Surveying Theories and Philosophies of Mathematics Education

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Preliminary Remarks

Any theory of thinking or teaching or learning rests on an underlying philosophy of knowledge. Mathematics education is situated at the nexus of two fields of inquiry, namely mathematics and education. However, numerous other disciplines interact with these two fields, which compound the complexity of developing theories that define mathematics education (Sriraman 2009a). We first address the issue of clarifying a philosophy of mathematics education before *attempting* to answer whether theories of mathematics education are constructible. In doing so we draw on the foundational writings of Lincoln and Guba (1994), in which they clearly posit that any discipline within education, in our case mathematics education, needs to clarify for itself the following questions:

(1) What is reality? Or what is the nature of the world around us?

This question is linked to the general ontological question of distinguishing objects (real versus imagined, concrete versus abstract, existent versus non-existent, independent versus dependent and so forth) (Sriraman 2009b).

(2) How do we go about knowing the world around us? [the methodological question, which presents possibilities to various disciplines to develop methodological paradigms] and,
(3) How can we be certain in the "truth" of what we know? [the epistemological question].

Even though the aforementioned criteria have been labelled by educational theorists as the building blocks of a paradigm (Ernest 1991; Lincoln and Guba 1994; Sriraman 2009a), others have argued that these could very well constitute the foundations of a philosophy for mathematics education (Sriraman 2008, 2009a).

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At the outset, it is also important to remind the community that Jean Piaget (cf. Piaget 1955) started from Emmanuel Kant's paradigm of reasoning or "thinking" and arrived at his view of cognition as a biologist viewing intelligence and knowledge as biological functions of organisms (Bell-Gredler 1986). Piaget's theories of knowledge development have been interpreted differently by different theorists, such as von Glasersfeld's notion of radical constructivism (von Glasersfeld 1984, 1987, 1989) or viewed through its interaction with the theories of Vygotsky by theorists like Paul Cobb and Heinrich Bauersfeld as social constructivism. However another major influence on these theories of learning and developing a philosophy of mathematics of relevance to mathematics education is Imre Lakatos' (1976) book *Proofs and Refutations* (Lerman 2000; Sriraman 2009a). The work of Lakatos has influenced mathematics education as seen in the social constructivists' preference for the "Lakatosian" conception of mathematical certainty as being subject to revision over time, in addition to the language games à la Wittgenstein "in establishing and justifying the truths of mathematics" (Ernest 1991, p. 42) to put forth a fallible and non-Platonist viewpoint about mathematics. This position is in contrast to the Platonist viewpoint, which views mathematics as a unified body of knowledge with an ontological certainty and an infallible underlying structure. In the last two decades, major developments include the emergence of social constructivism as a philosophy of mathematics education (Ernest 1991), the well documented debates between radical constructivists and social constructivists (Davis et al. 1990; Steffe et al. 1996; von Glasersfeld 1987) and recent interest in mathematics semiotics, in addition to an increased focus on the cultural nature of mathematics. The field of mathematics education has exemplified voices from a wide spectrum of disciplines in its gradual evolution into a distinct discipline. Curiously enough Hersh (2006) posited an analogous bold argument for the field of mathematics that its associated philosophy should include voices, amongst others, of cognitive scientists, linguists, sociologists, anthropologists, and last but not least interested mathematicians and philosophers!

Imre Lakatos and Various Forms of Constructivism

Proofs and Refutations is a work situated within the philosophy of science and clearly not intended for, nor advocates a didactic position on the teaching and learning of mathematics (Pimm et al. 2008; Sriraman 2008). Pimm et al. (2008) point out that the mathematics education community has not only embraced the work but has also used it to put forth positions on the nature of mathematics (Ernest 1991) and its teaching and learning (Ernest 1994; Lampert 1990; Sriraman 2006). They further state:

> We are concerned about the proliferating Lakatos personas that seem to exist, including a growing range of self-styled 'reform' or 'progressive' educational practices get attributed to him. (Pimm et al. 2008, p. 469)

This is a serious concern, one that the community of mathematics educators has not addressed. Generally speaking *Proofs and Refutations* addresses the importance of the role of history and the need to consider the historical development of mathematical concepts in advocating any philosophy of mathematics. In other words, the book attempts to bridge the worlds of historians and philosophers. As one of the early reviews of the book pointed out:

> *His (Lakatos') aim is to show while the history of mathematics without the philosophy of mathematics is blind, the philosophy of mathematics without the history of mathematics is empty*. (Lenoir 1981, p. 100) (italics added)

Anyone who has read *Proofs and Refutations* and tried to find other mathematical "cases" such as the development of the Euler-Descartes theorem for polyhedra, will know that the so called "generic" case presented by Lakatos also happens to be one of the few special instances in the history of mathematics that reveals the rich world of actually *doing* mathematics, the world of the working mathematician, and the world of informal mathematics characterized by conjectures, failed proofs, thought experiments, examples, and counter examples etc.

Reuben Hersh began to popularize *Proof and Refutations* within the mathematics community in a paper titled, "*Introducing Imre Lakatos*" (Hersh 1978) and called for the community of mathematicians to take an interest in re-examining the philosophy of mathematics. Nearly three decades later, Hersh (2006) attributed *Proofs and Refutations* as being instrumental in a revival of the philosophy of mathematics informed by scholars from numerous domains outside of mathematical philosophy, "in a much needed and welcome change from the foundationist ping-pong in the ancient style of Rudolf Carnap or Willard van Ormond Quine" (p. vii). An interest in this book among the community of philosophers grew as a result of Lakatos' untimely death, as well as a favourable review of the book given by W.V. Quine himself in 1977 in the *British Journal for the Philosophy of Science.* The book can be viewed as a challenge for philosophers of mathematics, but resulted in those outside this community taking an interest and contributing to its development (Hersh 2006). Interestingly enough, one finds a striking analogical development in voices outside of the mathematics education community contributing to its theoretical development. In one sense the theoretical underpinnings of mathematics education has developed in parallel with new developments in the philosophy of mathematics, with occasional overlaps in these two universes. Lakatos is an important bridge between these two universes.

Proofs and Refutations was intended for philosophers of mathematics to be cognizant of the historical development of ideas. Yet, its popularization by Reuben Hersh (and Philip Davis) gradually led to the development of the so called "maverick" traditions in the philosophy of mathematics, culminating in the release of Reuben Hersh's (2006) book *18 Unconventional Essays on the Nature of Mathematics*—a delightful collection of essays written by mathematicians, philosophers, sociologists, an anthropologist, a cognitive scientist and a computer scientist. These essays are scattered "across time" in the fact that Hersh collected various essays written over the last 60 years that support the "maverick" viewpoint. His book questions what constitutes a philosophy of mathematics and re-examines foundational questions without getting into Kantian, Quinean or Wittgensteinian linguistic quagmires. In a similar vein the work of Paul Ernest can be viewed as an attempt to develop a maverick philosophy, namely a social constructivist philosophy of mathematics (education). We have put the word education in parentheses because Ernest does not make any explicit argument for an associated pedagogy as argued by Steffe (1992).

