Integrals don't have anything to do with discrete math, do they?

Kal M. Karply*

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Abstract

Most people think of integration as belonging to the realm of continuous mathematics, far removed from its 'polar opposite', discrete mathematics. We present a few examples illustrating the ubiquity of integration, even in the discrete world. Our goal is to convince students to keep up on their analysis, even if they lean more to the mathematically discrete. One never knows when an integral might rise seemingly out of nowhere and play an interesting role in a discrete problem.

Keywords: gamma function, perfect matching, graph, bipartite graph, rook polynomial, derangement

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1 Introduction

When asked to elaborate on my mathematical interests, I invariably include discrete mathematics in my answer, which is usually met with a puzzled look. At that point, I'll often mention continuous mathematics as the branch most people are more familiar with, through examples such as distance, temperature, and time. This usually elicits a nod, at which point I explain that discrete math is a sort of 'polar opposite'.

This is all well and good if one is simply trying to give a layperson an approximate taste for the two mathematical flavours. But too often in my teaching, I've found that students drawn to discrete mathematics are quick to turn their backs on the continuous. Many forget their calculus training, and most try to avoid analysis courses altogether, as if their decision to take the discrete road somehow excuses them from any further meanderings down continuous paths.

The goal of this article is to convince mathematics students and their teachers that the worlds of discrete and continuous mathematics are not so very far apart. Though they may frequently feel like polar opposites, there are also times when they join to become one, like antipodal points in projective space. Therefore, any serious study of discrete math ought to include a healthy dose of the continuous, and vice versa.

*correspondence to `mark.kayll@umontana.edu`

Teaser

To whet the reader's appetite, we begin with a problem that can be solved using ideas in this article.

Problem 1 Give a *combinatorial proof* that

$$\int_0^\infty (t^3 - 6t^2 + 9t - 2)e^{-t}\, dt = 1; \tag{1}$$

i.e., count something that, on one hand, is easily seen to number the left side of (1) and on the other, the right.

For a delightful treatment of such proof techniques, see [4].

On first blush, Problem 1 may appear to be ill-posed—a *combinatorial* proof of an integral identity—what could this mean? Please read on.

Entities: continuous and discrete

After introducing our objects of study, we reveal some of their connections in Sections 2 and 3. In an attempt to make the article mostly self-contained, we include an Appendix (Section 4) containing some basic facts and other curiosities about these objects, including a solution to Problem 1.

Integrals

As hinted in the title, our continuous objects derive from integrals. Euler's *gamma function* can be defined, for $0 < x < \infty$, by

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{2}$$

One can check that this improper integral converges for such x; see, e.g., [2]. In fact, Γ need not be confined to the positive real numbers—it is possible to extend its definition so that Γ becomes a meromorphic function on the complex plane, with poles at the origin and each negative integer; see, e.g., [1, 11]—but we'll restrict our attention to the stated domain.

We also introduce certain 'probability moments' via integrals. For integers $n \geq 0$, the *nth moment* (of a Gaussian random variable with mean 0 and variance 1, i.e., a standard normal random variable) is defined by

$$\mathcal{M}_n := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty t^n e^{-t^2/2}\, dt.$$

These integrals also converge (see, e.g., [8]), and though probability language enters in their naming, we won't be making much use of this connection. Since we do need the fact that $\mathcal{M}_0 = 1$ (see Theorem 7), we present a standard proof of this identity in the Appendix (Lemma 8).

Graphs

Our discrete objects are graphs and several of their basic substructures. While we shall assume that the reader is familiar with these, we nevertheless introduce a few required elementary notions. Any standard graph theory text should suffice to close our expositional gaps; see, e.g., [5].

Recall that a *graph* $G = (V, E)$ consists of a finite set V (of *vertices*), together with a set E of unordered pairs $\{x, y\}$ (*edges*) with $x \neq y$ and both of $x, y \in V$. (Such graphs are called *simple* in [5].) A graph is *complete* if, for each pair x, y of distinct vertices, the edge $\{x, y\}$ appears in E. Figure 1 depicts the complete graphs with $1 \leq |V| \leq 6$ and introduces the standard notation K_n for the complete graph on $n \geq 1$ vertices.

