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Bharath Sriraman · Lyn English

Theories of mathematics education: Seeking New Frontiers

The inaugural monograph is based on the two highly acclaimed ZDM special issues on theories of mathematics education (issue 6/2005 and issue 1/2006), which stemmed from the revival of the *Theories of Mathematics Education* (TME group) organized by the editors. This monograph consists of articles with prefaces and commentaries from leading thinkers who have worked on theory building. It is as much summative as forward-looking by highlighting theories from psychology, philosophy and social sciences that continue to influence theory building, as well as providing new developments regarding feminist, complexity and critical theories of mathematics education. New chapters focus on neuroscience research and complexity theory for mathematics education.

This book's cast of authors include Paul Ernest, Stephen Lerman, Frank Lester, David Tall, John Pegg, Richard Lesh, Gerald Goldin, Luis Moreno Armella, Bharath Sriraman, Lyn English, Angelika Bikner-Ahsbahs, Guenter Toerner, Gabriele Kaiser and Guershon Harel, among others.

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ADVANCES IN MATHEMATICS EDUCATION

Theories of mathematics education: Seeking New Frontiers

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Chapter xx

Problem Solving for the 21st Century

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Mathematical problem solving has been the subject of substantial and often controversial research for several decades. We use the term, *problem solving,* here in a broad sense to cover a range of activities that challenge and extend one's thinking. In this chapter, we initially present a sketch of past decades of research on mathematical problem solving and its impact on the mathematics curriculum. We then consider some of the factors that have limited previous research on problem solving. In the remainder of the chapter we address some ways in which we might advance the fields of problem-solving research and curriculum development.

A BRIEF REFLECTION ON PROBLEM-SOLVING RESEARCH

In this section, we do not attempt to provide a comprehensive coverage of problem-solving research over past decades. There are several other sources that provide such coverage, including Lester and Kehle's (2003) work on the development of thinking about research on complex mathematical activity, Lesh and Zawojewski's (2007) research on problem solving and modeling, and English and Halford's (1995) work on problem solving, problem posing, and mathematical thinking.

Concerns about students' mathematical problem solving can be traced back as far as the period of *meaningful learning* (1930s and 1940s), where William Brownell (1945), for example, emphasized the importance of students appreciating and understanding the structure of mathematics. In a similar vein, Van Engen (1949) stressed the need to develop students' ability to detect patterns in similar and seemingly diverse situations. However, it was Polya's (1945) seminal work on how to solve problems that provided the impetus for a lot of problem-solving research that took place in the following decades. Included in this research have been studies on computer-simulated problem solving (e.g., Simon, 1978), expert problem solving (e.g., Anderson, Boyle, & Reiser, 1985), problem solving strategies/heuristics and metacognitive processes (e.g., Charles & Silver, 1988; Lester, Garofalo, & Kroll, 1989), and problem posing (ADD ENGLISH, SILVER ETC, GOLDENBERG &??). More recently there has been an increased focus on mathematical modeling in the elementary and middle grades, as well as interdisciplinary problem solving (ADD). The role of complexity and complex systems in the mathematics curriculum is just starting to be explored (Campbell, ADD; English, 2008, ADD), as is the role of educational neuroscience in helping us improve students' mathematics learning (Campbell, 2006).

A sizeable proportion of past research has focused primarily on *word problems* of the type emphasized in school textbooks or tests. These include "routine" word problems requiring application of a standard computational procedure, as well as "non-routine" problems involving getting from a given to a goal when the path is not evident. It is the latter problems with which students especially struggled. Polya's book, *How to Solve It* (1945), was thus a welcomed publication because it introduced the notion of heuristics and strategies– such as *work out a plan*, *identify the givens and goals, draw a picture, work backwards,* and *look for a similar problem*—tools of an "expert" problem solver. Mathematics educators seized upon the book, viewing it as a valuable resource for improving students' abilities to solve unfamiliar problems, that is, to address the usual question of "What should I do when I'm stuck?"

Despite the ground-breaking contribution of Polya's book, it seems that the teaching of heuristics and strategies has not made significant inroads into improving students' problem solving (Lesh & Zawojewski, 2007; Schoenfeld, 1992; Silver, 1985). Even back in 1979, Begle noted in his seminal book, *Critical Variables in Mathematics Education*:

> A substantial amount of effort has gone into attempts to find out what strategies students use in attempting to solve mathematical problems….no clear-cut directions for mathematics education are provided….In fact, there are enough indications that problem-solving strategies are both problem- and student-specific often enough to suggest that hopes of finding one (or few) strategies which should be taught to all (or most) students are far too simplistic. (p. 145)

Six years later, Silver's (1985) report was no more encouraging. His assessment of the literature showed that, even in studies where some successful learning had been reported, transfer of learning was unimpressive. Furthermore, improvement in problem solving usually occurred only when expert teachers taught lengthy and complex courses in which the size and complexity of the interventions made it unclear exactly why performance had improved. Silver suggested that these improvements could have resulted simply from the students learning relevant mathematical concepts, rather than from learning problem-solving strategies, heuristics, or processes.

Seven years on, Schoenfeld's (1992) review of problem-solving research also concluded that attempts to teach students to apply Polya-style heuristics and strategies generally had not proven to be successful. Schoenfeld suggested that one reason for this lack of success could be because many of Polya's heuristics appear to be "descriptive rather than prescriptive' (p. 353). That is, most are really just names for large categories of processes rather than being well-defined processes in themselves. Therefore, in an effort to move heuristics and strategies beyond basic descriptive tools to prescriptive tools, Schoenfeld recommended that problem-solving research and teaching should: (a) help students develop a larger repertoire of more specific problem-solving strategies that link more clearly to specific classes of problems, (b) foster metacognitive strategies (self-regulation or monitoring and control) so that students learn when to use their problem-solving strategies and content knowledge, and (c) develop ways to improve students' beliefs about the nature of mathematics, problem solving, and their own personal competencies.

Unfortunately, a decade after Schoenfeld's recommendations, Lester and Koehle (2003) drew similar conclusions regarding the impact of problem-solving research on classroom practice: "Teaching students about problem-solving strategies and heuristics and phases of problem solving…does little to improve students' ability to solve general mathematics problems" (p. 508). For many mathematics educators, such a consistent finding is disconcerting.