Does Lakatos' work have any direct significance for mathematics education? Can Lakatos' *Proofs and Refutations* be directly implicated for the teaching and learning of mathematics? We would argue that it cannot be directly implicated. However *Proofs and Refutations* may very well serve as a basis for a philosophy of mathematics, such as a social constructivist philosophy of mathematics, which in turn can be used a basis to develop a theory of learning such as constructivism. This is a position that Steffe (1992) advocated, which has gone unheeded. Les Steffe in his review of Ernest's (1991) *The Philosophy of Mathematics Education* wrote:

> Constructivism is sufficient because the principles of the brand of constructivism that is currently called "radical" (von Glasersfeld 1989) should be simply accepted as the principles of what I believe should go by the name Constructivism. It seems to me that the radical constructivism of von Glasersfeld and the social constructivism of Ernest are categorically two different levels of the same theory. Constructivism (radical), as an epistemology, forms the hard core of social constructivism, which is a model in what Lakatos (1970) calls its protective belt. Likewise, psychological constructivism is but a model in the protective belt of the hard-core principles of Constructivism. These models continually modify the hard-core principles, and that is how a progressive research program that has interaction as a principle in its hard core should make progress. It is a lot easier to integrate models in the protective belt of a research program that has been established to serve certain purposes than it is to integrate epistemological hard cores. (Steffe 1992, p. 184)

Theory Development

Our arguments on the relevance of Lakatos for mathematics education comes more from the view of doing research and being practitioners, both of which have to rest on an underlying philosophy and an associated theory of learning. The present diversity in the number of new theories used in mathematics education from domains like cognitive science, sociology, anthropology and neurosciences are both natural and necessary given the added complexity in teaching and learning processes/situations in mathematics. Even though theory development is essential for any field mathematics education has often been accused of "faltering" in theories (Steen 1999). The development of "universal" theoretical frameworks has been problematic for mathematics education. A research forum on this topic was organized by us at the 29$^{\text{th}}$ Annual meeting of the International Group for the Psychology of Mathematics Education (PME29) in Melbourne, which led to the two ZDM issues on theories that eventually became a basis for the present book. In one of the extended papers emanating from this research forum, Lester elaborated on the effect of one's philosophical stance in research:

> Cobb puts philosophy to work by drawing on the analyses of a number of thinkers who have grappled with the thorny problem of making reasoned decisions about competing theoretical perspectives." He uses the work of noted philosophers such as (alphabetically) John

> Dewey, Paul Feyerabend, Thomas Kuhn, Imre Lakatos, Stephen Pepper, Michael Polanyi, Karl Popper, Hilary Putnam, W.V. Quine, Richard Rorty, Ernst von Glasersfeld, and several others to build a convincing case for considering the various theoretical perspectives being used today "as sources of ideas to be appropriated and adapted to our purposes as mathematics educators. (Lester 2005, p. 461)

Having addressed some of the debates that dominated the theoretical underpinning of the field for nearly two decades, we now focus on alternative conceptions of theory development. As stated earlier, we have seen a significant increase in the conceptual complexity of our discipline, where we need to address myriad factors within a matrix comprising of people, content, context, and time (Alexander and Winne 2006; Sriraman 2009a). This complexity is further increased by ontological and epistemological issues that continue to confront both mathematics education and education in general, which unfortunately have not been directly addressed. Instead a utilitarian mix-and-match culture pervades the field given the fact that mathematics education researchers have at their disposal a range of theories and models of learning and teaching. Choosing the most appropriate of these, singly or in combination, to address empirical issues is increasingly challenging. The current political intrusion, at least in the USA, into what mathematics should be taught, how it should be assessed, and how it should be researched further complicates matters (e.g., Boaler 2008). Indeed, Lester (2005) claimed that the role of theory and philosophical bases of mathematics education has been missing in recent times, largely due to the current obsession with studying "what works"—such studies channel researchers along pathways that limit theoretical and philosophical advancement (p. 457).

On the other hand, if we compare the presence of theory in mathematics education scholarship today with its occurrence in past decades, it is clear that theory has become more prominent. Herein lies an anomaly, though. The elevation of theory in mathematics education scholarship could be considered somewhat contradictory to the growing concerns for enhancing the relevance and usefulness of research in mathematics education (Silver and Herbst 2007). These concerns reflect an apparent scepticism that theory-driven research can be relevant to and improve the teaching and learning of mathematics in the classroom. Such scepticism is not surprising, given that we have been criticized for inadequacy in our theoretical frameworks to improve classroom teaching (e.g., King and McLeod 1999; Eisenberg and Fried 2008; Lesh and Sriraman 2005; Lester 2005; Steen 1999). Claims that theoretical considerations have limited application in the reality of the classroom or other learning contexts have been numerous, both in mathematics education and in other fields (Alexander and Winne 2006; Sfard 1991). But we concur with Alexander and Winne (2006) that "principles in theory necessarily have a practical application" (p. xii); it remains one of our many challenges to clearly demonstrate how theoretical considerations can enhance the teaching and learning of mathematics in the classroom and beyond. One source of difficulty here lies in the language barriers that so many theories display—how can others interpret and apply our theoretical messages if the intended meaning is lost in a world of jargon? We explore the following but do not claim to have covered all that needs examining:

- Is there such a thing as *theory* in mathematics education?
- What are the changes in theory in recent decades and the impact on mathematics education?
- What are some European schools of thought on theory development, particularly the French School?
- What are the future directions and possibilities?

Many commentaries have been written on theory and mathematics education, including why researchers shift their dominant paradigms so often, whether we develop our own theories or borrow or adapt from other disciplines, whether we need theory at all, how we cope with multiple and often conflicting theories, why different nations ignore one another's theories, and so on (e.g., Cobb 2007; King and McLeod 1999; Steiner 1985; Steiner and Vermandel 1988). Steen's (1999) concerns about the state of mathematics education in his critique of the *ICMI study on Mathematics Education as a Research Domain: A search for Identity* (Sierpinska and Kilpatrick 1998) were reflected a decade later in Eisenberg and Fried's (2009) commentary on Norma Presmeg's reflections on the state of our field (see Presmeg 2009). Eisenberg and Fried (2009) claimed that, "Our field seems to be going through a new phase of self-definition, a crisis from which we shall have to decide who we are and what direction we are going." (p. 143). It thus seems an appropriate time to reassess theory in mathematics education, the roles it has played and can play in shaping the future of our discipline.

