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Conjecturing via reconceived classical analogy

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1. Introduction

Analogical reasoning, in general, refers to the ability to perceive and construct corresponding structural similarity in objects whose surface features are not necessarily similar (Richland, Holyoak, & Stigler, 2004, p. 37). Research has shown that children can use analogical reasoning to adapt to new novel contexts (Holyoak & Thagard, 1995), transfer representations across contexts (Novick, 1988), understand and solve word problems (Reed, Dempster, & Ettinger, 1985; Bassok, 2001), and solve comparison problems (English, 1999). However, unlike science education scant attention has been given in mathematics education to analogical reasoning as a concept-development skill (English & Sharry, 1996, p. 138).

Knowledge construction or concept development in mathematics education has been described as repeated abstraction (Boero et al., 2002). Analogical reasoning plays an important role in the process of abstraction through investigating similarities and discerning structures (Sriraman, 2004). However, there is growing concern that despite everyday usage, learners are unable to transfer analogical reasoning to learning situations (Dunbar, 2001; Lobato, 2003; Leech, Mareschal, & Cooper, 2008). Thus, mathematics education would benefit greatly from studies on the use of analogical reasoning as an instructional device for conjecturing in discourse rich mathematics classrooms. Our present study adds to the missing gaps in the existing line of research on analogical thinking.

2. Conceptual framework

2. 1. Conjecturing by analogies

Polya (1954) emphasized the value of conjecturing via analogies in mathematics learning, particularly how mathematicians utilized analogies when discovering new concepts or new problem solving methods. Numerous examples abound in the history of mathematics of mathematicians like Euler, Newton and others taking “daring” steps, i.e., reasoning by induction or analogy, such as Newton’s formula for binomial expansions with rational powers, and Euler’s astonishing closed sum of the infinite series $1 + ¼ + 1/9 + 1/16 +…+ 1/n^2 +…$, in which he applied finite methods to infinite cases. However Euler confidently guessed the sum of the series to be $\pi^2/6$, did not attribute this coincidence to chance and boldly conjectured that the sum of this series was indeed $\pi^2/6$, not to mention the fact that he later proved his conjecture to be true (Polya, 1954, pp.95-96).

Lakatos (1976) highlighted how conjecturing by analogies can contribute to mathematical discovery and some case studies utilizing a Lakatosian framework (Sriraman, 2006) indicate that conjecturing by analogies occurs in mathematics classrooms although this is more an exception than the rule. For instance in empirical studies based on observation of mathematics classrooms, Richland et al. (2004, p. 55) found only 2% of analogies were produced by students. As the researchers interpreted, “teachers may be failing to provide an important learning opportunity for students by maintaining control over the reasoning process” (Richland et al., p. 58). This warrants the necessity of research on facilitating analogy use in mathematics classrooms.

2. 2. Re-conception of *classical analogy*

There are three types of analogies that have been used in mathematics education: *classical analogy*, *problem analogy*, and *pedagogical analogy* (English, 2004). Among the three aforementioned types of analogies, *problem analogy* and *pedagogical analogy* were widely used as heuristics in mathematics learning. However, *classical analogy* problems are mainly applied measurements of intelligence and reasoning ability development exercises, rather than used as a domain-specific cognitive skill in mathematics learning. Whereas previous research on analogical reasoning in mathematics education has focused much on the use of *problem analogy* and *pedagogical analogy,* few researches have (e.g., English & Sharry, 1996; Zaslavsky, 2008; Lee, 2009; Cañadas, Deulofeu, Figueiras, Reid, & Yevdokimov, 2007; etc.) considered analogical reasoning in a more broad sense. As a result of this instructional approach, learners were able to voice diverse views on relational similarities. One new perspective on the transfer of learning called *actor-oriented transfer* (Lobato & Siebert, 2002), supports the use of analogical reasoning in a broader sense rather than just the application of the aforementioned three analogies. Based on empirical evidence, Lobato and Siebert claim that teacher should develop better understanding how students find or construct relational similarity to design instruction rather than just distinct surface or structure in similarity from his/her point of view and give static tasks to promote transfer (pp. 112-113).

Following *actor-oriented transfer* perspective, we changed a classical analogy problem into a more dynamic one by providing learners with opportunity to choose a target object and its property. Let’s start with analysing the task used in Cañadas et al. (2007):

> Given a triangle ABC and a point P inside the triangle construct the three lines from each vertex A, B, C to the point P. What can you say about the relationships between the lines and the sides of the triangle? (p. 59)

Note that the last part of the task asks students to make a conjecture about the relationships between “some” lines and sides of a triangle without giving any specific objects or relations. To solve the problem, learners need to select “some” lines and look for “a relation”, which is similar to what they already know using their language. For example, Cañadas et al. (2007) give the conjecture: “if two lines cut the sides in a 2:1 ratio, the third one will, too” (p. 60). This conjecture was based on an analogy from knowledge of what happens when lines cut the sides at midpoint. Though the conjecture is false, it can be assumed productive if it is used as a mediation to investigate other new properties of a triangle.