One explanation for the apparent failure of such teaching is that short lists of descriptive processes or rules tend to be too general to have prescriptive power. Yet, longer lists of prescriptive processes or rules tend to become so numerous that knowing when to use them becomes problematic in itself. We contend that knowing when, where, why, and how to use heuristics, strategies, and metacognitive actions lies at the heart of what it means to understand them (English, Lesh, & Fennewald, 2008). For example, in the very early phases of complex problem solving, students might typically not apply any specific heuristics, strategies, or metacognitive actions—they might simply brainstorm ideas in a random fashion. When progressing towards a solution, however, effective reasoning processes and problem-solving tools are needed—whether these tools be conceptual, strategic, metacognitive, emotional (e.g., beliefs and dispositions), or social (e.g., group-mediated courses of action). Again, students need to know which tools to apply, when to apply them, and how to apply them. Of course, such applications will vary with the nature of the problem-solving situation being addressed. We contend that recognizing the underlying structure of a problem is fundamental to selecting the appropriate tools to use. For example, the strategic tool, *draw a diagram*, can be effective in solving some problems whose structure lends itself to the use of this tool, such as combinatorial problems. However, the solver needs to know which *type* of diagram to use, *how to use it,* and *how to reason* systematically in executing their actions.

Another issue of concern is the traditional way in which problem solving has been implemented in many classrooms. Existing, long-standing perspectives on problem solving have treated it as an isolated topic. Problem-solving abilities are assumed to develop through the initial learning of basic concepts and procedures that are then practised in solving word "story" problems. Exposure to a range of problem-solving strategies and applications of these strategies to novel or non-routine problems usually follows. As we discuss later, when taught in this way, problem solving is seen as independent of, and isolated from, the development of core mathematical ideas, understandings, and processes.

As we leave this brief reflection on problem-solving research, we list some issues that we consider in need of further research with respect to the use of problem-solving heuristics, strategies, and other tools. As English et al., 2008 noted, we need to develop useful operational definitions that enable us to answer questions more fundamental than "Can we teach heuristics and strategies" and "Do they have positive impacts on students' problem-solving abilities?" We need to also ask: (a) What does it mean to “understand” problem-solving heuristics, strategies, and other tools? (b) How, and in what ways, do these understandings develop and how can we foster this development? (d) How can we reliably observe, document, and measure such development? and (e) How can we more effectively integrate core concept development with problem solving?

One wonders why these issues have not received substantial research in recent years, especially given the high status accorded to mathematical problem solving and reasoning in various national and international documents (e.g., NCTM, 2000; ADD). To add to this concern, there has been a noticeable decline in the amount of problem-solving research that has been conducted in the past decade. Recent literature that has its main focus on problem solving, or concept development through problem solving, has been slim.

A number of factors have been identified as contributing to this decline, which we address in the next section. These include the discouraging cyclic trends in educational policy and practices, limited research on concept development and problem solving, insufficient knowledge of students’ problem solving beyond the classroom, the changing nature of the types of problem solving and mathematical thinking needed beyond school, and the lack of accumulation of problem solving research (English et al., 2008; Lesh & Zawojewski, 2007).

LIMITING FACTORS IN PROBLEM-SOLVING RESEARCH

Pendulum Swings Fuelled by High-Stakes Testing

Over the past several decades, we have seen numerous cycles of pendulum swings between a focus on problem solving and a focus on “basic skills” in school curricula. These approximately 10-year cycles, especially prevalent in the USA but also evident in other nations, appear to have brought few knowledge gains with respect to problem solving development from one cycle to the next (English, 2008; English et al., 2008; Lesh & Zawojewski, 2007). Over the past decade or so, many nations have experienced strong moves back towards curricula materials that have emphasized basic skills. These moves have been fuelled by high-stakes national and international mathematics testing, such as PISA (Programme for International Student Assessment: http://www.pisa.oecd.org/) and TIMMS (Third International Mathematics and Science Study: http://timss.bc.edu/timss2003i/intl_reports.html).

These test results have led many nations to question the substance of their school mathematics curricula. Indeed, the strong desire to lead the world in student achievement has led several nations to mimic curricula programs from those nations that score highly on the tests, without well-formulated plans for meeting the specific needs of their student and teacher populations (Sriraman & Adrian, 2008). This teaching-for-the test has led to a "New Push for the Basics" as reported in the New York Times, November 14, 2006. Unfortunately, these new basics are not the basics needed for future success in the world beyond school. With this emphasis on basic skills, at the expense of genuine real-world problem solving, the number of articles on research in problem solving has declined. What is needed, as we flagged previously, is research that explores students' concept and skill development as it occurs through problem solving.

Limited Research on Concept Development through Problem Solving

We begin this section by citing again from Begle's (1979) seminal work:

> It is sometimes asserted that the best way to teach mathematical ideas is to start with interesting problems whose solution requires the use of ideas. The usual instructional procedure, of course, moves in the opposite direction. The mathematics is developed first and then is applied to problems….Problems play

> an essential role in helping students to learn concepts. Details of this role, and the role of problems in learning other kinds of mathematical objects, are much needed. (p. 72)

Unfortunately, it would appear that Begle's concerns are still applicable today. While we are not advocating that learning important mathematical ideas through problem solving is the *only* way to go, we nevertheless argue for a greater focus on problem-driven conceptual development. The usual practice involving routine word problems, which Hamilton (2007) refers to as the "concept-then-word problem" approach (p.4), engages students in a one- or two-step process of mapping problem information onto arithmetic quantities and operations. In most cases, the problem information has already been carefully mathematized for the students. Their goal is to unmask the mathematics by mapping the problem information in such a way as to produce an answer using familiar quantities and basic operations. If the majority of students' classroom mathematics experiences are of this nature, then their ability to solve problems in the real world will be compromised. Students at all grade levels need greater exposure to problem situations which promote the *generation* of important mathematical ideas, not just the application of previously taught rules and procedures.