Theory and Its Role in Mathematics Education

The increased recognition of theory in mathematics education is evident in numerous handbooks, journal articles, and other publications. For example, Silver and Herbst (2007) examined "Theory in Mathematics Education Scholarship" in the *Second Handbook of Research on Mathematics Teaching and Learning* (Lester 2007) while Cobb (2007) addressed "Putting Philosophy to Work: Coping with Multiple Theoretical Perspectives" in the same handbook. And a central component of both the first and second editions of the *Handbook of International Research in Mathematics Education* (English 2002, 2008a, 2008b) was "advances in theory development." Needless to say, the comprehensive second edition of the *Handbook of Educational Psychology* (Alexander and Winne 2006) abounds with analyses of theoretical developments across a variety of disciplines and contexts.

Numerous definitions of "theory" appear in the literature (e.g., see Silver and Herbst 2007). It is not our intention to provide a "one-size-fits-all" definition of theory per se as applied to our discipline; rather we consider multiple perspectives on theory and its many roles in improving the teaching and learning of mathematics in varied contexts.

At the 2008 *International Congress on Mathematical Education*, Assude et al. (2008) referred to theory in mathematics education research as dealing with the teaching and learning of mathematics from two perspectives: a *structural* and a *functional* perspective. From a structural point of view, theory is "an organized and coherent system of concepts and notions in the mathematics education field." The "functional" perspective considers theory as "a system of tools that permit a 'speculation' about some reality." When theory is used as a *tool*, it can serve to: (a) conceive of ways to improve the teaching/learning environment including the curriculum, (b) develop methodology, (c) describe, interpret, explain, and justify classroom observations of student and teacher activity, (d) transform practical problems into research problems, (e) define different steps in the study of a research problem, and (f) generate knowledge. When theory functions as an *object*, one of its goals can be the advancement of theory itself. This can include testing a theory or some ideas or relations in the theory (e.g., in another context or) as a means to produce new theoretical developments.

Silver and Herbst (2007) identified similar roles but proposed the notion of theory as a mediator between problems, practices, and research. For example, as a mediator between research and problems, theory is involved in, among others, generating a researchable problem, interpreting the results, analysing the data, and producing and explaining the research findings. As a mediator between research and practice, theory can provide a norm against which to evaluate classroom practices as well as serve as a tool for research to understand (describe and explain) these practices. Theory that mediates connections between practice and problems can enable the identification of practices that pose problems, facilitate the development of researchable problems, help propose a solution to these problems, and provide critique on solutions proposed by others. Such theory can also play an important role in the development of new practices, such as technology enhanced learning environments.

What we need to do now is explore more ways to effectively harmonize theory, research, and practice (Silver and Herbst 2007; Malara and Zan 2008) in a coherent manner so as to push the field forward. This leads to an examination of the extant theoretical paradigms and changes that have occurred over the last two decades. This was briefly discussed at the outset of this chapter.

Changes in Theoretical Paradigms

> Theories are like toothbrushes... everyone has their own and no one wants to use anyone else's. (Campbell 2006)

As several scholars have noted over the years, we have a history of shifting frequently our dominant paradigms (Berliner 2006; Calfee 2006; King and McLeod 1999). Like the broad field of psychology, our discipline "can be perceived through a veil of 'isms'" (Alexander and Winne 2006, p. 982; Goldin 2003). We have witnessed, among others, shifts from behaviourism, through to stage and level theories, to various forms of constructivism, to situated and distributed cognitions, and more recently, to complexity theories and neuroscience. For the first couple of decades of its life, mathematics education as a discipline drew heavily on theories and methodologies from psychology as is evident in the frameworks of most papers that appeared in journals like *Journal for Research in Mathematics Education* (JRME) and *Educational Studies in Mathematics* (ESM). According to Lerman (2000), the switch to research on the social dimensions of mathematical learning towards the end of the 1980s resulted in theories that emphasized a view of mathematics as a social product. Social constructivism, which draws on the seminal work of Vygotsky and Wittgenstein (Ernest 1994) has been a dominant research paradigm for many years. Lerman's extensive analysis revealed that, while the predominant theories used during this period were traditional psychological and mathematics theories, an expanding range from other fields was evident especially in PME and ESM. Psychosocial theories, including re-emerging ones, increased in ESM and JRME. Likewise, papers drawing on sociological and socio-cultural theories also increased in all three publications together with more papers utilizing linguistics, social linguistics, and semiotics. Lerman's analysis revealed very few papers capitalizing on broader fields of educational theory and research and on neighbouring disciplines such as science education and general curriculum studies. This situation appears to be changing in recent years, with interdisciplinary studies emerging in the literature (e.g., English 2007, 2008a, 2008b, 2009; English and Mousoulides 2009) and papers that address the nascent field of neuroscience in mathematics education (Campbell 2006).

Numerous scholars have questioned the reasons behind these paradigm shifts. Is it just the power of fads? Does it only occur in the United States? Is it primarily academic competitiveness (new ideas as more publishable)? One plausible explanation is the diverging, epistemological perspectives about what constitutes mathematical knowledge. Another possible explanation is that mathematics education, unlike "pure" disciplines in the sciences, is heavily influenced by unpredictable cultural, social, and political forces (e.g., D'Ambrosio 1999; Secada 1995; Skovsmose and Valero 2008; Sriraman and Törner 2008).

A critical question, however, that has been posed by scholars now and in previous decades is whether our paradigm shifts are genuine. That is, are we replacing one particular theoretical perspective with another that is more valid or more sophisticated for addressing the hard core issues we confront (Alexander and Winne 2006; King and McLeod 1999; Kuhn 1966)? Or, as Alexander and Winne ask, is it more the case that theoretical perspectives move in and out of favour as they go through various transformations and updates? If so, is it the voice that speaks the loudest that gets heard? Who gets suppressed? The rise of constructivism in its various forms is an example of a paradigm that appeared to drown out many other theoretical voices during the 1990s (Goldin 2003). Embodied mathematics made its appearance with the work of Lakoff and Núñez (2000), yet the bold ideas proposed in *Where Does Mathematics Come From*, received very little attention from mathematics education researchers in terms of systemic follow-ups in teaching, learning and researching. Similarly, even though Lev Vygotsky's (1978) work is cited in the vast literature in mathematics education that uses social constructivist frameworks, very little attention is paid to his cultural-historical activity theory, which has simultaneous orientation with embodied operations and the social dimensions allowing for a theorization of the intricate relationships between individual and social cognition (Roth 2007). In essence, the question we need to consider is whether we are advancing professionally in our theory development. Paradigms, such as constructivism, which became fashionable in mathematics education over recent decades, tended to dismiss or deny the integrity of fundamental aspects of mathematical and scientific knowledge. In essence, the question we need to consider is whether we are advancing professionally in our theory development. We debate these issues in the next sections.

Are We Progressing?

