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Commentary on the Cognitive aspects of Early Algebraization

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Abstract: *In this commentary to the nine chapters in the cognitive section of early algebraization, we synthesize and critically discuss common themes found in them such as components of non-formal algebraic thinking, the purported dichotomy between arithmetic and algebra; meaningful arithmetic, and generalizing ability, among others using the frameworks of William Brownell, Ernst Haeckl and Jean-Baptiste Lamarck.*

Keywords: abstraction; early algebraization; generalization; generalized arithmetic; recapitulation; cultural evolution; longitudinal algebra projects; history of mathematics

Introductory Remarks

The nine chapters that comprise section 3 of this book consist of 4 revised ZDM articles and 5 new chapters, which together explore the cognitive aspects of early algebraization. As spelled out in the preface of the first volume of the Advances in Mathematics Education series, "the purpose of a commentary is not only to elucidate ideas present in an original text, but…[t]o take them forward in ways not conceived of originally" (Sriraman & Kaiser, 2010, p. vi). Therefore, our aim in this commentary is to synthesize common themes found in the nine chapters of this section and to discuss the significance of the ideas and claims made in the chapters through different theoretical lenses. We add that a commentary cannot occur *in the void* meaning that it needs to be anchored in what is already existent in the literature. So for the sake of better inter-textuality in the existent literature (Sriraman, 2010) and preserving the integrity of what is already known in the field, we refer the reader to two related books released by Springer in the MEL[2] series, namely vol.43 focused on *Educational Algebra* (Filloy, Rojana and Puig, 2008) and vol.22 which explored *Perspectives on School Algebra* (Sutherland, Rojano, Bell and Lins, 2001) that contain complementary perspectives to those found in this commentary and book. The critically reflective question we asked ourselves is whether these nine chapters really represented *advances* in the domain of early algebraization or whether they were simply a regurgitation or recycling of old ideas in new clothing?

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[2] Mathematics Education Library Series, Springer.

Early algebraization versus meaningful arithmetic

There is no rigid definition of early algebraization per se found in the literature, which is acknowledged by the editors of the book (Cai & Knuth). Instead the term is used to refer to algebraic thinking initiated from the early grades (elementary school) via the use of student's informal knowledge, different modes of representations, pattern activities that lead to generalization, and building on natural linguistic and cognitive mechanisms by reflection, verbalization, articulation and sense making (Greer, 2008; Kaput, 1999). The question that arises in our mind when reading these chapters that address cognitive aspects of early algebraization is whether they are really addressing the notion of "meaningful arithmetic" as proposed by William Brownell (1895- 1977)? Amongst mathematics educators there is consensus that the traditional separation of arithmetic from algebra hinders the development of algebraic thinking in the later grades. Hence it makes sense to push for developing algebraic ideas in the earlier grades through various activities such as pattern finding and relationships in the curriculum without the formalism of notation or symbols, which collectively is termed *early algebraization.* We do not dispute this, however are these ideas really advancing our field, or are they a recycling of previous work? This is the focus of our discussion in this section.

Brownell (1947) was the chief spokesperson for the "meaningful" arithmetic. Meaningful arithmetic refers to instruction which is deliberately planned to teach arithmetical meanings and to make arithmetic more sensible to children through its mathematical relationships. Brownell categorized the meanings of arithmetic into the following groups.

1. A group consisting of a large list of basic concepts. For example: meanings of whole numbers, of fractions, ratios and proportions etc.
2. A second group consisting of arithmetical meanings which includes understanding of fundamental operations. Children must know when to add, subtract, multiply, and divide. They must also know what happens to the numbers used when a given operation is performed.
3. A third group of meanings consisting of principles that are more abstract. For example: relationships and generalizations of arithmetic, like knowing that 0 serves as an additive identity, the product of two abstract factors remains the same regardless of which factor is use as a multiplier, etc.
4. A fourth group of meanings that relates to the understanding of the decimal number system, and its uses in rationalizing computational procedures and algorithms. (Brownell, 1947).

Meaningful arithmetic is "deliberately planned to teach arithmetical meanings and to make arithmetic sensible to children through its mathematical relationships". Brownell emphasized that learning arithmetic through computations required continuous practice. He suggested that teaching meaningful arithmetic would reduce practice time, encourage problem solving, and develop independence in students and remarked that there was lack of research in 1947, on teaching and learning arithmetic meaningfully. He also doubted that quantitative research could address questions in the area of teaching and learning arithmetic. One could say that Brownell was clairvoyant for his time, emphasizing qualitative research in the age of behaviorism and he was noted for his use of a variety of techniques for gathering data, including extended interviews with individual children and teachers, as well as his careful, extensive and penetrating analyses of those data (Kilpatrick, 1992).