Unfortunately, we are lacking studies that address problem-driven conceptual development as it interacts with the development of problem-solving competencies (Cai, 2003; Lester & Charles, 2003; Schoen & Charles, 2003). For example, it is still not clear how concept development is expected to interact with the development of the relevant problem-solving tools we mentioned previously. This state of affairs is not helped by curriculum documents (e.g., NCTM, 2000, 2008, http://standards.nctm.org/document/chapter3/prob.htm) that treat problem solving as an isolated topic akin to algebra or geometry. We need better integration of problem solving within all topic areas across the mathematics curriculum, and we would argue, across disciplines. For example, primary school students can generate for themselves an understanding of basic statistical notions when they explore a modelling problem based on team selection for the Olympic Games (English, 2008X). The more we can incorporate genuinely real-world problems within the curriculum, the better our chances of enhancing students' motivation and competencies in mathematical problem solving. This is not an easy task, of course. Knowing which problems appeal to our technologically competent students and to students from different cultural background is the first challenge; being able to design or restructure such problems to maximize students' mathematical development is a second challenge. And many more challenges remain.

Limited Knowledge of Students' Problem Solving Beyond the Classroom

As we have highlighted, problem solving is a complex endeavor involving, among others, mathematical content, strategies, thinking and reasoning processes, dispositions, beliefs, emotions, and contextual and cultural factors. Studies of problem solving that embrace the complexity of problem solving as it occurs in school and beyond are not prolific. Although a good deal of research has been conducted on the relationship between the learning and application of mathematics in and out of the classroom (see, e.g., de Abreu, 2008; Nunes & Bryant, 1996; Nunes, Schliemann, & Carraher, 1993; Saxe, 1991), we still know comparatively little about students' problem-solving capabilities beyond the classroom. We need to know more about why students have difficulties in applying the mathematical concepts and abilities (that they presumably have learned in school) outside of school—or in other classes such as those in the sciences.

A prevailing explanation for these difficulties is the context-specific nature of learning and problem solving, that is, problem-solving competencies that are learned in one situation take on features of that situation; transferring them to a new problem in a new context poses challenges (ADD REF). On the other hand, we need to reassess the nature of the problem-solving experiences we present students, with respect to the nature of the content and how it is presented, the problem contexts and the extent of their real-world links, the reasoning processes likely to be fostered, and the problem-solving tools that are available to the learner. Given the changing nature of problem solving beyond school, we consider it important that these issues be addressed.

Lack of Accumulation of Problem-Solving Research

A further factor that appears to have stalled our progress in problem-solving research is our limited accumulation of knowledge in the field. For example, perspectives on mathematical models and modeling, which we address in the next section, vary across nations with insufficient recognition of, or communication between, the various research hubs addressing this important form of problem solving. The long-standing work on modeling in some European countries (e.g., Germany, Kaiser, ADD DATE) and the substantial research on interdisciplinary model-eliciting activities in the USA and Australia (e.g., Lesh, ADD; English, ADD, Galbraith and Stillman??, ADD) remain in many ways isolated from one another. While different hubs of research on models and modelling are making substantial advancements, such as improving engineering education (e.g., Zawojewski, Diefes-Dux, & Bowman, 2008), one wonders what further achievements could be made if the knowledge across hubs were more accumulative. Nevertheless, the research on models and modeling is becoming more interdisciplinary and is providing new opportunities for improving classroom problem solving.

Problem-solving research has also failed to accumulate adequately with respect to theory advancement and subsequent implications for the classroom, (Lesh, 2008). While we do not advocate the production of a "grand theory" of problem solving, we suggest that mathematics education researchers work more collaboratively in building a cohesive knowledge bank—one that can help us design more appropriate 21st century problems and one that can provide tools that enable us to more reliably observe, document, and assess important mathematical developments in our students.

ADVANCING THE FIELDS OF PROBLEM-SOLVING RESEARCH AND CURRICULUM DEVELOPMENT

The Nature of Problem Solving in Today's World

Although we have highlighted some of the issues that have plagued problem-solving research in past decades, there are emerging signs that the situation is starting to improve. We believe the pendulum of change is beginning to swing back towards problem solving on an international level, providing impetus for new perspectives on the nature of problem solving and its role in school mathematics (Lester & Kehle, 2003). For example, a number of Asian countries have recognized the importance of a prosperous knowledge economy and have been moving their curricular focus toward mathematical problem solving, critical thinking, creativity and innovation, and technological advances (e.g., Maclean, 2001; Tan, 2002). In refocusing our attention on problem solving and how it might become an integral component of the curriculum rather than a separate, often neglected, topic we explore further the following issues:

- What is the nature of problem solving in various arenas of today's world?

- What future-oriented perspectives are needed on the teaching and learning of problem solving including a focus on mathematical content development through problem solving?
- How does mathematical modeling contribute to a future-oriented curriculum?

As we indicated previously, the world is experiencing rapid changes in the nature of the problem solving and mathematical thinking needed beyond school. Indeed, concerns have been expressed by numerous researchers and employer groups that schools are not giving adequate attention to the understandings and abilities that are needed for success beyond school. For example, potential employees most in demand in mathematics/science related fields are those that can (a) interpret and work effectively with complex systems, (b) function efficiently and communicate meaningfully within diverse teams of specialists, (c) plan, monitor, and assess progress within complex, multi-stage projects, and (d) adapt quickly to continually developing technologies (Lesh, 2008).

Research indicates that such employees draw effectively on interdisciplinary knowledge in solving problems and communicating their findings. Furthermore, although they draw upon their school learning, these employees do so in a flexible and creative manner, often creating or reconstituting mathematical knowledge to suit the problem situation, unlike the way in which they experienced mathematics in their school days (Gainsburg, 2006; Hamilton, 2007; Lesh, in press; Zawojewski & McCarthy, 2007). In fact, these employees might not even recognize the relationship between the mathematics they learned in school and the mathematics they apply in solving the problems of their daily work activities. Furthermore, problem solvers beyond the classroom are often not isolated individuals but instead are teams of diverse specialists (Hutchins, 1995a, 1995b; Sawyer, 2007).

