König-Egerváry graphs are non-Edmonds

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16 October 2009

Abstract

König-Egerváry graphs are those whose maximum matchings are equicardinal to their minimum-order coverings by vertices. Edmonds [*J. Res. Nat. Bur. Standards Sect. B* **69B** (1965), 125–130] characterized the perfect matching polytope of a graph $G = (V, E)$ as the set of nonnegative vectors $\mathbf{x} \in \mathbb{R}^E$ satisfying two families of constraints: 'vertex saturation' and 'blossom'. Graphs for which the latter constraints are implied by the former are termed *non-Edmonds*. This note presents two proofs—one combinatorial, one algorithmic—of its title's assertion. Neither proof relies on the characterization of non-Edmonds graphs due to de Carvalho et al. [*J. Combin. Theory Ser. B* **92** (2004), 319–324].

Keywords: matching, perfect matching polytope, covering

2010 *MSC*: Primary 05C70; Secondary 05C35, 90C35, 68R10, 52B99

Introduction

Jack Edmonds [6] characterized the perfect matching polytope $\mathrm{PM}(G)$ of a graph $G = (V, E)$ as the set of nonnegative vectors $\mathbf{x} \in \mathbb{R}^E$ satisfying two families of constraints: 'vertex saturation' and 'blossom'. Sometimes, the latter constraints are implied by the former. For example, if G is bipartite, then a nonnegative $\mathbf{x} \in \mathbb{R}^E$ lies in $\mathrm{PM}(G)$ if and only if $\mathbf{x}$ satisfies the vertex saturation constraints. (This assertion is equivalent to the Birkhoff-von Neumann Theorem [1, 13], which states that every doubly-stochastic matrix is a convex combination of permutation matrices; see, e.g., [2, Exercise 17.4.5].) This note adds the so-called 'König-Egerváry' graphs to the class of graphs for which Edmonds' blossom constraints are redundant for determining their perfect matching polytopes. We present two proofs, neither relying on the characterization of 'non-Edmonds' graphs due to de Carvalho et al. [3]. The first has a combinatorial flavour while the second is algorithmic.

Notation and terminology

Our graphs are finite and contain neither loops nor parallel edges. The *matching number* ν of G is the maximum size of a matching in G. A graph is *König-Egerváry*, or *K-E*, if ν agrees with the minimum size of a covering of G (i.e., a set of vertices meeting every edge of G). K-E graphs were introduced a generation ago—see [5, 12]—as a generalization of bipartite graphs, the latter having been proved to be K-E independently by König [9] and Egerváry [8]. The *coboundary* $\partial(S)$ of a subset $S \subseteq V$ is the set of edges of G with exactly one end in S; if $S = \{v\}$, then we abbreviate $\partial(S)$ by $\partial(v)$. The *incidence vector* $\boldsymbol{\chi}_M \in \mathbb{R}^E$ of a set $M \subseteq E$ is the $0,1$-vector with $\boldsymbol{\chi}_M(e) = 1$

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if and only if $e \in M$. The *perfect matching polytope* PM(G) of G is the convex hull of the set of incidence vectors of the perfect matchings of G. We shall find it necessary to have a more precise statement of *Edmonds' constraints* on a vector $\mathbf{x} \in \mathbb{R}^E$ than we gave above:

$$\begin{array}{llll} \text{(i)} & \mathbf{x} \geq \mathbf{0} & & (\textit{nonnegativity}); \\ \text{(ii)} & \displaystyle\sum_{e \in \partial(v)} \mathbf{x}(e) = 1 \text{ for each } v \in V & & (\textit{vertex saturation}); \text{ and} \\ \text{(iii)} & \displaystyle\sum_{e \in \partial(S)} \mathbf{x}(e) \geq 1 \text{ for each odd subset } S \subseteq V & & (\textit{blossom}). \end{array}$$

As noted in the introduction, Edmonds [6] characterized PM(G) as the set of those $\mathbf{x} \in \mathbb{R}^E$ satisfying (i), (ii), and (iii). A graph G is *non-Edmonds* whenever PM(G) is characterized only by Edmonds' constraints (i) and (ii), i.e., whenever an $\mathbf{x} \in \mathbb{R}^E$ lies in PM(G) if and only if $\mathbf{x}$ satisfies (i) and (ii); this terminology follows [11]. Please see [2] for any omitted definitions.

The result

Theorem *If G is König-Egerváry, then G is non-Edmonds.*

Proof. Let $G = (V, E)$ be a K-E graph, and suppose that $\mathbf{x} \in \mathbb{R}^E$ satisfies Edmonds' constraints (i) and (ii). To show that G is non-Edmonds, it is enough to show that $\mathbf{x} \in$ PM(G). We proceed under the assumption that PM(G) $\neq \varnothing$; the (simpler) case when PM(G) $= \varnothing$ can be handled using similar techniques and is omitted.