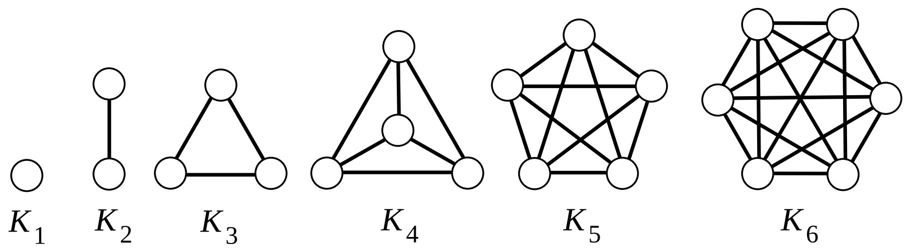
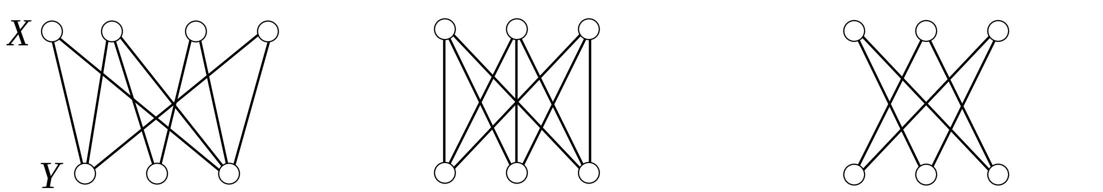


Figure 1: complete graphs on up to six vertices

The second graph family of primary interest in this article is the collection of *bipartite* graphs G, i.e., those for which the vertex set admits a partition $V = X \oplus Y$ into nonempty sets X, Y such that each edge of G is of the form $\{x, y\}$, with $x \in X$ and $y \in Y$. One often forms a mental picture of a bipartite graph by imagining two rows of dots—a row for X and a row for Y—together with a collection of line segments xy joining an $x \in X$ to a $y \in Y$ whenever $\{x, y\} \in E$. The next two definitions depend on a fixed positive integer n. The bipartite graph $(X \oplus Y, E)$ for which $|X| = |Y| = n$ and E consists of all n^2 possible edges between X and Y is called *complete bipartite* and denoted by $K_{n,n}$. A bipartite graph $(X \oplus Y, E)$ with $|X| = |Y| = n$ is called a *spanning subgraph* of $K_{n,n}$. Figure 2 depicts a few small bipartite graphs.



(a) a bipartite graph (b) the complete bipartite graph $K_{3,3}$ (c) a spanning subgraph of $K_{3,3}$

Figure 2: bipartite and complete bipartite graphs

A first brush between continuous and discrete

For the gamma function (2), it is easy to check that $\Gamma(1) = 1$, and integration by parts yields the recurrence

$$\Gamma(x + 1) = x\Gamma(x), \tag{3}$$

valid for positive real numbers x. By mathematical induction, it follows that each non-negative integer n satisfies $\Gamma(n+1) = n!$; i.e., the gamma function generalizes the factorial function to the real numbers.

Given this generalization, a natural first question to ponder might be: What values does Γ take on at half-integers? The reader might enjoy showing that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{4}$$

and then using (3) to prove that

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$$

whenever n is a non-negative integer. (Corollary 9 in the Appendix provides a key step in this exercise.) The ease in determining Γ at half-integers belies the dearth of known exact values; for example, $\Gamma(1/3)$ and $\Gamma(1/4)$ are unknown—see [11].

What good, we might ask, is a continuous version of the factorial function? One answer is that a careful study of Γ can be used to establish Stirling's Approximation for $n!$:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \tag{5}$$

published by James Stirling in 1730. The estimate (5), involving two of the most famous mathematical constants and invoking only basic algebraic operations, is no doubt beautiful. Moreover, it is useful any time one wants to gain insight into the growth-rate of functions involving factorials. For example, using (5), one easily shows that

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}},^{1}$$

and so learns something about the asymptotics of the *Catalan numbers* $\binom{2n}{n}/(n+1)$ (see, e.g., [15] for more on this pervasive sequence).