Identifying and understanding the differences between school mathematics and the work-place is critical in formulating a new perspective on problem solving. One of the notable findings of studies of problem solving beyond the classroom is the need to master mathematical modeling. Many new fields, such as nanotechnology, need employees who can construct basic yet powerful constructs and conceptual systems to solve the increasingly complex problems that confront them. Being able to adapt previously constructed mathematical models to solve emerging problems is a key component here.

Future-Oriented Perspectives on the Teaching and Learning of Problem Solving

In proposing future-oriented perspectives on problem solving we need to offer a more appropriate definition of problem solving, one that does not separate problem solving from concept development as it occurs in real-world situations beyond the classroom. We adopt here the definition of Lesh and Zawojewski (2007):

> A task, or goal-directed activity, becomes a problem (or problematic) when the "problem solver" (which may be a collaborating group of specialists) needs to develop a more productive way of thinking about the given situation (p. 782).

Thinking in a productive way requires the problem solver to interpret a situation mathematically, which usually involves progression through iterative cycles of describing, testing, and revising mathematical interpretations as well as identifying, integrating, modifying, or refining sets of mathematical concepts drawn from various sources (Lesh & English, 2005; Lesh & Zawojewski, 2007). We contend that future-oriented perspectives on problem solving should transcend current school curricula and national standards and should draw upon a wider range of research across disciplines (English, 2008; Beckmann, 2009; Lesh, 2008).

A core component of any agenda to advance the teaching and learning of problem solving is the clarification of the relationships and connections between the development of mathematical content understanding and the development of problem-solving abilities, as we have emphasized earlier in this chapter. If we can clarify these relationships we can inform curriculum development and instruction on ways in which we can use problem solving as a powerful means to develop substantive mathematical concepts. In so doing, we can provide some alternatives to the existing approaches to teaching problem solving. These existing approaches include instruction that assumes the required concepts and procedures must be taught first and then practiced through solving routine "story" problems that normally do not engage students in genuine problem solving (primarily a content-driven perspective). Another existing approach, which we have highlighted earlier, is to present students with a repertoire of problem solving heuristics/strategies such as "draw a diagram," "guess and check," "make a table" etc. and provide a range of non-routine problems to which these strategies can be applied (primarily a problem-solving focus). Unfortunately, both these approaches treat problem solving as independent of, or at least of secondary importance to, the concepts and contexts in question.

A rich alternative to these approaches is one that treats problem solving as integral to the development of an understanding of any given mathematical concept or process (Lesh & Zawojewski, 2007). Mathematical modelling is one such approach.

Mathematical Modelling

Our world is increasingly governed by complex systems. Financial corporations, education and health systems, the World Wide Web, the human body, and our own families are all examples of complex system. In the 21st century, such systems are becoming increasingly important in the everyday lives of both children and adults.

Educational leaders from different walks of life are emphasizing the need to develop students' abilities to deal with complex systems for success beyond school. These abilities include: interpreting, describing, explaining, constructing, manipulating, and predicting complex systems (such as sophisticated buying, leasing, and loan plans); working on multi-phase and multi-component projects in which planning, monitoring, and communicating are critical for success; and adapting rapidly to ever-evolving conceptual tools (or complex artifacts) and resources (English, 2002; Gainsburg, 2006; Lesh & Doerr, 2003).

With the proliferation of complex systems have come new technologies for communication, collaboration, and conceptualization. These technologies have led to significant changes in the forms of mathematical thinking that are needed beyond the classroom. For example, workers can offload important aspects of their thinking so that some functions become easier (such as information storage, retrieval, representation, or transformation), while other functions become more complex and difficult (such as interpretation of data and communication of results). Computational processes alone are inadequate here—the ability to interpret, describe, and explain data and communicate results of data analyses is essential (Hamilton, 2007; Lesh, 2007; Lesh, Middleton, Caylor & Gupta, 2008).

Significant changes in the types of problem-solving situations that demand the above forms of mathematical thinking have taken place (Hamilton, 2007; Lesh, 2007). For example, in just a few decades, the application of mathematical modelling to real-world problems has escalated. Traffic jams are modeled and used in traffic reports; political unrests and election situations are modeled to predict future developments, and the development of internet search engines is based on different mathematical models designed to find more efficient ways to undertake searches. Unfortunately, research on mathematical problem solving in school has not kept pace with the rapid changes in the mathematics and problem solving needed beyond school. In particular, opportunities for students to engage in mathematical modeling from a young age have been lacking. Yet it is increasingly recognized that modelling provides students with a "sense of agency" in appreciating the potential of mathematics as a critical tool for analyzing important issues in their lives, their communities, and in society in general (Greer, Verschaffel, & Mukhopadhyay, 2007). Indeed, new research is showing that modelling promotes students' understanding of a wide range of key mathematical and scientific concepts and "should be fostered at every age and grade…as a powerful way to accomplish learning with understanding in mathematics and science classrooms" (Romberg et al., 2005, p. 10). Students' development of potent models should be regarded as among the most significant goals of mathematics and science education (Lesh & Sriraman, 2005; Niss, Blum, & Galbraith, 2007).

Mathematical modeling has traditionally been reserved for the secondary and tertiary levels, with the assumption that primary school children are incapable of developing their own models and sense-making systems for dealing with complex situations (Greer Greer, Verschaffel, & Mukhopadhyay, 2007). However, recent research (e.g., English, 2006; English & Watters, 2005) is showing that younger children can and should deal with situations that involve more than just simple counts and measures, and that entertain core ideas from other disciplines.

The terms, *models*, and *modeling*, have been used variously in the literature, including in reference to solving word problems, conducting mathematical simulations, creating representations of problem situations (including constructing explanations of natural phenomena), and creating internal, psychological representations while solving a particular problem (e.g., Doerr & Tripp, 1999; English

& Halford, 1995; Gravemeijer, 1999; Greer, 1997; Lesh & Doerr, 2003; Romberg et al., 2005; Van den Heuvel-Panhuizen, 2003). As Kaiser and Sriraman (2006) highlighted, a homogeneous understanding of modeling and its epistemological backgrounds does not exist within the international community, yet one can find global commonalities in the teaching and learning of mathematical modeling. In particular, the development of detailed descriptions of students' mathematical modelling processes, the identification of the blockages they face and how they overcome these, and the associated challenges in fostering students' modelling abilities are common issues in global studies of modeling (Kaiser, Blomhoj, & Sriraman, 2006).