The freedom to create a target object and a relation is not apparent within a classical analogy form because under the condition, the relation between "A" and "B" and "C" are all pre-determined by experts. As a result, there is not much room for students to engage in similarity-making activity that are familiar to them. By leaving the base object's attribute "B" and the target "C" and its corresponding property "D" in a classical analogy to the discretion of a learner, the learner may manage his/her personal journey of knowledge construction based on learner's own similarity-making action and its results as each participant did in hypothetical journey of discovery described in Lakatos (1976). Therefore, an analogy that requires learners to authentically search for the "B", "C", and "D" terms of a classical analogy, whichwe call *Open Classical Analogy* (hereinafter referred to as OCA), could be used as an instructional device in mathematics classrooms.

2.3. OCA type problems

Requiring learners to look for "B", "C", and "D" terms, we can design three types of OCA problems depending which terms are given by teachers and are open for learners' discretion. First, a teacher may opt to start with "A" and "C" terms of a classical analogy:

> Consider a tetrahedron is similar to a triangle. Conjecture a property of a tetrahedron analogous to the property of a triangle you already know. Explain your answer.

A teacher could also choose to provide students with another two terms, "A" and "B":

> The interior angles of a triangle sum to two right angles. Conjecture a similar property of any geometric figure that is analogous to a triangle. Explain your answer.

Finally, it is also possible to start with providing students with only "A" term:

> Select a geometric figure that is analogous to a triangle and make a conjecture its property that is analogous to a property of a triangle. Explain your answer.

In general, we can modify many problems in school mathematics to OCA type problem. For example, the aforementioned problem given in Cañadas et al. (2007) can be rephrased in the form of OCA problem where "A" and "B" terms are given:

> Three median lines of a triangle meet in a single point. Select other line and conjecture a property that is analogous to the given property. Explain your answer.

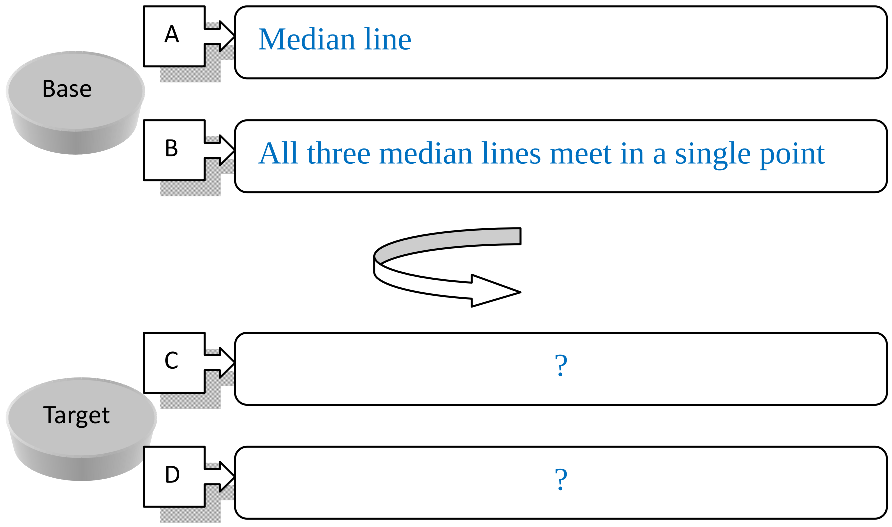

Fig. 1. The second type of OCA problem

2.4. Nature of knowledge construction through OCA problem solving

Three epistemic actions elaborated in Hershkowitz et al., (2001) such as "recognizing", "building-with", and "constructing" are expected to happen during knowledge construction via OCA problem solving. In the OCA problem solving process, "recognizing" involves recalling conceptual aspects of a base object and "building-with" consists of combining existing conceptual aspects to select a target or to make a conjectural statement. "Constructing" consists of assembling knowledge artefacts to produce a new structure of a target object as well as a base object. For the case shown in Fig. 1, one should recognize that "median line of a triangle" is related to the special point called "midpoint." Then, one should combine these ideas together and build "a relational concept" such as each median line "*divides*" one side of a triangle "*into two equal parts*." The relational concept can be found in other familiar object of a triangle, i.e., angle, then, one can construct a similar statement as a conjecture, i.e., "all three angle bisectors meet in a single point." Thus, conjecturing a similar property of a target object heavily depends on one's building ability of a relational concept of a base object.

For the first and the second types of OCA problems, parts of actions such as finding property of a base object or looking for an analogous object are not necessary. We hypothesize that there are three different levels of OCA problem solving. The students in the first level might focus only on "*the perceptual or surface similarity*" of objects as Gentner & Rattermann (1991) pointed out and use it as a similarity idea for OCA problem solving. Second level of OCA problem solving involves recognition or creation of concept, which we call "*transitional similarity*," which is relational but not connected to justification context. The highest level of OCA problem solving is supported by "*relational similarity*", at which learner can construct new concepts or new properties of a target object in the form of conjectures to be verified.