Goldin (2003) expressed a number of sentiments about the chasms that have opened up over the years between mathematicians, mathematics educators, and classroom practitioners. Our own views resonate with his heart-felt, personal observations and experiences that have left him "profoundly sceptical of the sweeping claims and changing fashions that seemed to characterize educational research" (p. 175). Goldin also cites a vulnerable group for which popular paradigms of the day can be very restrictive to their growth as researchers, namely doctoral students and recent doctoral graduates. Indeed, Goldin makes a plea to our young researchers to be proactive in instigating "a major change of direction in the mathematics education field" (pp. 175–176). We agree with his claim that:

> It is time to abandon, knowledgeably and thoughtfully, the dismissive fads and fashions—the 'isms'—in favour of a unifying, non-ideological, scientific and eclectic approach to research, an approach that allows for the consilience of knowledge across disciplines. (p. 176)

Such an approach would help establish the much-needed basis for a sound intellectual relationship between the disciplines of mathematics education research and mathematics. To date, scholars from allied disciplines do not seem to value one another's contributions in their efforts to improve mathematics learning. As a consequence, we do not seem to be *accumulating* the wealth of knowledge gained from numerous studies (Lesh and Sriraman 2005). We applaud Goldin's (2003) call for mathematics education researchers to incorporate within their studies the most appropriate and useful constructs from many different theoretical and methodological approaches "but *without* the dismissals" (p. 198). As pointed out earlier, the two dominant philosophies that arose in the 80's and 90's were radical constructivism (see von Glasersfeld 1984) and social constructivism (Ernest 1991). With a very instrumental view of mathematics—understandably—the classical "Stoffdidaktik" tradition in Germany asserts the need to continually develop the pedagogy of mathematics. However there were some inherent problems in each of these philosophies as pointed out by Goldin (2003)

> Social constructivism pointed to the importance of social and cultural contexts and processes in mathematics as well as mathematics education, and postmodernism highlighted functions of language and of social institutions as exercising power and control. And 'mind-based mathematics' emphasized the ubiquity and dynamic nature of metaphor in human language, including the language of mathematics. Unfortunately, in emphasizing its own central idea, each of these has insisted on excluding and delegitimizing other phenomena and other constructs, even to the point of the words that describe them being forbidden—including central constructs of mathematics and science—or, alternatively, certain meanings being forbidden to these words. Yet the ideas summarized here as comprising the 'integrity of knowledge' from mathematics, science, and education are not only well-known, but have

> proven their utility in their respective fields. There are ample reasoned arguments and supporting evidence for them. (p. 196)

The need to draw upon the most applicable and worthwhile features of multiple paradigms has been emphasized by numerous researchers in recent years. We are now witnessing considerable diversity in the theories that draw upon several domains including cognitive science, sociology, anthropology, philosophy, and neuroscience. Such diversity is not surprising given the increasing complexity in the teaching and learning processes and contexts in mathematics.

Are theories in mathematics education being reiterated or are they being reconceptualized (Alexander and Winne 2006), that is, are we just "borrowing" theories from other disciplines and from the past, or are we adapting these theories to suit the particular features and needs of mathematics education? A further question—are we making inroads in creating our own, unique theories of mathematics education. Indeed, should we be focusing on the development of a "grand theory" for our discipline, one that defines mathematics education as a field—one that would give us autonomy and identity (Assude et al. 2008)?

Over a decade ago, King and McLeod (1999) emphasized that as our discipline matures, it will need to travel along an independent path not a path determined by others. Cobb (2007) discusses "incommensurability" in theoretical perspectives and refers to Guerra (1998) who used the implicit metaphor of theoretical developments as a "*relentless march of progress*." The other metaphor is that of "*potential redemption*." Cobb thus gives an alternate metaphor, that of "*co-existence and conflict*", namely "The tension between the march of progress and potential redemption narratives indicates the relevance of this metaphor." (Cobb 2007, p. 31)

Home-Grown Theories versus Interdisciplinary Views

We now discuss the issue of "borrowing" theories from other disciplines rather than developing our own "home-grown" theories in mathematics education (Steiner 1985; Kilpatrick 1981; Sanders 1981). We agree with Steiner (1985) that Kilpatrick's and Sanders' claims that we need more "home-grown" theories would place us in "danger of inadequate restrictions if one insisted in mathematics education on the use of home-grown theories" (p. 13). We would argue for theory building for mathematics education that draws upon pertinent components of other disciplines. In Steiner's (1985) words:

> The nature of the subject [mathematics education] and its problems ask for *interdisciplinary approaches* and it would be wrong not to make meaningful use of the knowledge that other disciplines have already produced about specific aspects of those problems or would be able to contribute in an interdisciplinary cooperation. (p. 13)

> Actually interdisciplinary does not primarily mean borrowing ready-made theories from the outside and adapting them to the condition of the mathematical school subject. There exist much deeper interrelations between disciplines. (p. 13)

> Mathematics education has not sufficiently reflected and practiced these indicated relations between disciplines. Rather than restricting its search for theoretical foundations to *home-*

> *grown theories* it should develop more professionality in formulating *home-grown demands* to the cooperating disciplines. (p. 14)

Since Steiner's and Kilpatrick's papers we have witnessed considerable diversity in the number of new theories applied to mathematics education. Silver and Herbst (2007) argue that we should aspire to build such a theory. They write "This type of theory responds to a need for broad schemes of thought that can help us organize the field and relate our field to other fields, much in the same way as evolutionary theory has produced a complete reorganisation of biological sciences." … "It can also be seen as a means to aggregate scholarly production within the field" (p. 60).

Silver and Herbst (2007) claim that this has long been the goal of some pioneers in our field such as H.G. Steiner. They write

> The development of a grand theory of mathematics education could be useful in providing warrants for our field's identity and intellectual autonomy within apparently broader fields such as education, psychology, or mathematics. In that sense, *a grand theory could be helpful to organize the field,* imposing something like a grand translational or relational scheme that allows a large number of people to see phenomena and constructs in places where others only see people, words, and things. A grand theory of the field of mathematics education could seek to spell out what is singular (if anything) of *mathematics education as an institutional field* or perhaps seek to spell out connections with other fields that may not be so immediately related and that establish the field as one among many contributors to an academic discipline. (p. 60)

We however do not agree with the claims of Silver and Herbst for the following reason. In Sriraman and English (2005), we put forth an argument on the difficulty of abstracting universal invariants about what humans do in different mathematical contexts, which in turn, are embedded within different social and cultural settings; this suggests that it is a futile enterprise to formulate grand theories. At this point in time such a grand theory does not appear evident, and indeed, we question whether we should have such a theory. As we indicate next, there are many levels of theory and many "adapted" theories that serve major functions in advancing our field. The issue of a grand theory is one for ongoing debate.