Sixty three years after Brownell's recommendations, we see ample research evidence in the nine chapters of this section that the algebra underpinning the arithmetic operations can be made accessible to students without the need for sophisticated notation that impedes understanding. Two chapters report on the results of longitudinal research projects. For instance, the chapter by Britt and Irwin on *Algebraic thinking with and without algebraic representation: a pathway to learning* reports on substantial gains in introducing algebraic thinking within arithmetic in the New Zealand Numeracy Project with 4-7 year olds, with the gains remaining in a follow up large scale study with the same students at the age of 11-12 on a 21-item test consisting of various algebraic properties. Similarly Cai, Moyer, Wang and Nie report the findings of the LieCal[3] Project on the algebraic development of middle school students (grades 6-8) exposed to reform based curricula (CMP) versus a traditional curricula (non CMP). They found that CMP students performed better on generalization tasks than their peers in the traditional (non CMP) track. The CMP curriculum used a functional approach and emphasized conceptual understanding whereas the non CMP curriculum took a structural approach and emphasized procedural understanding. Interestingly enough in this rather massive study, both groups performed equally well on equation solving!

Generalized Arithmetic, Generalizing, Generalization

Amongst mathematicians, algebra is typically understood as generalized arithmetic and some would qualify this characterization by saying it is *meaningful* generalized arithmetic. In other words there is a significant shift (abstraction) from talking about basic arithmetic operations concretely, to talking about operations arbitrarily. There is an even greater shift (abstraction) when one views numbers as a special case of polynomial evaluations. However mathematics educators would object to such characterizations as being top-down, i.e., viewing particulars through general lenses as opposed to discovering the general via particulars (Mason, 1992; Sriraman, 2004) and argue that algebraic *thinking* which fosters the elements of abstraction and generalization inductively from the early grades through contextual and rich mathematical activities is more valuable than the deductive approach.

The theme of generalization and generalizing ability recurs in the chapter by Cooper and Warren (*Years 2 to 6 students' ability to generalize: models, representations and theory for teaching/learning*), and the chapter by Rivera and Becker (*Formation of pattern generalization involving linear figural patterns among middle school students: Results of a three-year study*). The former chapter by Cooper and Warren is the only one in this entire section to cite the work of Dienes (1961) and push his ideas of multiple embodiments fostering abstraction and generalization, by exploring the interrelations between generalization and verbal/visual comprehension of context, and arguing for the value of communicating commonalties seen across different representations. The latter chapter delineates the different nuances of generalization such as constructive versus deconstructive generalizations, and the role of sociocultural mediation in fostering/facilitating verbalization of generalizations in teaching

[3] Longitudinal Investigation of the Effect of Curriculum on Algebra Learning

experiments. The different patterning activities found in these chapters serve well to validate the claims made by these authors.

Many of the chapters in the cognitive section try to move beyond or work their way around the debate that Algebra is generalized arithmetic. However Radford's chapter on *Grade 2 students' non-symbolic algebraic thinking* tackles this issue head on by trying to draw a clear distinction between what is arithmetic and what is algebraic. This is accomplished by focusing on non-symbolic means of expression, going back to the ideas of other ways (e.g., linguistic, gestural, figural) of expressing algebraic notions as opposed to expressing them symbolically. This chapter brings to the foreground the need to have a historical perspective on the development of mathematical ideas (algebraic or otherwise). Diophantus' *Arithmetica* is a misnomer and deals with solving algebraic equations with integer co-efficients. Does that mean Algebra as we define it today preceded Arithmetic or that they developed concurrently, just as the notion of Integrals preceded the rigorous development of convergence of sequences and limits but the approximation of areas and volumes contained the intuitive notions that were later formalized? We leave these questions for the reader to ponder over. But we foray briefly into the role of history of mathematics in mathematics education before we conclude our commentary. In doing so, we highlight the need to change the dominant discourses present in arithmetic-algebra dichotomy, and adopt a historical-cultural perspective in addition to a strictly cognitive perspective. The commentary offered by Balacheff (2001) on *Perspectives in School Algebra* offers another perspective on the didactical dilemma/distinction between symbolic arithmetic and algebra.