OCA problems present learners with the chance not only to analyze a base object but also to construct a target object based on recognized or produced similarity. The similarity ideas learners depend on during an OCA problem solving differs from the ideas used in a classical analogy problem solving in two aspects. First, it is a learners' creation not an experts', which prompts learners to employ knowledge they already know. Second, similarity ideas utilized during an OCA problem solving are often *provisional similarity* ideas, which can be replaced by better ideas or revised. In this case, learners are more involved in identifying or investigating resemblances among geometric objects with their own knowledge. OCA problem usage can also serve as instructional devices to activate learners' prior knowledge and provoke learners to expand their knowledge scope.

Reflective abstraction is facilitated with perspective changing on constructs by analogical reasoning (English &Sharry, 1997). This "perspective changing" has been expressed as *objectification* by Dubinsky (1991) *or reification* by Sfard (1991): *objectification or reification of relational similarity*. An instructional approach, called "*focusing phenomena*", employed in Lobato, J., Ellis, A. B., & Muñoz (2003), is useful to highlight some mathematical activities to facilitate objectification or reification of relational similarity. As the teacher did in Lobato et al., (2003), "directing students' attention to *particular aspects* of mathematical activity" (p. 3, emphasis added) would enable students to become aware of a hidden relation and utilize them while weakening others.

3. Method

3.1. Participants

Three high-achieving 8th graders (Kim, Park, and Choi, 14 years old) were selected as study participants to investigate the possibility that students can manage knowledge construction during OCA problem solving. Kim, Park, and Choi were selected out of seventeen students within the year long talent development program run by Seoul National University. They had previous exposure to the solving of linguistic analogies and geometric pattern analogies from attempting IQ tests.

One three-hour-long lesson where students were engaged in an OCA problem solving and hourlong clinical interview both before and after lessons with the selected students was conducted. The instructor, whom we call Seo, worked in middle school for 7 years had an interest in creativity development through student knowledge construction, participated in this research. She followed the teaching perspective that students are able to build meaningful conjectures when challenged to create meaningful analogies through focusing on relational similarities. She had previous experience in such research oriented activities. She was aware of the goal for OCA problem solving as "seeing the structure of the base object" and "build a structure of new object (the target)" based on the emergent structure of the base object. For the conceived goal, she encouraged students to "revisit" what they have learned or discovered on the base object and to "project them" to the assumed target object and its property. Her intervention can be characterized by "reflective discourse" (Cobb & Boufi,1997, p. 258) claim.

3.2. The task

Among the three types of OCA problems, we chose the most open one. In particular, students are given only “A” term of a classical analogy and required to create the other three terms. The base object for the task was a triangle, learned since students’ primary school years.

> Select a geometric figure that is analogous to a triangle and make a conjecture its property that is analogous to a property of a triangle. Explain your answer.

Students are familiar with many aspects of a triangle such as definition, measurement attributes, properties in the form of mathematical propositions. Thus, students were expected to activate their prior knowledge of a triangle to find relational concepts as similarity ideas and make a good conjecture of the selected target object. The task was modified to other types of OCA giving additional information depending on students’ reactions. For example, in case students did not use prior knowledge, then, the teacher gave a target object or one specific property (See Sec. 2.3. for details). Students’ answers were evaluated, whether it is complete or not;correct or incorrect, if it had any potential to reveal the hidden structure of a triangle or to create a new structure of their choices for target objects.

3.3. Data

Data about students’ previous knowledge of triangle were collected from the one hour-long pre-lesson interview. Students were asked to explain “what they know about a triangle” including definitions, attributes, and properties. That was to make sure that they had relevant knowledge to construct conjectures of geometrical objects selected to be target objects. Then, students were encouraged to recall “how they have learned about triangle” since primary school. Finally, students were asked to think about “how the process of constructing mathematical knowledge looks like.” The famous “eureka” episode of Archimedes was used as a motivating example of analogical reasoning when he found the way to examine if base metal had been substituted for gold in the crown that had been commissioned by his king.

Extensive videobased data was collected on similarity-making activities including processes while solving the given OCA problem.. Students were given self-monitoring and self-controlling responsibilities, which is supported and detailed by Zimmerman (1998, 2000, and 2002), while they generated meaningful conjectures. Students were required to consolidate analogy based on their own similarity-making activities. In sum, the OCA task solving for this study was planned according to the following stages: (a) OCA introduction as a thinking format or a discovering tool of mathematics, (b) target object and property searching through their own similarity-recognizing or similarity-making, (c) evaluation and improvement of initial analogies, and (d) consolidation of analogies through justification.