Our argument is supported by the work of a core group of researchers in the domain of models and modelling, which follows. Lesh and Sriraman (2005) put forth a much harsher criticism of the field when it comes to developing theories. They claimed that the field, having developed only slightly beyond the stage of continuous theory borrowing, is engaged in a period in its development which future historians surely will describe as something akin to the *dark ages*—replete with inquisitions aimed at purging those who do not vow allegiance to vague philosophies (e.g., "constructivism"—which virtually every modern theory of cognition claims to endorse, but which does little to inform most real life decision making issues that mathematics educators confront and which prides itself on not generating testable hypotheses that distinguish one theory from another)—or who don't pledge to conform to perverse psychometric notions of "scientific research" (such as pretest/posttest designs with "control groups" in situations where nothing significant is being controlled, where the most significant achievements are not being tested, and where the teaching-to-the-test is itself is the most powerful untested component of the "treatment"). With the exception of small schools of mini-theory development that occasionally have sprung up around the work a few individuals, most research in mathematics education appears to be ideology-driven rather than theory-driven or model-driven. Ideologies are more like religions than sciences; and, the "communities of practice" that subscribe to them tend to be more like cults than continually adapting and developing learning communities (or scientific communities). Their "axioms" are articles of faith that are often exceedingly non-obvious—and that are supposed to be believed without questioning. So, fatally flawed ideas repeatedly get recycled. Their "theorems" aren't deducible from axioms; and, in general, they aren't even intended to inform decision-making by making predictions. Instead, they are intended mainly to be after-the-fact "cover stories" to justify decisions that already have been made. They are accepted because they lead to some desirable end, not because they derive from base assumptions (Lesh and Sriraman 2005).

Lesh and Sriraman (2005) further criticize the closed mindedness of the field towards new ideas. They write:

> New ideas (which generally are not encouraged if they deviate from orthodoxy) are accepted mainly on the basis of being politically correct—as judged by the in-group of community leaders. So, when basic ideas don't seem to work, they are made more-and-more elaborate—rather than considering the possibility that they might be fundamentally flawed. Theories are cleaned up bodies of knowledge that are shared by a community. They are the kind of knowledge that gets embodied in textbooks. ... They emphasize formal/deductive logic, and they usually try to express ideas elegantly using a single language and notation system. The development of theory is absolutely essential in order for significant advances to be made in the thinking of communities (or individuals within them). ... [B]ut, theories have several shortcomings. Not everything we know can be collapsed into a single theory. For example, models of realistically complex situations typically draw on a variety of theories. Pragmatists (such as Dewey, James, Pierce, Meade, Holmes) argued that it is arrogant to assume that a single "grand theory" will provide an adequate basis for decision-making for most important issues that arise in life (Lesh and Sriraman 2005). Instead, it is argued that it might be better for the field to develop models of thinking, teaching and learning, which are testable and refine-able over time (see Lesh and Sriraman, this volume for a schematic of the interaction between theories and models).

European Schools of Thought in Mathematics Education

The field of mathematics education when viewed through its developments in Europe from the turn of the 19th century can be "simplistically" thought of in the following terms. Its origins lay in the classical tradition of Felix Klein onto the structuralist agenda influenced by the Bourbaki and Dieudonné at the Royaumont seminar in France, followed by Freudenthal's reconception of mathematics education with emphasis on the humanistic element of doing mathematics. The approaches of Klein and Dieudonné steeped in an essentialist philosophy gave way to the pragmatic approach of Freudenthal. Skovsmose (2005) critiqued the French tradition of mathematic didactics as being "socio-political blind" ... "with such research not supporting teachers in interpreting ... the politics of public labeling" (p. 3). An interpretation of the effect of the essentialist view on mathematics didactics traditions in Germany is thoroughly described in Sriraman and Törner (2008). Inspite of the criticism of Skovsmose (2005), unlike the dominant *discourse of confusion* that seems to characterize the Anglo-American spheres of mathematics education research, the French research paradigm is surprisingly homogenous, with a body of theories to advance their programmes of research noteworthy for its consistency in theory, methodology, and terminology.

Didactique des Mathématiques—The French Tradition

The term "*Didactique des Mathematiques*" (henceforth DdM) is the study of the process of the dissemination of mathematical knowledge, with more emphasis on the study of teaching. The French term also encompasses the study of the transformations produced on mathematical knowledge by those learning it in an institutional setting. DdM as a field of science lies at the intersection of mathematics, epistemology, history of mathematics, linguistic psychology and philosophy. As is the case in Germany, research in DdM occurs within specific departments in the institutionalized setting of universities, with international networks of collaborators and regular conferences.

We briefly outline the historical origins of the French tradition because it is substantially older than the Anglo-American traditions. In terms of the roots of mathematics education in philosophy, numerous writings on the history of didactic traditions (Kaiser 2002; Pepin 1998) suggest that humanism played a major role as the general philosophy of education in both England, the Netherlands, Scandinavia and Germany. On the other hand the French educational philosophy mutated from humanism to an "encyclopedic" tradition (or Encyclopaedism[1]) as seen in the massive works of Denis Diderot (1713–1784), Charles Monstequieu (1689–1755), Francois Voltaire (1694–1778), Jean Jacques Rousseau (1712–1778) and many others who were instrumental in paving the way for the French revolution. It is particularly interesting that many of these philosophers took a deep interest in the fundamental questions of learning which are still unresolved today.

Rousseau outlined a comprehensive philosophy of education in the *Emile*. Rousseau theorized that there was one developmental process common to all humans, its earliest manifestation was seen in children's curiosity which motivated them to learn and adapt to the surroundings. A detailed discussion of these works is beyond the scope of this chapter but it helps establish the encyclopaedic roots of the French traditions. Just as politics and philosophy have been deeply intertwined in French society, so have philosophy and education. The French educational system was grounded on the principles of égalité (equality) and laicité (secularism) with mathematics as one of the many subjects important to develop a person's rational faculties (see Pepin 1998, 1999a, 1999b). A documented concern for improving mathematics education has been present for over a hundred years as seen in the

[1] The definition of the word Encyclopaedism in the online dictionary (wordreference.com) suggests that the word means eruditeness, learnedness, scholarship and falls within the same categorical tree as psychology, cognition (knowledge, noesis), content, education and letters.

founding of the journal *L'Enseignment Mathématique* in 1899 by Henri Fehr and Charles-Ange Laisant. Furinghetti (2003) in her introduction to the monograph celebrating 100 years of this journal wrote:

> The idea of internationalism in mathematics education was crucial to the journal right from its very beginning…the two editors had proposed in 1905 to organize an international survey on reforms needed in mathematics education, asking in particular opinions on the conditions to be satisfied by a complete-theoretical and practical-teaching of mathematics in higher institutions. (p. 12)