Our real purpose is to refute the first part of this article's title, and as we move in that direction, we can't resist sharing a couple more fun facts about Γ; these, moreover, emphasize the stature of Γ in the gallery of basic mathematical functions. First, as long as neither x nor $1-x$ belongs to the 'forbidden' set $\mathcal{F} := \{1, 0, -1, -2, -3, \ldots\}$, we have

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)},$$

which generalizes (4). This 'complement formula' was first proved by Euler; see, e.g., [1, 11] for modern proofs. Second, if

$$\zeta(x) := \sum_{k=1}^{\infty} \frac{1}{k^x}$$

denotes the Riemann zeta function, then whenever $x \notin \mathcal{F}$, we have

$$\zeta(x)\Gamma(x) = \int_0^{\infty} \frac{t^{x-1}}{e^t - 1}\, dt, \tag{6}$$

[1] Here and in (5), the notation means that the ratio of the left to the right side tends to 1 as $n \to \infty$.

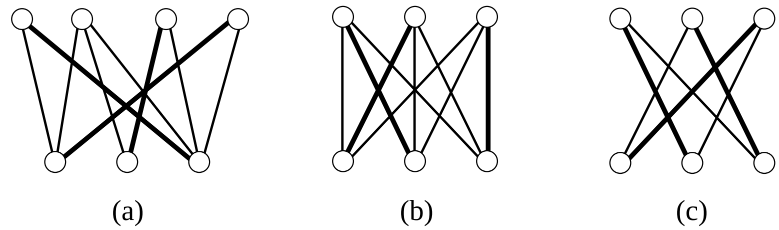


Figure 3: Matchings in the graphs of Figure 2 indicated by bold edges; those in (b) and (c) are perfect.

which bares a striking resemblance to (2); again, see [1, 11] for proofs. Because of ζ's central role in connecting number theory to complex analysis, the relation (6) opens deeper connections of Γ to number theory (beyond those stemming from the factorial function). Viewing number theory as falling within the discrete realm, we see in (6) further evidence of the fallacy of this article's title.

2 Counting perfect matchings in $K_{n,n}$

A *matching* M in a bipartite graph $G = (X \oplus Y, E)$ is a subset $M \subseteq E$ such that the edges in M are pairwise disjoint. We think of M as 'matching up' some members of X with some members of Y. If every $x \in X$ appears in some $e \in M$, and likewise for Y, then we call M a *perfect* matching. It is a simple exercise to show that if G contains a perfect matching, then $|X| = |Y|$, so that G is a spanning subgraph of some $K_{n,n}$. Figure 3 highlights one matching within each of the graphs in Figure 2.

Given a bipartite graph G, we might be interested to know how many perfect matchings it contains; we use $\Xi(G)$ to denote this number.[2] Let's warm up by asking for the value of $\Xi(K_{n,n})$; a moment's reflection shows that for each integer $n \geq 1$, the answer is $n!$. (To see this, continue to denote the 'bipartition' by (X, Y), and notice that the perfect matchings of $K_{n,n}$ are in one-to-one correspondence with the bijections between X and Y.) Since $n! = \Gamma(n + 1)$, we have proven

Proposition 2 $\Xi(K_{n,n}) = \int_0^\infty t^n e^{-t}\, dt.$ □

If we replace $K_{n,n}$ by a different bipartite graph, how must we modify the formula in Proposition 2? It turns out that a so-called 'rook polynomial' should replace the polynomial t^n.

2.1 Rook polynomials

Given a graph G and an integer r, we denote by $\mu_G(r)$ the number of matchings in G containing exactly r edges.

[2]We chose this notation because the Greek letter Xi (Ξ) resembles a perfect matching in a graph of order six.

Example 3 ***The graph $G = K_{3,3} - \{\{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}\}$***
This is the graph in Figure 2(c). Since the empty matching contains no edges, we have $\mu_G(0) = 1$; since each singleton edge forms a matching, we have $\mu_G(1) = 6$, and since G contains two perfect matchings, we have $\mu_G(3) = 2$. Fixing a vertex x, we see that there are three matchings of size two using either of the edges incident with x and three more two-edge matchings not meeting x; thus $\mu_G(2) = 9$.

Now suppose that G is a spanning subgraph of $K_{n,n}$. The *rook polynomial* of G is defined by

$$R_G(t) := \sum_{r=0}^{n} (-1)^r \mu_G(r) t^{n-r}.$$

See [10] or [14] for the etymology of this term.