Finally, data about what the participants reflected on their similarity-making activities and conjecturing were collected from the hour-long post-lesson interview. While watching the recorded activities, students were asked to explain what they “newly” came to know about triangle, geometric objects that they selected as target objects, and properties and so on. In addition, the subjects were asked to explain “how and why” they focused on particular objects or properties. The data from the post-lesson interview were to investigate the possibility for students to be aware of the goal and the use of OCA problem solving in knowledge construction.

4. Analysis and Findings

Three subjects, Park, Kim, and Choi were identified to have relevant and qualitatively the same prior knowledge about geometric objects such as triangle, polygons, and polyhedrons by the data from pre-lesson interview. One exceptional difference was Park's ability to use formal language when explain properties of triangle. He used symbols and conventional expressions for proving. Thus, three subjects were assumed to be ready for OCA task solving. However, the overall pictures of their performances were quite different as described in this section. This section is organized into two parts: (a) emerged similarities and conjectures, and (b) reflective discourses to shift student attention while OCA problem solving.

4.1. Emerged similarities and conjectures

4. 1. 1. Perceptual or surface similarities

Right from the onset students leapt directly into the process of looking for perceptually similar figures without reflecting on or careful classification of the properties of a triangle(Gentner and Rattermann, 1991). For example, one subject, Park, suggested the target could be a figure made by transforming the three sides of a triangle into three curves because the figures would "look similar." Then, Park created target figures by rotating a triangle. The two suggested targets generated through this technique were a conic and another figure made by rotating the triangle - the longest side of the triangle was used as the axis of rotation. Kim and Choi, chose a pyramid as the target figure, justifying their choices with the claim that a pyramid contains a triangle as in Kim's comment "I *can see* triangles in both figures."

As predicted, perceptual similarities were not linked to meaningful conjectures. Students meandered around shapes of some geometric objects not conceptual relations or properties.

4.1.2. Transitional similarities

The teacher intended to engage students in focusing phenomena through enlightening students' critical sense of their similarity-making activities. While recalling definitions of geometric figures and concepts about a triangle, students focused on some "common words" in definitions of geometric figures as follows:

- Quadrilateral is a <u>polygon</u> with *four* <u>vertices</u>, *four* sides, and one face
- N-polygon is a <u>polygon</u> with *n* <u>vertices</u>, *n* sides, and one face
- Tetrahedron is <u>a polyhedron</u> with *four* <u>vertices</u>, *six* sides, and *four* faces.

Kim claimed "any polygon" could be a potential target object since all polygons contain similar elements– "all are *composed of* vertices, sides, and faces." Choi, focussed on elements of a triangle, asserted that a tetrahedron is analogous to a triangle since both figures are constructed in accordance with the same principle: a triangle is made by "connecting" one side to a vertex that is not on the side, and a tetrahedron is made by "connecting" one face to a vertex that is not on the face.

Choi, after investigating the number of elements in geometric figures, characterized a triangle as the figure with “the smallest number of vertices, edges, and faces” among plane figures. He concluded a tetrahedron to be an appropriate target object because it has “the smallest number of vertices, edges, and faces among solid figures.” Furthermore, he asserted that an arbitrary polyhedral could “decompose into” a tetrahedron just as an arbitrary polygon could “decompose into” a triangle. Noteworthy is his focus on concepts “unrelated” to a triangle that could be used in analogies. Choi explained his analogy between a triangle and a tetrahedron using the idea that both figures “do not have” parallel lines. He also added that the “non-existence” of a diagonal line was common to both a triangle and a tetrahedron.

It was interesting to see students revealed the new aspects of a triangle described as above that they have not focused before, but the emerged relational concepts such as “non-existence of diagonal line” or “the smallest number of vertices and edges” were only re-conceptualizations of the known object. There were no conjecture generations, so no justification context emerged. Hence, they were classified as *transitional similarities*.

4.1.3. Relational similarities

After a few transitional similarities emerged, the teacher, again, encouraged students to share their target object choices and explain the reasons behind their choices after having analyzed a triangle, polygons in general, and a tetrahedron during the creation of an analogy for the OCA task. Target choices by other students made Park focus on each element in a triangle separately. His investigation led to two separate categorizations of triangles: triangles that could be characterized by the relative lengths of their sides and triangles that could be characterized by the measure of their interior angles (See Fig. 3).

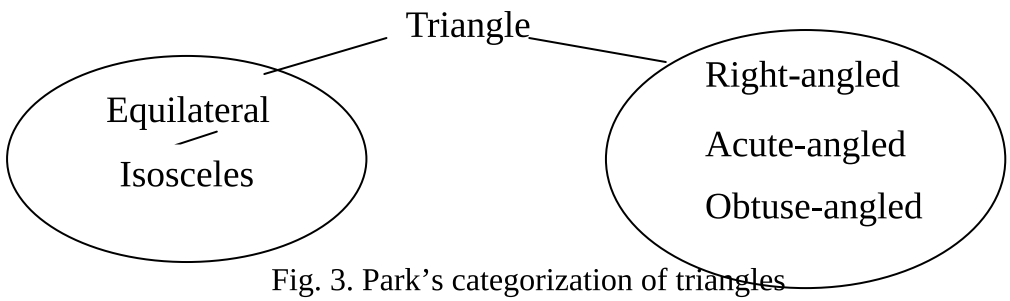

Fig. 3. Park’s categorization of triangles

Park attempted a similar *classification* of quadrilaterals based on the relative lengths of sides and the relative measure of interior angles (See Fig. 4). He claimed the same “classification criteria”; i.e., the relative lengths of sides and measure of interior angles can support similarity between a triangle and a quadrilateral.