The journal also initiated the study of mathematical creativity. This is a very important event as it brought into relevance the field of psychology and the attention of Jean Piaget and mathematicians within the fold (see Furinghetti 2003, pp. 36–37). The historical influence of prominent French mathematicians on mathematics education is seen particularly in textbooks used, the structure and focus of the content, and the unique characteristics of teacher training. For instance, entry into teacher education programs is extremely competitive and includes substantial course work in university level mathematics, much more in comparison to universities in the U.S. and Germany. The system in France is highly centralized with only a small proportion of students gaining entry into engineering programs and researcher or teacher training programs typically at the secondary level. The inference here is that these students are exposed to higher level mathematics content for a prolonged time period irrespective of whether they want to be teachers or researchers. From the point of view of mathematics education research, the influence of prominent mathematicians and philosophers on subsequent epistemologies of mathematics education is best evident in the fact that the works of Henri Poincaré (1908) and Léon Brunschwicg (1912) influenced subsequent works of Bachelard (1938), Jean Piaget (1972) and Dieudonné (1992). The emphasis of the French mathematics curriculum at all levels on logical reasoning, encouraging elements of proof, developing mathematical thinking and facilitating discovery contains elements from the writings of Piaget, Poincaré and Dieudonné.

The Royaumont Seminar

"For example, it is well known that Euclidean geometry is a special case of the theory of Hermitian operators in Hilbert spaces"—Dieudonné

It has become fashionable to criticize formal treatments of mathematics in the current post-constructivist phase of mathematics education research as well as to point to the shortcomings and failings of New Math. However the New math period was crucial from the point of view of sowing the seeds of reform in school curricula at all levels in numerous countries aligned with the United States in the cold war period as well as initiated systemic attempts at reforming teacher education. In fact many of the senior scholars in the field today owe part of their formative experiences as future mathematicians and mathematics educators to the New Math period. However the fundamental ideas of New Math were based on the massive work of the Bourbaki. The Bourbaki were a group of mostly French mathematicians, who began meeting in the 1930s and aimed to write a thorough (formalized) and unified account of all mathematics, which could be used by mathematicians in the future (see Bourbaki 1970). The highly formal nature of mathematics textbooks following the Boubarki tradition is evident in examples such as the "bourbakized" definition of $2^{\sqrt{2}}$ as the supremum of a suitable set of rational powers of 2 (Sriraman and Strzelecki 2004).

It is commonly agreed that New Math was one of outcomes of the Bourbakists, who systematized common threads from diverse mathematical domains into a coherent whole and influenced policy makers in the 1950's and early 1960's to attempt an analogous logical math program for schools (Pitman 1989). The mathematical community became interested in mathematics education stimulated by both their war-time experiences as well the new importance that mathematics, science, and technology had achieved in the public eye. This resulted in mathematicians and experts from other fields designing curriculums for schools (e.g *School Mathematics Study Group or SMSG*). One must understand that the intentions of mathematicians like Max Beberman and Edward Begle was to change the mindless rigidity of traditional mathematics. They did so by emphasizing the *whys* and the *deeper structures* of mathematics rather than the *hows* but it in hindsight with all the new findings on the difficulties of changing teacher beliefs it seems futile to impose a top-down approach to the implementation of the New Math approach with teacher "upgrades" via summer courses on university campuses. The global impact of New Math as a result of the *Royaumont Seminar* is *not one* that is well documented in the literature, particularly the huge influence it had on changes in mathematics content taught in schools (Dieudonné 1961; Moon 1986). Given no mention of this seminar in extant mathematics education histories constructed (Bishop 1992; Kilpatrick 1992) we deem it important to fill this gap in the literature.

The prominent French mathematician and Bourbakist, Jean Dieudonné played a significant role in initiating these changes. The Royaumont Seminar was held in 1959 in France (OEEC 1961), organized chiefly by the Organization for European Economic Co-operation and attended by 18 nations (including Germany, France and Italy), catalyzed New Math into a more global "Western" phenomenon. Dieudonné, who chaired one of the three sections of this seminar, made his famous declaration that "Euclid must go" (see Dieudonné 1961). The subsequent report released in 1961 led to the systematic disappearance of Euclidean geometry from the curricula of most participating countries. In fact the original SMSG materials included Euclidean geometry. Thus, the influence of prominent Bourbakists on New Math in Europe was instrumental in changing the face of mathematics education completely.

In spite of the history presented in the previous section, not every prominent French mathematician was enamored by New Math's promise of modernizing mathematics. In his address to the 2nd International Congress of Mathematics Education, René Thom (1923–2002) was unsparing in his criticism:

> Mathematics having progressed, so we are told, considerably since Cauchy, it is strange that in many countries the syllabuses have not done likewise. In particular, it is argued that the introduction into teaching of the great mathematical 'structures' will in a natural way simplify this teaching, for by doing so, one offers the universal schemata which govern mathematical thought. One will observe that neither of these two objectives is, to be precise 'modern' nor even recent. The anxiety about teaching mathematics in a heuristic or

> creative way does not date from yesterday (as Professor Polya's contribution to congress thought shows). It is directly descended from the pedagogy of Rousseau and one could say without exaggeration that modern educators could still be inspired by the heuristic pedagogy displayed in the lesson that Socrates gave to the small slave of Menon's.[2] As for the advancement of mathematics which would necessitate a re-organisation of syllabuses, one needs only point to the embarrassment and uncertainty of modern theorists in dating the alleged revolution which they so glibly invoke: Evariste Galois, founder of group theory; Weierstrass, father of rigour in analysis; Cantor, creator of set theory; Hilbert, provider of an axiomatic foundation for geometry; Bourbaki, systematic presenter of contemporary mathematics, so many names are called forth at random, and with no great theoretical accuracy, to justify curricular reform. (Thom 1973, pp. 194–195)

One direct inference to be made from Thom's criticism was that mathematics reform initiated by New Math was not anchored in any mathematics education/didactics research per se, and was simply being done on a whim by invoking individuals in history who had made seminal contributions to mathematics which resulted in what is now called modern mathematics. Parallel to the birth of *Mathematikdidaktik* as a separate academic discipline in Germany in the 1970's, in France the society of researchers engaged in DdM was founded in the 1973. Guy Brousseau and Gérard Vergnaud are widely regarded as the founders of this society. Among the systemic research initiatives engaged in by this group is the adaptation of the specific grammar (definitions, theoretical constructs etc.) from Brousseau's (1997) theory of didactical situations (TDS) as a theoretical framework in mathematics education research, as well as the significant extension of Brousseau's theory by Yves Chevallard into the anthropological theory of didactics (ATD). These theoretical developments are further described in the next sections of the chapter. The role of serendipity in the evolution of ideas is seen in the fact that Brousseau adapted Bachelard's (1938) theory of epistemological obstacles into the setting of education, particularly the researching of teaching. Vergnaud, a student of Jean Piaget, on the other hand, was extending Piaget's work on cognitive psychology into a theory of learning, and his work is widely known in the literature.

