesthetic appeal for the arts/music and for mathematics.
On the other hand, mathematics is far less forgiving than the arts. Just about any “out of the ordinary” feeling inspired by the arts or music is popularly tied with the aesthetic of the subject. To appreciate mathematics takes a very particular and intense effort, as well as a great deal of training.

For this reason, the arts have never inspired me in my creative endeavors in mathematics.

M2: Yes, in so far as I think of form and structure. Some art and music is chaotic--mathematics usually tries to describe chaos, not just sense it or absorb it. The arts have not inspired me in my creative work in mathematics.

M5: (a) Yes. In both cases one gets access to worlds, that otherwise would not be accessible.
(b) No, I have not been inspired by music or the like in a more or less direct way.

M6: As my examination of music and arts is less intensive than that of mathematics, the emotionality of my feelings for inspiring works in music or arts follows another curve as for mathematical inspiration.

M7: These are separate worlds for me.

6. Concluding remarks

The results of our study show, that the aesthetic appeal seems to play a crucial role in the creative work of contemporary mathematicians. As Nathalie Sinclair (see Sinclair, 2001, 2004, 2006 a, b, 2008) has pointed out, the question of "aesthetics" has been regarded as either frivolous or elitist in both mathematics and mathematics education. The elitist perception of the aesthetic dimension in mathematics education is attributable to the fact that most descriptions of "mathematical beauty" found in the literature come from eminent mathematicians. However classroom based research has shown that students are capable of appreciating the beauty inherent in mathematics (see Brinkmann, 2000, 2004a,b, 2006). In our study, it was found that the aesthetic component need not necessarily derive or be connected to a theorem or proof that the mathematician is currently working on, which can more often be one of sustained trial and frustration, described by one of the mathematicians as "trench war", but aesthetics is often present in appreciation of other results, reading elegantly presented material in books as well as listening to lectures from peers.

Aesthetics has also been relegated by some mathematics education researchers as a small component of the affective dimension of learning, when in fact it intertwines with both the cognitive and the affective components, and as indicated by our study an important aspect of creativity. Sinclair (2009) also presents a convincing argument that aesthetics may very well be the missing gap in numerous failed attempts at motivating students. Given the numerous reform movements in mathematics education that have occurred in many parts of the world, and the call to view school students as budding mathematicians and to get them engaged in mathematical thinking, it is ironic that the aesthetics has not received more attention by the community of mathematics educators. The five pedagogical principles outlined earlier give one possible way in which creativity can be fostered in the classroom. John Mason in his numerous writings has often offered insights into "*the lived experience of mathematical thinking*", and we think we are justified in claiming that aesthetics forms an important aspect of this lived experience!

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