Example 3 (cont'd)
Based on our observations in the first part of this example, we see that

$$R_G(t) = t^3 - 6t^2 + 9t - 2.$$

Example 4 ***Empty graphs***
If G is the empty graph on $2n$ vertices (i.e. $|V| = 2n$ and $E = \varnothing$), then

$$\mu_G(r) = \begin{cases} 0 & \text{if } r > 0 \\ 1 & \text{if } r = 0, \end{cases}$$

so that $R_G(t) = t^n$; we're getting a little ahead of ourselves, but this is the polynomial appearing in the integrand in Proposition 2.

Example 5 ***Perfect matchings***
If G consists of n pairwise disjoint edges (i.e., G is induced by a perfect matching), then one can easily see that $\mu_G(r) = \binom{n}{r}$ for $0 \le r \le n$. Thus, the Binomial Theorem shows that $R_G(t) = (t-1)^n$.

Continuing to let G denote a spanning subgraph of $K_{n,n}$, we now define its *bipartite complement* $\widetilde{G}$; this graph shares the vertex set of G and has for edges all the edges of $K_{n,n}$ that are not in G. We're ready to state a generalization of Proposition 2.

Theorem 6 (Godsil [9]; Joni and Rota [13]) *If G is a spanning subgraph of $K_{n,n}$, then*

$$\Xi(G) = \int_0^\infty R_{\widetilde{G}}(t) e^{-t}\, dt.$$

The proof of Theorem 6 is beyond our scope, but we'll present an application in Section 2.2; [7] is a recent proof. Theorem 6 generalizes Proposition 2 because the bipartite complement of $K_{n,n}$ is the empty graph on $2n$ vertices; see Example 4. Further generalizations of Theorem 6 are discussed in [10].

2.2 An application to derangements

A *derangement* σ of a set S is a permutation of S admitting no fixed points; i.e., $\sigma\colon S \to S$ is a bijection such that $\sigma(x) \neq x$ for each $x \in S$. Counting the number of derangements of a finite set is a standard problem in introductory combinatorics and probability texts. We'll let $\mathcal{D}_n$ denote the set of derangements of $\{1, 2, \ldots, n\}$ and $d_n = |\mathcal{D}_n|$. We can easily determine these parameters for the smallest few values of n; Table 1 displays the results. We leave it as an exercise to show that $d_5 = 44$ and (for the punishment gluttons) $d_6 = 265$.

n	d_n	$\mathcal{D}_n$
1	0	$\varnothing$
2	1	$\{21\}$
3	2	$\{231, 312\}$
4	9	$\{2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321\}$

Table 1: derangement numbers and their corresponding derangements for $1 \leq n \leq 4$

But what is the pattern? Perhaps surprisingly, one way to obtain a general expression for d_n is to invoke Theorem 6.

Consider the bipartite graph G obtained from $K_{n,n}$ by removing the edges of a perfect matching; say, $G = K_{n,n} - \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_n, y_n\}\}$. Notice that each perfect matching in G corresponds to exactly one derangement of $\{1, 2, \ldots, n\}$ and vice versa. Thus, $d_n = \Xi(G)$. Since the bipartite complement of G is the graph considered in Example 5, Theorem 6 implies that

$$d_n = \int_0^\infty (t-1)^n e^{-t}\, dt.$$

If we separate the integral and change variables on the first subinterval, an evaluation of Γ presents itself:

$$\begin{aligned} d_n &= \int_1^\infty (t-1)^n e^{-t}\, dt + \int_0^1 (t-1)^n e^{-t}\, dt \\ &= \int_0^\infty x^n e^{-(x+1)}\, dx + \int_0^1 (t-1)^n e^{-t}\, dt \\ &= e^{-1}\Gamma(n+1) + E_n, \end{aligned} \tag{7}$$

where we now view the second integral as an error term E_n. It turns out that E_n doesn't contribute much to d_n; since $e^{-t} < 1$ on the interval $(0, 1)$, we obtain

$$|E_n| \leq \int_0^1 \left|(t-1)^n e^{-t}\right| dt < \int_0^1 (1-t)^n\, dt = \frac{1}{n+1}.$$

This shows that for each $n \geq 1$, the error $|E_n| < 1/2$, and it follows from (7) that d_n is the integer closest to $e^{-1}\Gamma(n+1)$, i.e., to $n!/e$.