In our research we have remained with the definition of mathematical models advanced by Doerr and English (2003), namely, models are "systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behavior of some other familiar system" (p.112). From this perspective, modelling problems are realistically complex situations where the problem solver engages in mathematical thinking beyond the usual school experience and where the products to be generated often include complex artefacts or conceptual tools that are needed for some purpose, or to accomplish some goal (Lesh & Zawojewski, 2007).

Modelling as an Advance on Existing Classroom Problem Solving

A focus on modelling in the mathematics curriculum provides an advance on existing approaches to the teaching of mathematics in the primary classroom in several ways. First, the quantities and operations that are needed to mathematize realistic situations often go beyond what is taught traditionally in school mathematics. The types of quantities needed in realistic situations include accumulations, probabilities, frequencies, ranks, and vectors, while the operations needed include sorting, organizing, selecting, quantifying, weighting, and transforming large data sets (Doerr & English, 2001; English, 2006; Lesh, Zawojewski, & Carmona, 2003). Modelling problems provide children with opportunities to generate these important constructs for themselves.

Second, mathematical modelling offers richer learning experiences than the typical classroom word problems ("concept-then-word problem," Hamilton, 2007, p.4). In solving such word problems, children generally engage in a one- or two-step process of mapping problem information onto arithmetic quantities and operations. In most cases, the problem information has already been carefully mathematised for the children. Their goal is to unmask the mathematics by mapping the problem information in such a way as to produce an answer using familiar quantities and basic operations. These word problems constrain problem-solving contexts to those that often artificially house and highlight the relevant concept (Hamilton, 2007). They thus preclude children from creating their own mathematical constructs out of necessity. Indeed, as Hamilton (2007) notes, there is little evidence to suggest that solving standard textbook problems leads to improved competencies in using mathematics to solve problems beyond the classroom.

In contrast, modelling provides opportunities for children to elicit their own mathematics as they work the problem. That is, the problems require children to make sense of the situation so that they can mathematize it themselves in ways that are meaningful to them. This involves a cyclic process of interpreting the problem information, selecting relevant quantities, identifying operations that may lead to new quantities, and creating meaningful representations (Lesh & Doerr, 2003).

A third way in which modelling is an advance on existing classroom practices is that it explicitly uses real-world contexts that draw upon several disciplines. In the outside world, modelling is not just confined to mathematics—other disciplines including science, economics, information systems, social and environmental science, and the arts have also contributed in large part to the powerful mathematical models we have in place for dealing with a range of complex problems (Steen, 2001; Lesh & Sriraman, 2005; Sriraman & Dahl, in press). Unfortunately, our mathematics curricula do not capitalize on the contributions of other disciplines. A more interdisciplinary and unifying model-based approach to students' mathematics learning could go some way towards alleviating the well-known "one inch deep and one mile wide" problem in many of our curricula (Sabelli, 2006, p. 7; Sriraman & Dahl, 2009; Sriraman & Steinthorsdottir, 2007). There is limited research, however, on ways in which we might incorporate other disciplines within the mathematics curriculum.

A fourth way in which modelling advances existing practices is that it encourages the development of generalizable models. The research of English, Lesh, and Doerr (e.g., Doerr & English, 2003; Doerr & English, 2006; English & Watters, 2005, Lesh et al., 2003) has addressed sequences of modelling activities that encourage the creation of models that are applicable to a range of related situations. Students are initially presented with a problem that confronts them with the need to develop a model to describe, explain, or predict the behavior of a given system (*model-eliciting problem*). Given that re-using and generalizing models are central activities in a modelling approach to learning mathematics and science, students then work related problems (*model-exploration* and *model-application problems*) that enable them to extend, explore, and refine those constructs developed in the model-eliciting problem. Because the students' final products embody the factors, relationships, and operations that they considered important, important insights can be gained into their mathematical and scientific thinking as they work the sequence of problems.

Finally, modelling problems are designed for small-group work where members of the group act as a “local community of practice” solving a complex situation (Lesh & Zawojewski, 2007). Numerous questions, issues, conflicts, revisions, and resolutions arise as students develop, assess, and prepare to communicate their products to their peers. Because the products are to be shared with and used by others, they must hold up under the scrutiny of the team and other class members.

An Example of an Interdisciplinary Mathematical Modelling Problem

As previously noted, mathematical modelling provides an ideal vehicle for interdisciplinary learning as the problems draw on contexts and data from other domains. Dealing with “experientially real” contexts such as the nature of community living, the ecology of the local creek, and the selection of national swimming teams provides a platform for the growth of children's mathematisation skills, thus enabling them to use mathematics as a “generative resource” in life beyond the classroom (Freudenthal, 1973).

One such interdisciplinary problem *The First Fleet*, which has been implemented in fifth-grade classrooms in Brisbane, Australia (English, 2007), complemented the children's study of Australia’s settlement and incorporated ideas from the curriculum areas of science and studies of society and environment.