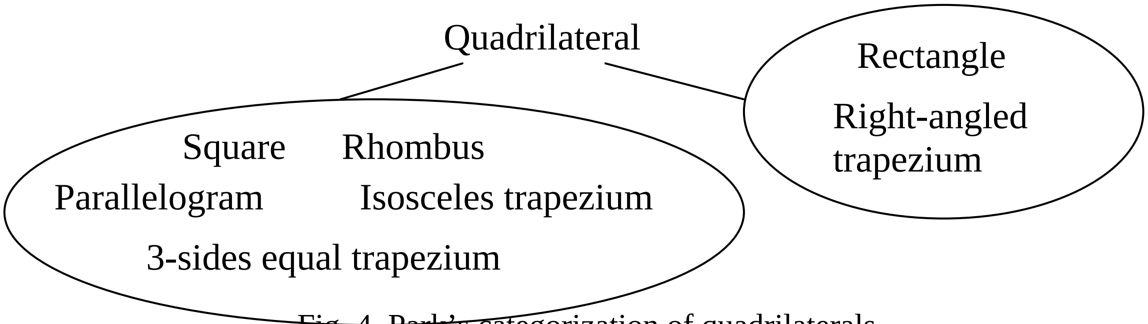

Fig. 4. Park's categorization of quadrilaterals

Similar classifications of tetrahedrons were attempted by Park and Kim. Park argued that a regular tetrahedron corresponded to an equilateral triangle. He also coined terminology like "right-angled tetrahedron" for a tetrahedron that contained three right-angled triangles joined at one vertex and "isosceles tetrahedron" for a tetrahedron that contained three isosceles triangles sides.

The two foci used in the previous classification activities of geometric figures; i.e., side and angle, were analyzed from a different perspective. Kim argued that there were two different kinds of "geometric actions" related to lengths of sides and angle measures. The first was addition or division of the lengths of segments, and the second was addition or division of angle measurements. His three statements, which are very similar in structure, are listed below.

- *The interior angles* of a triangle always *add up* to 180 degrees
- *The interior angles* of a quadrilateral always *add up* to 360 degrees
- *The interior angles* of a tetrahedron always *add up* to "*some*" degrees

He also constructed the following three similarly structured statements.

- *The exterior angles* of a triangle always *add up* to 360 degrees
- *The exterior angles* of a quadrilateral always *add up* to 360 degrees
- *The exterior angles* of a tetrahedron always *add up* to "*some*" degrees

The final statements from both analogies are excellent conjectures despite being incomplete. After examining his analogies, Kim pondered the degrees needed to complete his last two statements. Kim and Choi eventually reached a conclusion regarding angle degrees. The two subjects reached quite different conclusions (See section 4. 2.3).

While attempting to generate analogies for the OCA task, Choi focused on area and its measurement in a triangle. He found explanation about the volume measurement of a tetrahedron based on how he learned about the area measurement of a triangle. During the post-task interview, he explained his thought process.

> I noticed similarity that I never realized before. I took the idea that *triangles between parallel lines* with a common base have the same area; then, I analogically determined that *tetrahedrons between parallel planes* with a common base "*must*" have the same volume. It was an exciting moment when I found out my analogy was right.

The circumcenter and the incenter of a triangle were used to solve the OCA task, too. Kim after tying to project these centers onto a quadrilateral easily found that not all quadrilaterals have those centers. He produced the following statements.

- Three perpendicular side bisectors of a triangle meet at a single point and four perpendicular side bisectors of a square and a rectangle meet at a single point
- Three angle bisectors of a triangle intersect at a single point and four angle bisectors of a square and a rhombus intersect at a single point

Both Kim and Choi tried to generate an analogy between a triangle and a tetrahedron using the existence of two centers. They summarized their work with the statements below.