From Haeckel to Lamarck to Early Algebraization

The emphasis on the important role of the history of mathematics in mathematics education research is one that has been sporadically addressed by the community. Furinghetti & Radford (2002) traced the evolution of *Haeckel's (1874) law of recapitulation* from the point of view that parallelism is inherent in how mathematical ideas evolve and the cognitive growth of an individual (Piaget & Garcia, 1989). In other words the difficulties or reactions of those who encounter a mathematical problem can invariably be traced to the historical difficulties during the development of the underlying mathematical concepts. The final theoretical product (namely the mathematical theorem or object), the result of the historical interplay between phylogenetic and ontogenetic developments of mathematics, where phylogeny is recapitulated by ontogeny, has an important role in pedagogical considerations (Bagni, Furinghetti, & Spagnolo, 2004; Furinghetti & Radford, 2002; Sriraman & Törner, 2008). Psychological constructs as well as the study and formation of intellectual mechanisms are not as tenable as the clearly dated and archived transformations of mathematics in its historical development. Further, the apparent free use of Haeckel's recapitulation theory as the link between the psychological and historical domains is in need of re-examination. It is a well known fact that Haeckel's law in its original form was rejected by the community of biologists and has been transformed numerous times by some, over the last 100 years to better explain the relationship between phylogeny and ontogeny in different species. However in mathematics education we are referring to psychological recapitulation or the use of recapitulation metaphors to explain the evolution of mathematical ideas (see Furinghetti & Radford, 2008). A neo-Lamarckian perspective needs to be introduced into the recapitulation discussion for the following reasons. Recapitulation cannot be applied or transposed directly to the study of didactical problems because it does not take into account the influence of experience (or more broadly culture). Just as Jean-Baptiste Lamarck proposed in vain to his peers in 1803, that hereditary characteristics may be influenced by culture, we need to take into account how culture influences the mutation of historical ideas. Gould (1979) wrote that

> *Cultural evolution has progressed at rates that Darwinian processes cannot begin to approach…[t]his crux in the Earth's history has been reached because Lamarckian processes have finally been unleashed upon it. Human cultural evolution, in strong opposition to our biological history, is Lamarckian in character. What we learn in one generation, we transmit directly by teaching and writing. Acquired characters are inherited in technology and culture. Lamarckian evolution is rapid and accumulative…*

The teaching and learning of mathematics bears strong evidence to this Lamarckian nature. Indeed, what took Fermat, Leibniz and Newton a 100 years is taught and often digested by students in one year of university Calculus. Any higher level mathematics textbook is a cultural artifact which testifies to rapid accumulation and transmission of hundreds (if not thousands of years) of knowledge development. So, evolutionary epistemologists have now begun to accept the fact that for humans, cultural evolution in a manner of speaking is neo-Lamarckian (Callebaut, 1987, Gould, 1979).

The neo-Lamarckian perspective becomes evident in the research reported by Izsak in his chapter on *Representational competence and algebraic modeling*. In this chapter, the reader confronts students with a significant and complex "substrata of knowledge" and criteria developed from the culture of learning they were previously exposed to that resulted in them inventing their own private inscriptions and criteria for evaluating representations. Developing notational competence in students, i.e., the ability to adapt their inscriptions to fit the external representations they are confronted with, and progress to the necessity for uniformity in notation for the purposes of communication is an inference we draw from this chapter. The history of algebra as seen in the development of the theory of equations shows that the notation (or private inscriptions) developed by Galois on general methods for the solvability of equations by radicals, took his peers a significant amount of time to decipher, nearly 15 years after his death (until 1846), when Louiville published it in his journal commenting on Galois' solution, " . . . as correct as it is deep of this lovely problem: Given an irreducible equation of prime degree, decide whether or not it is soluble by radicals".

In the chapter by Ellis on *Algebra in the Middle School*, the informal or cultural notion of comparing quantities is used to scaffold the building of quantitative relationships and the sophisticated idea of covariation in younger students. In spite of creating relevant, contextual and quantitatively rich situations, Ellis reports that the "students unique interactions with an interpretations of real world situations remind us that these contexts are not a panacea. Introducing a quantitatively rich situation does not guarantee that students build quantitative relationships…[s]tudents may focus on any number of features in a problem situation…[t]herefore teachers play an important role in shaping a classroom discussion,…"

Knuth et al in their reprint of the 2005 ZDM article suggest that students "pre-algebraic" experiences are crucial in laying the foundation for the study of more formal algebra. These authors view the middle school grades as the link from early algebraic reasoning to more complex and abstract reasoning. Five years later, the studies from New Zealand, Australia, Canada and the U.S reported in this section, indicate that algebraic thinking can be cultivated from the very early grades on if teachers are cognizant of non-symbolic modes of reasoning. It is a testament to our development as a field that the seemingly divergent ideas of Brownell, Haeckel and Lamarck converge in our understanding of early algebraization, which can no longer be viewed as a neologism but a clearly defined term!

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