Working in small groups, students in three 5th-grade classes completed *The First Fleet* problem during the second year of their participation in a three-year longitudinal study of mathematical modelling. The problem comprised several components. First, the students were presented with background information on the problem context, namely, the British government’s commissioning of 11 ships in May, 1787 to sail to “the land beyond the seas.” The students answered a number of “readiness questions” to ensure they had understood this background information. After responding to these questions, the students were presented with the problem itself, together with a table of data listing 13 key environmental elements to be considered in determining the suitability of each of five given sites (see Appendix). The students were also provided with a comprehensive list of the tools and equipment, plants and seeds, and livestock that were on board the First Fleet. The problem text explained that, on his return from Australia to the UK in 1770, Captain James Cook reported that Botany Bay had lush pastures and well watered and fertile ground suitable for crops and for the grazing of cattle. But when Captain Phillip arrived in Botany Bay in January 1788 he thought it was unsuitable for the new settlement. Captain Phillip headed north in search of a better place for settlement. The children's task was as follows:

Where to locate the first settlement was a difficult decision to make for Captain Phillip as there were so many factors to consider. If you could turn a time machine back to 1788, how would you advise Captain Phillip? Was Botany Bay a poor choice or not? Early settlements occurred in Sydney Cove Port Jackson, at Rose Hill along the Parramatta River, on Norfolk Island, Port Hacking, and in Botany Bay. Which of these five sites would have been Captain Phillip’s best choice? Your job is to create a system or model that could be used to help decide where it was best to anchor their boats and settle. Use the data given in the table and the list of provisions on board to determine which location was best for settlement. Whilst Captain Phillip was the first commander to settle in Australia many more ships were planning to make the journey and settle on the shores of Australia. Your system or model should be able to assist future settlers make informed decisions about where to locate their townships.

The children worked the problem in groups of 3-4, with no direct teaching from the teachers or researchers. In the final session, the children presented group reports on their models to their peers, who, in turn, asked questions about the models and gave constructive feedback.

The students completed the problem in four, 50-minute sessions with the last session devoted to group reports to class peers on the models created. In the next section we illustrate the cyclic development of one group of students (Mac's group) as they worked the problem. Models developed by other groups are described in English (2009).

Cycles of development displayed by one group of children

Mac's group commenced the problem by prioritizing the elements presented in the table.

Cycle 1: Prioritising and assessing elements

Mac began by expressing his perspective on solving the problem: "So, to find out, OK, if we're going to find the best place I think the most important thing would be that people need to stay alive." The group then proceeded to make a prioritised list of the elements that would be most needed. There was substantial debate over which elements to select, with fresh water, food (fishing and animals), protective bays, and soil and land being chosen. However, the group did not remain with this selection and switched to a focus on all 13 elements listed in the table of data.

The students began to assess the elements for the first couple of sites by placing a tick if they considered a site featured the element adequately and a cross otherwise. The group then began to aggregate the number of ticks for each site but subsequently reverted to their initial decision to just focus on the most essential elements ("the best living conditions to keep the people alive"). Still unable to reach agreement on this issue, the group continued to consider all of the elements for the remaining sites and rated them as “good” and “not so good.” The students explained that they were looking for the site that had “the most good things and the least bad things.”

Cycle 2: Ranking elements across sites

Mac’ group then attempted a new method: they switched to ranking each element, from 1 (“best”) to 5, across the five sites, questioning the meaning of some of the terminology in doing so. The group also questioned the number of floods listed for each site, querying whether it represented the number of floods per year or over several years. As the students were ranking the first few elements, they examined the additional sheet of equipment etc. on board the First Fleet to determine if a given site could accommodate all of the provisions and whether anything else would be needed for the settlement. The group did not proceed with this particular ranking system, however, beyond the first few elements.

Cycle 3: Proposing conditions for settlement and attempting to operationalise data

The group next turned to making some tentative recommendations for the best sites, with Mac suggesting they create conditions for settlement:

> …like if you had not much food and not as many people you should go to Norfolk Island; if you had a lot of people and a lot of food you should go to Sydney Cove or um Rosehill, Parramatta.

The group then reverted to their initial assessment of the elements for each site, totalling the number of ticks (“good”) and crosses (“bad”) for each site. In doing so, the students again proposed suggested conditions for settlement:

> And this one with the zero floods (Norfolk Island), if you don’t have many people that’s a good one cause that’s small but because there’s no floods it’s

> also a very protected area. Obviously, so maybe you should just make it (Norfolk Island) the best area.

Considerable time was devoted to debating conditions for settlement. The group then made tentative suggestions as to how to operationalise the “good” and the “bad.” Bill suggested finding an average of “good” and “bad” for each site but his thinking here was not entirely clear and the group did not take up his suggestion:

> We could find the average, I mean as in like, combine what’s bad, we add them together; we can combine how good we think it might be out of 10. Then we um, could divide it by how many good things there is [sic] and we could divide it by how many bad things there is [sic].

A suggestion was then offered by Marcy: “Why don’t you just select what’s the best one from there, and there, and there,” to which Bill replied, “That’s a good idea, that’s a complete good idea… That’s better than my idea! But how are we going to find out….” The group was becoming bogged down, with Mac demanding “Order, order!” He was attempting to determine just where the group was at and asked Bill to show him the table he was generating. However, Mac had difficulty in interpreting the table: “I can’t understand why you’re doing cross, cross, tick, tick, cross, cross . ..,” to which Bill replied, “Maybe we just combine our ideas.” The group then turned to a new approach.

Cycle 4: Weighting elements and aggregating scores

This new cycle saw the introduction of a weighting system, with the students assigning 2 points to those elements they considered important and 1 point to those elements of lesser importance (“We’ve valued them into points of 1 and 2 depending on how important they are”). Each site was then awarded the relevant points for each element if the group considered the site displayed the element; if the site did not display the element, the relevant number of points was subtracted. As the group explained:

> The ones (elements) that are more important are worth 2 points and the ones that aren't are 1. So if they (a given site) have it you add 2 or 1, depending on how important it is, or you subtract 2 or 1, if they don't have it.

The students totalled the scores mentally and documented their results as follows (1 refers to Botany Bay, 2 to Port Jackson, and so on):

1 - 12 + 10 = - 2

2 - 9 + 13 = 4

3 - 5 + 17 = 12

4 - 7 + 15 = 8

5 - 9 + 13 = 4

Cycle 5: Reviewing models and finalising site selection

The group commenced the third session of working the problem (the following morning) by reflecting on the two main models they had developed to determine the best site, namely, the use of ticks ("good") and crosses ("bad") in assessing elements for each site and trying to operationalise these data, and the weighting of elements and aggregating of scores. Mac commenced by reminding his group of what they had found to date:

> Yesterday we, um, OK, the first thing we did yesterday showed us that the fifth one (Norfolk Island) was the best place, second one (weighting of elements) we did told us … showed us that number three (Rosehill, Parramatta) was the best. So it's a tie between number three and number five. So it's limited down to them, work it out. Hey guys, are you even listening?