- Three perpendicular side bisectors of a triangle meet at a single point and four "unknown" perpendicular bisectors of a tetrahedron meet at a single point
- Three angle bisectors of a triangle intersect at a single point and four "unknown" angle bisectors of a tetrahedron meet at a single point

Kim and Choi, independently proceeded to generate an analogy for the OCA task using similar foci. Next is a list of their conjectures regarding the conditions for a quadrilateral and a tetrahedron:

- The lengths of four sides and two angle measures determine a quadrilateral
- If the four sides of a quadrilateral have the same length as the four sides of another quadrilateral and one angle has the same measurement, the two quadrilaterals are congruent"
- If four corresponding sides of two quadrilaterals are in proportion and their angles have the same measurement, the two quadrilaterals are similar
- The lengths of six sides determine a tetrahedron
- If the six sides of a tetrahedron have the same length as the six sides in another tetrahedron, the two tetrahedrons are congruent
- If six corresponding sides of two tetrahedrons are in proportion, the two tetrahedrons are similar

4. 2. Focusing phenomena while conjecturing

Several attention shifts across foci appeared while students made conjectures for the OCA task. Mainly, the teacher promoted attention shifts by encouraging students to share their conjectures.

4.2.1. Focusing on the structure and goal of OCA

Though the teacher, Seo, specifically addressed the necessity of selecting a target, a triangle property, and conjecturing of a target property that corresponded to the selected triangle property, Park could not comprehend the task at hand. The teacher invited students to consider the structure of the OCA after Park selected a target figure created by transforming the three sides of a triangle into three curves based on perceptual similarity. The discourse between Seo and Park is given below.

45. Park: A triangle has three sides. (Pointing to his drawing of a triangle on the blackboard) I transformed this to this. (Pointing to his drawing of a figure with three curves)
46. Seo: Park, what is your analogy, then?
47. Park: It is this figure.
48. Seo: What analogy are you making with this figure? Can you explain your analogy using your solution?
49. Park: I mean, this figure is similar to a triangle.
50. Seo: Can you explain why it is similar to a triangle?
51. Park: Because I can see shape similarity.
52. Seo: So, *how would you write out your analogy*?
53. Park: Wait a minute, I only have the A and C terms.
54. Seo: What else do you need?
55. Park: The “B” and “D” terms.
56. Seo: What would “B” be here?
57. Park: It has to be a property of a triangle.

Seo encouraged students to reflect and share emerged relations between a triangle and assumed targets to facilitate students to be aware of sprit or goal of OCA as “*knowledge expansion or extrapolation by conjecturing*.” After examining his initial constructs, Kim stated, "pondering figures analogous to a triangle, I learnt I need to pay greater attention to good mathematical properties. There must be a good property that can *bind* a triangle to the new figure and, at the same time, simultaneously *distinguish* it." Choi assessed his initial target objects as being too superficial, and as a result, deemed them useless. Emerged and potential similarities and found in this vignette.

121. Seo: On what basis did you evaluate your construct and analogy? Would you mind sharing your *criteria* for analogy evaluation?

122. Kim: The target *should be interesting and meaningful and new*.

123. Seo: What do you mean by that?

124. Kim: My initial figure, the one I created by rotation, was neither interesting nor meaningful because I couldn't find anything related to the properties of a triangle. I mean, I should have been able to use something I already knew about a triangle.

125. Choi: I had similar thoughts. If I do nothing to *generate* a new object, I can't encounter anything interesting.

4.2.2. Focusing on relations originated from geometric actions

Being directed to reflect on their initial similarity-making activities by the teacher, Kim and Choi became aware of the necessity to recognize *relational concepts(similarities)* necessary for conjecturing. In other words, students shifted their main attention from the whole image (virtual and conceptual) of a base object to the conceptual elements and the relations between them originated from geometric actions such as "finding minimal conditions" to judge determinance, congruence, and similarity. Below are part of Kim's comments.

> The conditions for congruence (of a triangle) are SSS, SAS, and ASA, so I considered the conditions for congruence in other geometric figures. There must be conditions for congruence, for instance, SSSS can definitely work. Hmm, no, it might be SASSS. No, it still doesn't work. It is quite different from a triangle. What if I change a quadrangle to a parallelogram? Wait a minute. SAS is a condition for congruence. Yes, I can find all the counterparts for the conditions for congruence of a triangle, but not for a scalene quadrilateral.

He also maintained that if the lengths of six sides are given, a tetrahedron is determined. Noteworthy is Kim's word choice. He often used expressions such as "definitely", "doesn't work", "for the case of", "if", "explain", "can find all the counterparts", and so on, which are related to justification of created mathematical conjectures. Likewise, Choi also developed similar vocabulary showing the occurrence of spontaneous interaction between conjecturing and justifying. Hence, it is clear that Kim and Choi were consciously aware of the mechanism to find or make relations in geometry focusing on special actions and the necessity to mathematically justify them.

4.2.3. Focusing on corresponding

There was an interesting discussion on corresponding issue when Kim and Choi make a conjecture about the sum of interior angles in a tetrahedron:

498. Seo: Would you please share what you've discovered?

(Kim and Choi come to the blackboard and write their findings for all to see)

Table 3
Conjectures by Kim and Choi

Kim	Choi
The sum of the interior angles of a tetrahedron is *720 degrees*	The sum of the interior angles of a tetrahedron is *constant*

499. Seo: How interesting! Look everyone, Kim and Choi have just written two different conjectures. Let's invite Kim and Choi to *enlighten us on the reasons for their conjectures*? Who'd like to go first?