Let $M = \{v_i w_i\}_{i=1}^{\nu}$ be a perfect matching and C a covering of G such that $|M| = |C|$. Because M is a matching and C covers M, of necessity C is comprised of exactly one end chosen from each $e \in M$; without loss of generality, we may take $C = \{v_1, \ldots, v_\nu\}$. As each $e \in E$ has at least one end in C, the set $W := \{w_1, \ldots, w_\nu\}$ is stable in G. Let us call any edge $e \in E$ of the form $e = v_i w_j$, for some $i, j \in \{1, \ldots, \nu\}$, a *crossing edge* of G and denote by $F \subseteq E$ the set of crossing edges.

We claim that $\mathbf{x}$ is supported within F. To see why, sum the values $\mathbf{x}(e)$ for crossing edges e in two ways. On one hand, since $\mathbf{x}$ satisfies (ii) and W is stable, we have

$$\nu = \sum_{i=1}^{\nu} \sum_{e \in \partial(w_i)} \mathbf{x}(e) = \sum_{i=1}^{\nu} \sum_{e \in \partial(w_i) \cap F} \mathbf{x}(e). \tag{1}$$

On the other hand, summing these same values from the perspective of the v_i's and taking into account the fact that $\mathbf{x}$ satisfies (i) and (ii), we find that the right side of (1) is

$$(\nu =) \quad \sum_{i=1}^{\nu} \sum_{e \in \partial(v_i) \cap F} \mathbf{x}(e) \leq \sum_{i=1}^{\nu} \sum_{e \in \partial(v_i)} \mathbf{x}(e) = \nu. \tag{2}$$

Since the extremes in (2) coincide, the inequality here is really an equality, so that $\sum_{i=1}^{\nu} \sum_{e \in \partial(v_i) \smallsetminus F} \mathbf{x}(e) = 0$, and the nonnegativity of $\mathbf{x}$ now implies that $\sum_{e \in \partial(v_i) \smallsetminus F} \mathbf{x}(e) = 0$ for each $i \in \{1, \ldots, \nu\}$, whence $\mathbf{x}(e) = 0$ for each $e \notin F$; i.e., the claim holds.

Next define a $\nu \times \nu$ (nonnegative real) matrix $B = (b_{ij})$ by

$$b_{ij} \;=\; \begin{cases} \mathbf{x}(v_i w_j) & \text{if } v_i w_j \in F; \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Notice that B is doubly stochastic: for each $i \in \{1, \ldots, \nu\}$, we have

$$\sum_{j=1}^{\nu} b_{ij} = \sum_{e \in \partial(v_i) \cap F} \mathbf{x}(e) = \sum_{e \in \partial(v_i)} \mathbf{x}(e) = 1$$

(since $\mathbf{x}$ is supported within F and satisfies (ii)), and, likewise, each $j \in \{1, \ldots, \nu\}$ satisfies $\sum_{i=1}^{\nu} b_{ij} = 1$. By the Birkhoff-von Neumann Theorem [1, 13] (see [2, Exercise 16.2.19] or [4] for a proof), we may write

$$B = \sum_{k=1}^{N} \lambda_k P_k \tag{4}$$

for some permutation matrices $\{P_k\}_{k=1}^{N}$ and some nonnegative real numbers $\{\lambda_k\}_{k=1}^{N}$ with $\sum_{k=1}^{N} \lambda_k = 1$. Since the nonzero values in B correspond to edges of G (indeed, to crossing edges), the 1-entries of each P_k also have this property. Moreover, the property of being a permutation matrix implies that each P_k corresponds to a perfect matching M_k in G. Finally, the definition (3) of B and the relation (4) together show that

$$\mathbf{x} = \sum_{k=1}^{N} \lambda_k \boldsymbol{\chi}_{M_k},$$

so that $\mathbf{x} \in \mathrm{PM}(G)$ as desired. □

Algorithmic proof

In our second proof, we continue to assume that $\mathrm{PM}(G) \neq \varnothing$. We present an algorithm that accepts as input an arbitrary vector $\mathbf{c} \in \mathbb{R}^E$ and produces a perfect matching M^* of G and a vector $\mathbf{y}^* \in \mathbb{R}^V$ such that

$$\mathbf{y}^*(u) + \mathbf{y}^*(v) \geq \mathbf{c}(uv) \text{ for each } uv \in E \tag{5}$$

and, with $\mathbf{x}^* := \boldsymbol{\chi}_{M^*}$, such that

$$\sum_{v \in V} \mathbf{y}^*(v) = \mathbf{c}^T \mathbf{x}^*. \tag{6}$$

If A is the vertex-edge incidence matrix of G, then (5) verifies that $\mathbf{y}^*$ is a feasible solution to the LP dual of the linear programming problem

$$\left.\begin{aligned} \max\ & \mathbf{c}^T \mathbf{x} \\ \text{s.t. } & A\mathbf{x} = \mathbf{1} \\ & \mathbf{x} \geq \mathbf{0}; \end{aligned}\right\} \tag{7}$$