Theory of Didactical Situations (1970–): Guy Brousseau's (1981, 1986, 1997, 1999a, 1999b) theory of didactical situations (TDS) is a holistic theory. Simply put TDS studies the complexity inherent in any situation involving the interaction of teacher-student-content (a three-way schema). Broadly speaking TDS attempts to single out relationships that emerge in the interaction between learners-mathematics—the milieu. The milieu typically includes other learners, the concepts learned by students as well as prior conceptual machinery present in the student's repertoire and available for use. The interesting thing about TDS is the fact that its conceiver began his career as an elementary school teacher in Southwestern France and attributed the foundational ideas of his theory to his formative experiences as a practicing teacher in the 1950's. Much later, when reflecting on the origins of his theory Brousseau (1999b) stated:

> This three-way schema is habitually associated with a conception of teaching in which the teacher organizes the knowledge to be taught into a sequence of messages from which the

[2]Thom is referring to the Fire Dialogues of Plato.

> student extracts what he needs. It facilitates the determination of the objects to be studied, the role of the actors, and the division of the study of teaching among sundry disciplines. For example, mathematics is responsible for the content, the science of communication for the translation into appropriate messages, pedagogy and cognitive psychology for understanding and organizing the acquisitions and learnings of the student.

At this juncture, we will also point out the fact that Brousseau developed TDS with some practical ends in mind, that is, to ultimately be able to help teachers redesign/engineer mathematical situations and classroom practice so as to facilitate understanding. Again, in Brousseau's (1999b) own words:

> The systematic description of didactical situations is a more direct means of discussing with teachers what they are doing or what they could be doing and of considering a practical means for them to take into account the results of research in other domains. A theory of situations thus appeared as a privileged means not only of understanding what teachers and students are doing, but also of producing problems or exercises adapted to knowledge and to students, and finally a means of communication between researchers and with teachers.

TDS is very much a constructivist approach to the study of teaching situations (Artigue 1994) and "founded on the constructivist thesis from Piaget's genetic epistemology" (Balacheff 1999, p. 23). It could be thought of as a special science complete with theoretical considerations and methodological examples for a detailed study of mathematics teaching within an institutional setting. TDS includes a specific grammar with specific meanings for terms such as *didactical situation, adidactical situation, milieu, didactical contract* etc. Taken in its entirety TDS comprises all the elements of what is today called situated cognition. The only difference is that TDS is particularly aimed at the analysis of teaching and learning occurring within an institutional setting. The most significant contribution of TDS to mathematics education research is that it allows researchers from different theoretical traditions to utilize a uniform grammar to research, analyze and describe teaching situations. One example of this possibility is seen in the recent special volume of *Educational Studies in Mathematics* (2005, vol. 59, nos. 1–3) in which 9 empirical studies conducted in Europe used the "classroom situation" (in its entirety) as the unit of analysis. Such a uniform approach was made possible largely because of the utilization of Brousseau's TDS and Chevallard's ATD (next section) as the common theoretical framework. However the research sites at which these studies were conducted were predominantly in France, and Spain, which have historically used these frameworks.

Anthropological theory of Didactics (ATD): The Anthropological theory of didactics (ATD) is the extension of Brousseau's ideas from within the institutional setting to the wider "Institutional" setting. Artigue (2002) clarifies this subtlety by saying that:

> The anthropological approach shares with "socio-cultural" approaches in the educational field (Sierpinska and Lerman 1996) the vision that mathematics is seen as the product of a human activity. Mathematical productions and thinking modes are thus seen as dependent on the social and cultural contexts where they develop. As a consequence, mathematical objects are not absolute objects, but are entities which arise from the practices of given institutions. The word "institution" has to be understood in this theory in a very broad sense ... [a]ny social or cultural practice takes place within an institution. Didactic institutions are those devoted to the intentional apprenticeship of specific contents of knowledge. As regards the objects of knowledge it takes in charge, any didactic institution develops specific

> practices, and this results in specific norms and visions as regards the meaning of knowing or understanding such or such object. (p. 245)

The motivation for proposing a theory much larger in scope than TDS was to move beyond the cognitive program of mathematics education research, namely classical concerns (Gascón 2003) such as the cognitive activity of an individual explained independently of the larger institutional mechanisms at work which affect the individuals learning. Chevallard's (1985, 1992a, 1992b, 1999a) writings essentially contend that a paradigm shift is necessary within mathematics education, one that begins within the assumptions of Brousseau's work, but shifts its focus on the very origins of mathematical activity occurring in schools, namely the institutions which produce the knowledge (K) in the first place. The notion of didactical transposition (Chevallard 1985) is developed to study the changes that K goes through in its passage from scholars/mathematicians → curriculum/policymakers → teachers → students. In other words, Chevallard's ATD is an "epistemological program" which attempts to move away from the reductionism inherent in the cognitive program (Gascón 2003). Bosch et al. (2005) clarify the desired outcomes of such a program of research:

> ATD takes mathematical activity institutionally conceived as its primary object of research. It thus must explicitly specify what kind of general model is being used to describe mathematical knowledge and mathematical activities, including the production and diffusion of mathematical knowledge. The general epistemological model provided by the ATD proposes a description of mathematical knowledge in terms of mathematical *praxeologies* whose main components are *types of tasks (or problems), techniques, technologies*, *and theories*. (pp. 4–5)

It is noteworthy that the use of ATD as a theoretical framework by a large body of researchers in Spain, France and South America resulted in the inception of an International Congress on the Anthropological Theory of Didactics (held in 2005 in Baeza, Spain and Uzès, France, 2007). The aim of this particular Congress and future congresses is to propose a cross-national research agenda and identify research questions which can be systematically investigated with the use of ATD as a framework. The French tradition, while theoretically well anchored has not completely addressed its impact on practice, and as Skovsmose (2005) has pointed out, has turned a blind eye to the socio-political reality of teachers and students. Have other regions (UK and North America in particular) made strides in this important area?

Impact of Theories on Practice

Why do we need theories? Various roles are given including those by Silver and Herbst (2007) and Hiebert and Grouws (2007):

> Theories are useful because they direct researchers' attention to particular relationships in, provide meaning for the phenomena being studies, rate the relative importance of the research questions being asked, and place findings from individual studies within a larger context. Theories suggest where to look when formulating the next research questions and provide an organizational scheme, or a story line, within which to accumulate and fit together individual sets of results. (p. 373)

They also discuss the challenges and benefits of developing theories in that theories "allow researchers to understand what they are studying" (p. 394).