After considerable debate, Mac concluded, “OK, we’re doing a tie-breaker for number three and number five.” The group proceeded to revisit their first model, assigning each tick one point and ignoring the crosses. On totalling the points, Mac claimed that Rosehill, Parramatta, was the winning site. Bill expressed concern over the site’s record of 40 floods and this resulted in subsequent discussion as to whether Parramatta should be the favoured site. The children finally decided on Norfolk Island because it was flood-free and because it was their choice using their first model.

Students’ learning in working *The First Fleet* problem

As discussed previously, modelling problems engage students in multiple cycles of interpretations and approaches, suggesting that real-world, complex problem solving goes beyond a single mapping from givens to goals. Rather, such problem solving involves multiple cycles of interpretation and re-interpretation where conceptual tools evolve to become increasingly powerful in describing, explaining, and making decisions about the phenomena in question (Doerr & English, 2003). In *The First Fleet* problem, students displayed cycles of development in their mathematical thinking and learning as they identified and prioritized key problem elements, explored relationships between elements, quantified qualitative data, ranked and aggregated data, and created and worked with weighted scores—before being formally introduced to mathematisation processes of this nature.

Interdisciplinary modelling problems can help unify some of the myriad core ideas within the curriculum. For example, by incorporating key concepts from science and studies of society and the environment *The First Fleet* can help students appreciate the dynamic nature of environments and how living and non-living components interact, the ways in which living organisms depend on others and the environment for survival, and how the activities of people can change the balance of nature. *The First Fleet* problem can also lead nicely into a more in-depth study of the interrelationship between ecological systems and economies, and a consideration of ways to promote and attain ecologically sustainable development.

Finally, the inherent requirement that children communicate and share their mathematical ideas and understandings, both within a small-group setting and in a whole-class context, further promotes the development of interdisciplinary learning. These modelling problems engage students in describing, explaining, debating, justifying, predicting, listening critically, and questioning constructively—which are essential to all discipline areas.

Mathematical Modelling with Young Learners: A Focus on Statistical Reasoning

Limited research has been conducted on mathematical modelling in the early school years, yet it is during these informative years that important foundations for future learning need to be established. One such foundation is that of statistical reasoning.

Across all walks of life, the need to understand and apply statistical reasoning is paramount. Statistics underlie not only every economic report and census, but also every clinical trial and opinion poll in modern society. One has to look no further than non-technical publications such as *Newsweek* or daily newspapers to see the variety of graphs, tables, diagrams, and other data representations that need to be interpreted. Our unprecedented access to a vast array of numerical information means we can engage increasingly in democratic discourse and public decision making—that is, provided we have an appropriate understanding of statistics and statistical reasoning. Research has indicated, however, that many university students and adults have limited knowledge and understanding of statistics (e.g., Meletiou-Mavrotheris, Paparistodemou, & Stylianou, 2009; Rubin, 2002).

Young children are very much a part of our data-driven society. They have early access to computer technology, the source of our information explosion. They have daily exposure to the mass media where various displays of data and related reports can easily mystify or misinform, rather than inform, their young minds. It is thus imperative that we rethink the nature of children's statistical experiences in the early years of school and consider how best to develop the powerful mathematical and scientific ideas and processes that underlie statistical reasoning (Langrall, Mooney, Nisbet, & Jones, 2008). Indeed, several recent articles (e.g., Franklin & Garfield, 2006; Langrall et al., 2008) and policy documents have highlighted the need for a renewed focus on this component of early mathematics learning. For example, the USA National Council of Teachers of Mathematics (NCTM) identified data analysis and probability as its "Focus of the Year" for 2007-2008 (September, 2007), while the Australian Association of Mathematics Teachers (AAMT) and the Early Childhood Australia (ECA) have jointly called for a more future-oriented focus on mathematics education in the early childhood years (0-8 years; 2006; http://www.aamt.edu.au). In Europe, the *Enhancing the Teaching and Learning of Early Statistical Reasoning in European Schools* project (http://www.earlystatistics.net/) has developed an innovative professional development program for the teaching and learning of statistical reasoning at the elementary and middle school levels.

One approach to developing future-oriented statistical experiences for young learners is through data modelling. Such modelling engages children in extended and integrative experiences in which they generate, test, revise, and apply their own models in solving problems that they identify in their world. Data modelling differs from traditional classroom experiences with data in several ways, including;

- The problems children address evolve from their own questions and reasoning;
- There is a move away from isolated tasks with restricted data (e.g., recording and comparing children's heights as a “stand-alone” task) to comprehensive thematic experiences involving multiple data considerations in both mathematical and scientific domains;
- The components of data modelling involve foundational statistical concepts and processes that are tightly interactive (as indicated in Figure 11.1), rather than rigidly sequential, and that evolve over time;
- Identifying and working with underlying mathematical and scientific structures is a key feature;
- Children generate, test, revise, and apply their own models in solving problems in their world.

Figure 11.1 displays the essential components of data modelling. The starting point for developing statistical reasoning through data modelling is with the world and the problems it presents, rather than with any preconceived formal models. Data modelling is a developmental process (Lehrer & Schauble, 2005) that begins with young children’s inquiries and investigations of meaningful phenomena (e.g., exploring the growth of flowering bulbs under different conditions), progressing to deciding what aspects are worthy of attention and how these might be measured (e.g., identifying attributes such as amount of water and sunlight, soil conditions, and subsequently the height of plants at different growth points in the different conditions), and then moving towards structuring, organising, analysing, visualising, and representing data (e.g., measuring and comparing plant heights in each condition at identified growth points; organizing and displaying the data in simple tables, graphs, diagrams; and analyzing the data to identify any relationships or trends). The resultant model, which provides a solution to the children's original question/s, is repeatedly tested and revised, and ultimately allows children to draw (informal) inferences and make recommendations from the original problem and later, similar problems (e.g., applying their models to establishing an appropriate class garden). Children's generation, testing, and revision of their models, which lie at the core of statistical reasoning, is an important developmental process.