the relation (6) and our remarks below confirm that $\mathbf{y}^*$ is in fact dual-optimal. Since the constraints of (7) are Edmonds' constraints (i) and (ii), this will suffice to prove that

$$\mathrm{PM}(G) = \{\mathbf{x} \in \mathbb{R}^E \colon \mathbf{x} \text{ satisfies (i) and (ii)}\},$$

i.e., that G is non-Edmonds. Indeed, for every primal-feasible $\mathbf{x} \in \mathbb{R}^E$, we have

$$\mathbf{c}^T \mathbf{x} \ \leq\ \mathbf{y}^{*T} A \mathbf{x} \ =\ \sum_{v \in V} \mathbf{y}^*(v) \ =\ \mathbf{c}^T \mathbf{x}^*, \tag{8}$$

where the relations are justified, respectively, by (5), by the primal-feasibility of $\mathbf{x}$, and by (6). Now (8) shows that $\mathbf{x}^*$, an element of $\mathrm{PM}(G)$, is optimal for (7), whose feasible region contains $\mathrm{PM}(G)$. Since our algorithm accepts an arbitrary $\mathbf{c} \in \mathbb{R}^E$ and optimizes the corresponding linear programming problem (7) at an $\mathbf{x}^* \in \mathrm{PM}(G)$, it follows that $\mathrm{PM}(G)$ coincides with the feasible region of (7).

Here is our algorithm.

Input:	a K-E graph $G = (V, E)$ with $\mathrm{PM}(G) \neq \varnothing$ and a vector $\mathbf{c} \in \mathbb{R}^E$
Output:	a perfect matching M^* of G and a vector $\mathbf{y}^* \in \mathbb{R}^V$ satisfying (5) and (6) when $\mathbf{x}^* = \boldsymbol{\chi}_{M^*}$

1. Find a perfect matching M and a covering C of G such that $|M| = |C|$; let $F \subseteq E$ denote the set of crossing edges, relative to C.
2. Find a perfect matching M^* of G and a vector $\mathbf{y} \in \mathbb{R}^V$ such that

$$\mathbf{y}(u) + \mathbf{y}(v) \geq \mathbf{c}(uv) \text{ for each } uv \in F, \tag{9}$$

and

$$\sum_{v \in V} \mathbf{y}(v) = \mathbf{c}^T \mathbf{x}^*. \tag{10}$$

3. Choose $U \in \mathbb{R}$ sufficiently large so that $\mathbf{y}^* \in \mathbb{R}^V$ defined by

$$\mathbf{y}^*(v) = \begin{cases} \mathbf{y}(v) + U & \text{if } v \in C \\ \mathbf{y}(v) - U & \text{if } v \in V \smallsetminus C \end{cases}$$

satisfies (5).

4. Output $(M^*, \mathbf{y}^*)$.

Correctness and efficiency

Step 1 can be accomplished using Edmonds' matching algorithm [7] (to find M), together with the first algorithm presented in [5] (to find C). Deming's algorithm actually delivers a maximum stable set of vertices, but we may obtain our desired (minimum) covering by complementation. In Step 2, first note that the spanning subgraph $H = (V, F)$ of G is bipartite, and since C is a covering of G, all perfect matchings of G are contained in F. Therefore, Step 2 amounts to finding a primal-dual solution for the Assignment Problem on H, using the objective function of (7). Kuhn's algorithm [10] accomplishes this; see, e.g., [4, p. 400]. Step 3 requires an inspection of each non-crossing edge, so that $\mathbf{y}^*$ effectively boosts $\mathbf{y}$ enough to extend (9) to (5). Since $|C| = |V \smallsetminus C|$—as in our first proof—this modification does not change the dual objective function as we pass from $\mathbf{y}$ to $\mathbf{y}^*$; i.e.,

$$\sum_{v \in V} \mathbf{y}^*(v) = \sum_{v \in V} \mathbf{y}(v) = \mathbf{c}^T \mathbf{x}^*,$$

whence (6) is also satisfied.

Though it makes no difference to the correctness of the second proof, it is nevertheless satisfying to know that our algorithm runs in polynomial-time. This follows because each required algorithm is likewise efficient: Edmonds' matching algorithm; Deming's algorithm; Kuhn's algorithm; and finally, the one-time inspection of each non-crossing edge.

The second proof follows a pleasing template for establishing that the convex hull of a chosen set is determined by a given set of constraints. We have reason to believe that this approach is the one that would make Jack himself happy.

Acknowledgements

This note was written while the author was on sabbatical at Université de Montréal and a visitor at the Concordia Computational Combinatorial Optimization Laboratory (ConCoCO). Many thanks to Rama Murty for giving the talk that inspired the result, to Geňa Hahn for lending ears and eyes for the first presented version, and to Vašek Chvátal, who, on seeing our Theorem's first proof, generously contributed the second.

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