Similar sentiments are also found in Cobb's (2007) chapter in the Second National Council of Teachers of Mathematics Handbook. Cobb writes "Proponents of various perspectives frequently advocate their viewpoint with what can only be described as ideological fervor, generating more heat than light in the process" (p. 3). He questions the "repeated attempts to that have been made in mathematics education to derive instructional prescriptions directly from background theoretical perspectives." "The difficulty is not with the background theory, but with the relation that is assumed to hold between theory and instructional practice." ... "central tenets of a descriptive theoretical perspective are transformed directly into instructional prescriptions" (pp. 3–5). Cobb (2007) then argues that "research, theorizing, and indeed philosophising as distinct forms of practice rather than activities whose products provide a viable foundation for the activities of practitioners" (p. 7) and sees mathematics education as a design science and proposes criteria analogous to those outlined by other proposals of reconceptualising the entire field as a design science (Lesh 2007, 2008; Lesh and Sriraman 2005). Cobb (2007) suggests we adapt ideas from a range of theoretical sources and act as Bricoleurs. Bricologe "offers a better prospect of mathematics education research developing an intellectual identity distinct from the various perspectives on which it draws than does the attempt to formulate all-encompassing schemes" (Cobb 2007, p. 31). Schoenfeld (2000) proposes standards for judging theories, models and results in terms of their descriptive and explanatory powers. He writes "researchers in education have an intellectual obligation to push for greater clarity and specificity and to look for limiting cases or counterexamples to see where the theoretical ideas break down" (p. 647).

Closing Summary

Mathematics education as a field of inquiry has a long history of intertwinement with psychology. As evidenced in this chapter various theories and philosophies have developed often in parallel that have informed and propelled the field forward. One of its early identities was as a happy marriage between mathematics (specific content) and psychology (cognition, learning, and pedagogy). However as we have attempted to show in this chapter, the field has not only grown rapidly in the last three decades but has also been heavily influenced and shaped by the social, cultural and political dimensions of education, thinking, and learning. In a sense, the past of the field is really in front of us, meaning that having experienced repetitive cycles of development with some consolidation and syntheses of different theories and philosophies, the time has come to move forward. The social, cultural and political dimensions are more important and prescient for the field given the fact that there exists an adequate theoretical and philosophical basis. However to some the socio-political developments are a source of discomfort because they force one to re-examine the fundamental nature and purpose of mathematics education in relation to society. The social, cultural, and political nature of mathematics education is undeniably important for a host of reasons such as:

- Why do school mathematics and the curricula repeatedly fail minorities and first peoples in numerous parts of world?
- Why is mathematics viewed as an irrelevant and insignificant school subject by some disadvantaged inner city youth?
- Why do reform efforts in mathematics curricula repeatedly fail in schools? Why are minorities and women under-represented in mathematics and science related fields?
- Why is mathematics education the target of so much political/policy attention?

The traditional knowledge of cultures that have managed to adapt, survive and even thrive in the harshest of environments (e.g., Inuits in Alaska/Nunavut; Aboriginals in Australia, etc.) are today sought by environmental biologists and ecologists. The historical fact that numerous cultures successfully transmitted traditional knowledge to new generations suggests that teaching and learning were an integral part of these societies, yet these learners today do not succeed in the school and examination system. If these cultures seem distant, we can examine our own backyards, in the underachievement of African-Americans, Latino, Native American, the Aboriginals in Australia and socio-economically disadvantaged groups in mathematics and science. It is easy to blame these failures on the inadequacy of teachers, neglectful parents or the school system itself, and rationalize school advantage to successful/dominant socio-economic groups by appealing to concepts like special education programs, equity and meritocracy (see Brantlinger 2003). We tackle these issues more in depth in the concluding chapter of this book (see Sriraman, Roscoe, English, chapter Politizing Mathematics Education: Has Politics Gone Too Far? Or Not Far Enough?).

In the second edition of the *Handbook of Educational Psychology* (Alexander and Winne 2006) Calfee called for a broadening of horizons for future generations of educational psychologists with a wider exposure to theories and methodologies, instead of the traditional approach of introducing researchers to narrow theories that jive with specialized quantitative (experimental) methodologies that restrict communication among researchers within the field. Calfee also concluded the chapter with a remark that is applicable to mathematics education:

> Barriers to fundamental change appear substantial, but the potential is intriguing. Technology brings the sparkle of innovation and opportunity but more significant are the social dimensions—the Really Important Problems (RIP's) mentioned earlier are grounded in the quest of equity and social justice, ethical dimensions perhaps voiced infrequently but fundamental to the disciple. Perhaps the third edition of the handbook will contain an entry for the topic. (Calfee 2006, pp. 39–40)

Five years ago, Burton (2004) proposed an epistemological model of "coming to know mathematics" consisting of five interconnecting categories, namely the person and the social/cultural system, aesthetics, intuition/insight, multiple approaches, and connections, grounded in the extensive literature base of mathematics education, sociology of knowledge and feminist science, in order to address the challenges of objectivity, homogeneity, impersonality, and incoherence. Burton (2004) proposed we view mathematics as a socio-cultural artifact, part of a larger cultural system as opposed to the Platonist objective view. In order to substantiate her epistemological model, Burton drew extensively on the work of Lakoff and Núñez (2000) on embodiment, and Rotman (2000) on semiotics. Roth's (2009) *Mathematical Representation at the Interface of Body and Culture* presents a convergence of numerous ideas that have intersected with mathematics education but have not been properly followed up in terms of their significance for the field. Roth's book fills a major void in our field by giving a masterfully edited coherent synthesis of the ongoing work on embodiment and representations in mathematics, grounded in cultural-historical activity theory. It presents a strong case that much progress can and has been made in mathematics education.

Similarly the work of researchers within the networking theories group founded by Angelika Bikner-Ahsbahs presents huge strides forward in ways in which theoretical frameworks can be made to interact with one another in a systemic fashion. The ZDM issue on networking theories is a significant product of value to the field (see Prediger et al. 2008b). Another development is Anna Sfard's (2008) *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Sfard's book holds the promise of removing existing dichotomies in the current discourses on thinking, and may well serve as a common theoretical framework for researchers in mathematics education. Last but not least critical mathematics education has been gaining momentum in the last two decades with a canonical theoretical basis in neo-Marxist and/or the Frankfurt schools of philosophy—it remains to be seen whether more mathematics education researchers embrace the centrality and importance of this work. Skovsmose (2005) discusses critically the relations between mathematics, society and citizenship. According to him, critical mathematics give challenges connected to issues of globalization, content and applications of mathematics, mathematics as a basis for actions in society, and of empowerment and mathematical literacy (mathemacy). In earlier writings Skovsmose (1997, 2004) argued that if mathematics education can be organized in a way that challenges undemocratic features of society, then it could be called critical mathematics education. However he lamented that this education did not provide any recipe for teaching!

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