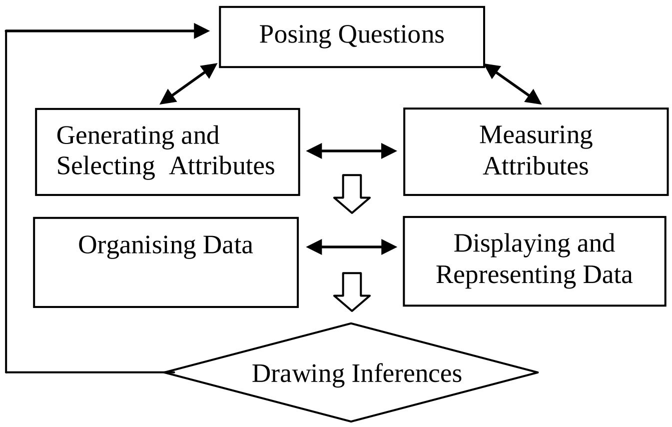

Figure 11.1. Components of data modelling (adapted from Lehrer & Schauble, 2004).

Data modelling experiences target powerful mathematical and scientific concepts and processes that need to be nurtured from a young age, such as:

- Problem solving and problem posing;
- Working and reasoning with number, including identifying patterns and relationships;
- Identifying features of and changes in living things and their interactions with the environment;
- Identifying, measuring, and comparing attributes;
- Developing an understanding of evidence;
- Collecting, organizing, analyzing, evaluating, and representing data;

- Identifying and applying basic measures of distance and centre;
- Making and testing conjectures and predictions; and
- Reflecting on, communicating, discussing, and challenging mathematical and scientific arguments.

The early school years comprise the educational environment where all children should begin a meaningful development of these core concepts and processes (Baroody, Lai, & Mix, 2006; Charlesworth & Lind, 2006; Ginsburg et al., 2006). However, as Langrall et al. (2008) note, even the major periods of reform in elementary mathematics do not seem to have given most children access to the deep ideas and key processes that lead to success beyond school.

To illustrate how the data modelling components displayed in Figure 11.1 might be developed in a classroom, we consider an activity centered on the school playground. Following initial whole-class discussions on the nature and design of playground environments, questions such as the following could arise for a series of investigations: (a) Is our own playground fun and safe? (b) How might we make our playground more exciting and safer for us? (c) Are we taking care of the plants and wildlife in our play areas? (d) What could we do to have more wildlife around? Effective questions suggest fruitful courses of action and contain the seeds of emerging new questions, so this initial phase is often revisited throughout a cycle of inquiry (Lehrer et al., 2002). In developing their models to answer their questions, children would normally cycle iteratively through the following phases as they work the investigation.

Refining questions and identifying attributes. For the first question “Is our own playground fun and safe?” children need to determine which attributes to consider, such as the nature, extent, location, and popularity of selected playground equipment; the number of sheltered and open play areas; and the distribution of bins for safe food disposal. Identifying and defining variables is an important, developmental process incorporating a fundamental understanding of sampling (Watson & Moritz, 2000).

Measuring attributes and recording initial data. Here, for example, children might decide to keep a tally of the number of children on each item of play equipment in different time periods, measure and tabulate the approximate distances between the items, tally the number of rubbish bins in a given area, measure the bins' distance from each other and from the eating areas, and estimate and record the number of children in the eating area using the bins.

Organising, analysing, interpreting, and representing their data. Children would then need to consider how they could utilise all of their data to help answer their initial question. For example, they might decide to draw simple pictures or bar graphs or tables to show that one item of play equipment is the most popular at morning recess but is also very close to another popular item; or that some bins are close to the eating areas while others are not. An important process here is children's ability to objectify their data (Lehrer et al., 2002), that is, treating data as objects in their own right that can be manipulated to discover relationships and identify any trends.

Developing data-based explanations, arguments, and inferences, and sharing these with their peers. Children might conclude, for example, that their data suggest the playground is fun for their peers (a wide range of equipment that is very popular at all play times) but is not sufficiently safe (e.g., equipment too close; inadequate number and distribution of rubbish bins). After testing and revising their models that address their initial question, children would share these with their peers during class presentations, explaining and justifying their representations, inferences, and arguments. The children's peers would be encouraged to ask questions and provide constructive feedback on their overall model (such model sharing and feedback provides rich opportunities for further conceptual development: English, 2006; Hamilton, Lesh, Lester, & Yoon, 2007.) A subsequent activity would involve the children in using their models to answer the second question, “How might we make our playground more exciting and safer for us?”

CONCLUDING POINTS

We have argued in this chapter that research on mathematical problem solving has stagnated for much of the 1990s and the early part of this century. Furthermore, the research that has been conducted does not seem to have accumulated into a substantive, future-oriented body of knowledge on how we can effectively promote problem solving within and beyond the classroom. In particular, there has been limited research on concept development through problem solving and we have limited knowledge of students’ problem solving beyond the classroom.

The time has come to consider other options for advancing problem-solving research and curriculum development. One powerful option we have advanced is that of mathematical modelling. With the increase in complex systems in today’s world, the types of problem-solving abilities needed for success beyond school have changed. For example, there is an increased need to interpret, describe, explain, construct, manipulate, and predict complex systems. Modelling problems, which draw on multiple disciplines, provide an ideal avenue for developing these abilities. These problems involve simulations of appealing, authentic problem-solving situations (e.g., selecting sporting teams for the Olympic Games) and engage students in mathematical thinking that involves creating and interpreting situations (describing, explaining, communication) at least as much as it involves computing, executing procedures, and reasoning deductively. We have argued that such problems provide an advance on existing classroom problem solving, including the provision of opportunities for students to generate important constructs themselves (before being introduced to these in the regular curriculum) and to create generalisable models.

Further research is needed on the implementation of modelling problems in the elementary school, beginning with kindergarten and first grade. One area in need of substantial research is the development of young children's statistical reasoning. Young children have daily exposure to mass media and live in the midst of our data-driven and data-explosive society. We need to ensure that they are given opportunities to develop early the skills and understandings needed for navigating and solving the problems they will increasingly face outside of the classroom.

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