500. Kim: A tetrahedron always has four faces. The faces are all triangles. The sum of the interior angles of a triangle is 180 degrees, and there are four triangles. Therefore, the sum of the interior angles of a tetrahedron is 720 degrees.

501. Seo: Any questions for Kim?

502. Choi: I have a question. How do you know the angles you are talking about are indeed the interior angles of a tetrahedron?

503. Kim: (Thinking it over)

504. Seo: Choi, would you mind elaborating? Why are you asking that question?

505. Choi: I mean, there is only one kind of angle in a triangle, but, but, there are, as I found, *three different kinds of angles* in a tetrahedron. I think we have to decide which angle is the counterpart for the interior angle of a triangle before we make an analogy.

506. Seo: Yes, I see. That's an interesting point. Does anyone want to respond to Choi's comment?

507. Choi: Among the three differing types of angles, the angle composed of three faces at each vertex is the one we're looking for, I think, because the interior angle of a triangle was defined as the angle composed of two sides at each vertex.

508. Kim: I agree, but how do we measure that angle? It seems impossible.

509. Choi: There must be a way; I can't explain it right now.

510. Seo: Then, can you explain your claim that it is constant?

511. Choi: (After drawing a triangle inscribed in a circle) If you draw a triangle in a circle (pointing to his drawing of a triangle, especially the points of the triangle that touch the rim of the circle), the sum of the inscribed angle equals half of the central angle. That's half of 360 or 180, so the sum of the interior angles of a triangle is always 180, which never changes, I mean, stays constant. (Draws a tetrahedron inscribed in a sphere) Also, if you measure this angle, (points to the angle formed from the three faces of the vertex) this angle can be considered an inscribed angle, so it's half of the central angle even though I don't know the actual degree of the central angle. Hence, I can analogically say the sum of these angles is constant just like the triangle case.

Justification vocabulary and ideas used in the above discourse relate to students' corresponding conjectures. Kim's explanation for why the sum of the interior angles is 720 degrees focused on justification of the conjecture itself (line 500). He concerned himself with verifying the reason the sum totals 720 degrees based on the fact that all tetrahedral are composed of four triangles and the angle sum of each triangle is 180 degrees. Choi questioned the analogy justification employed by Kim. For Choi, the angles that Kim measured did not correspond to the angles of a triangle, which totaled 180 degrees. He proclaimed it necessary to define a 3-dimensional interior angle that corresponds to a 2-dimensional interior angle. At the post-lesson interview, Kim and Choi made references to the dihedral angle sum though they could not use it to complete the justifications for their conjectures and analogies.

5. Discussion and Conclusions

The central theme of the paper concerned students' conjecturing via OCA problem solving as knowledge construction. The detailed analysis of the constructs and processes of several similarity-making and conjecturing activities supported the following conclusions.

First, relational similarity between objects is not recognized by simply recalling or applying but by activating learners' use of their own prior knowledge through intentional focusing. In this study, employing the format or the structure of a classical analogy was effective to facilitate students' attention shift from surface similarity to transitional and relational similarity. As described in Sec. 4.1.1, the subjects tended to look for perceptual similarity without reflecting on or careful analyzing the properties of a triangle in the beginning stage. For example, Park's initial target figure was made by transforming three sides of a triangle into three curves. This selection is based on a kind of "make up" strategy – change a part of the given problem situation without any proper understanding or goal, which was coined term in Lavy and Bershadsky (2003). Park's conjecturing and analogizing unpacks Lavy and Berhadsky's thesis. in his realization that he overlooked the necessity to think about some relations within the concept of a base object and came to pay attention on the relationships among conceptual elements not perceptual overall feature of a base object.

While paying attention to relationships between elements, students investigated the meaning or the results of specific geometric actions such as "*dividing* sides or angles into equal parts", "*classifying* geometric objects", "*finding* angle measures sum", "*investigating* invariant attributes such as area of triangles between parallel lines", "*looking for* minimal condition to judge determinance, congruency or similarity of geometric figures", "*deciding* on object existence or generality", and so on. All the relations students found or created came from the awareness of these geometric actions. For example, Kim noticed that the interior angle measures of a triangle "add up" to 180 degrees as described in Sec. 4.1.3. He, then, mapped this to a tetrahedron, and made a statement: The interior angles of a tetrahedron "add up to some" degrees. This statement provoked him to pursue the exact degrees as well as its justification as discussed in Sec. 4.2.3. To sum up, OCA problem solving gives a learner opportunity to experience "intentional" recognizing or creating relations while actively reflecting or re-doing geometric actions that he or she already did on a base object.

Second, conjecturing can be linked to knowledge construction only when it is done with student awareness of the necessity and the method of justification. Surface similarity between geometric objects was not linked to conjecturing activities as mentioned in Sec. 4.1.1. When students concerned only on surface similarities, they stuck to find a target object rather than conjecturing any properties. Transitional similarities such as "non-existence of diagonals" in a triangle and a tetrahedron were quite interesting to see since they are also new relations for learners. However, transitional similarities are only about descriptions not about conjecturing and its validation, either. Relational similarity, on the other hand, led students to conjectures to be tested or justified. For Kim, as detailed in Section 4.1.3, the judgment condition for determination of a quadrangle was not the exact projection from, so called, the *SSS* condition of a triangle. Realizing the complicated relationships between the components and the need to verify similarity in relations, he actively analyzed hypothetical situations and necessary conditions. In Sec. 4.2.3., Choi also concerned about conjecturing and justification simultaneously, neither just transformed some part of known properties to build a conjecture nor throwing a sentence without any responsibility of its validation. Thus, his conjecture from OCA problem solving could have potential to become a theorem, which can naturally form *cognitive unity* (Garuti, Boero, & Lemut, 1998).

Third, knowledge structure of a familiar object can be discovered by intentionally alienating it. In fact, the base object in this study, a triangle is so familiar for learners to think differently as pointed out in Mariotti & Fischbein (1997). Interestingly, the subjects often investigated of a triangle by converse reasoning. For example, although Kim knew the proof for constructing a circumcenter of a triangle, when inferring about the circumcenter of a tetrahedron, as described in Section 4.2.3, he relied on the reasoning about the target object and applied this reasoning to a triangle conversely. Once inferences had been made on his target object, the process was applied to the base object conversely to reflect knowledge he had from a different perspective. Therefore, seeing or constructing the structure of a base object through OCA problem solving can be regarded as a heuristics to view familiar object from a new perspective or deal with it in a new way.

Fourth, Aspiration of innovation is one of the main driving forces of knowledge construction through conjecturing. As described in the empirical context section, the class teacher assumed that students could build knowledge via conjecturing if they were fully engaged in OCA problem solving. With this assumption, she directed reflective discourse on the process and the constructs of OCA problem solving in a sense that Cobb & Boufi (1997, p. 258) suggested. This was helpful for Kim and Choi to grasp the goal of OCA solving and to implement meaningful conjecturing as reported in the result section. While pursuing innovation, Kim and Choi, kept continuously searching for interesting targets and related properties to explore mathematical meaning. This aspiration for innovation became their evaluation criterion for conjectures. For example, Kim's comment after reflecting on his initial analogy, "I don't like it because it is too similar to the base object," explicitly reveals his pursuit of innovation (See Sec. 4.2.1 for more details). Including this, commentary such as "I like it because it is very new", "I use it because it comes from the most important property", "I like it because it's simple, yet, cool like the known property", and "It looks great since I had never thought about it like that before" reveal personal feeling about the constructs. I would call this "mathematical taste" that were considered hidden important factor of productivity in mathematics research (Nirenberg, 2002; Tao, 2007).

However, the teacher's efforts to focus on the goal or the spirit of OCA relatively didn't work for Park. In Park's journey of conjecturing, it was hard to say that he was pursuing of innovation though he produced some properties of a quadrilateral and a tetrahedron. Park's relatively poor performance can be interpreted as partly due to his lack of productive mathematical taste. This phenomenon warrants the necessity to develop an alternative way of awakening the purpose or the role of OCA as well as to develop appropriate way for cultivating student productive mathematical taste.

The limitation of this research resides with the participants. Student participants constitute high-achieving or more advanced learners. The study group was not constructed of differing level achievers..

Reasoning and conjecturing by analogy is a fundamental human trait. One encounters excellent examples of this propensity to "analogize" in ancient Greek philosophy. If an ancient Greek philosopher were asked: why do we create analogies? The answer would simply be to create a framework by which we could better understand the dimensions of human experience (Sriraman, 2005). An important finding of English (2004) was that teachers must understand analogies themselves and know how to use them effectively (and also know which analogies are appropriate and which aren't when it come to their use). They sometimes have to make the relationships explicit for the child. The OCA framework we have developed through reflective discourse practices by the teacher Seo, illustrate that analogies arising in mathematics are quite different from those arising in a discipline such as the life sciences where spontaneous analogies work well because children have a much larger a priori linguistic base, whereas in mathematics children's pre-existing knowledge base is limited. This necessitates that both practitioners and researchers are sensitive to the major role that the knowledge base plays in the use of analogies for mathematics learning. The present study makes an important contribution for following this line of mathematical thinking initiated by the likes Newton, Euler and Polya. Further research is needed on a more typical classroom-type group of students. Another limitation relates to the limited content area. The focus of this paper was a triangle. Further studies involving a variety of OCA problems in different content areas are encouraged to verify the possibility of including OCA in mathematics learning. Finally, it will be necessary to not only identify but also clarify the kinds of norms or teaching interventions essential for effective integration of OCA into mathematics lessons.

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