Remarks

The novel derivation of d_n presented above is due to Godsil [10]. More typical approaches (e.g. [6, 12])—that apply either the Principle of Inclusion-Exclusion or generating functions—lead to a more 'standard' expression $d_n = n! \sum_{k=0}^{n} (-1)^k/k!$ for the derangement numbers.

Via the MacLaurin series for $1/e$, the latter expression is easily seen to be equivalent to the 'integer closest to $n!/e$' description obtained above.

3 Counting perfect matchings in K_n

A *matching* M in a graph $G = (V, E)$ is defined as it is in a bipartite graph, and, as before, if each $v \in V$ is an end of some $e \in M$, then M is called *perfect*. Figure 4 displays all of the perfect matchings admitted by K_4 and some of those admitted by K_6. The bracketed numbers in Figure 4(b) indicate how many different perfect matchings result under the action of successive rotation by 60°; in this way, all $15 = 2 + 3 + 6 + 3 + 1$ perfect matchings of K_6 are obtained.

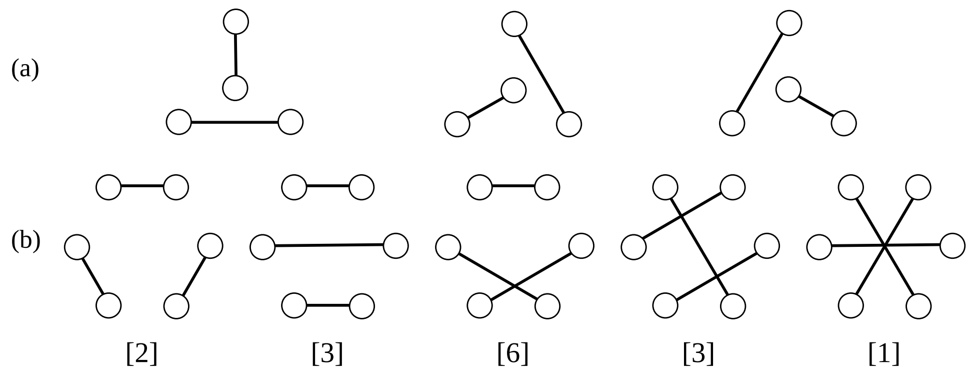


Figure 4: (a) all three perfect matchings in K_4; (b) five of fifteen perfect matchings in K_6

Following our line of questioning in Section 2, we can ask how many perfect matchings are contained in K_n. Since matchings pair off vertices, the question is interesting only when n is even; say $n = 2m$ for an integer $m \geq 1$. Let $V := V(K_{2m}) = \{1, 2, \ldots, 2m\}$. To determine a matching M, it is enough to decide, for each vertex $i \in V$, with which vertex i is paired under M. There are $(2m - 1)$ choices for pairing with vertex 1. Having formed this pair, say $\{1, j\}$, it remains to decide how to pair the remaining $(2m - 2)$ vertices. Selecting one of these, say k, there are $(2m - 3)$ choices for pairing with vertex k, namely, any member of $V \setminus \{1, j, k\}$. Continuing in this fashion and applying the multiplication rule of counting, we find that

$$\Xi(K_n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (2m-1)(2m-3)\cdots 5 \cdot 3 \cdot 1 & \text{if } n = 2m \text{ for an integer } m \geq 1. \end{cases} \tag{8}$$

Notice that the 'skip factorial' appearing in (8) can also be written as

$$\Xi(K_{2m}) = \frac{(2m)!}{2^m m!}, \tag{9}$$

which leads to an alternate way to count $\Xi(K_{2m})$: think of determining a matching by permuting the elements of V in a horizontal line (in $(2m)!$ ways) and then simply grouping the vertices into pairs from left to right. Of course, this over-counts $\Xi(K_{2m})$—by a factor of $m!$ since the resulting m matching edges are ordered, and by a factor of 2^m since each edge itself imposes one of two orders on its ends. After correcting for the over-counting, we arrive at (9) and thus have a second verification of (8).

As a final refutation of our title, we'll show that $\Xi(K_n)$ can also be expressed as an integral.

Theorem 7 (Godsil [9]; Azor et al. [3]) $\Xi(K_n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^n e^{-t^2/2}\, dt.$

Proof. The right side of the identity is the moment $\mathcal{M}_n$. We'll determine a couple of initial values of $\mathcal{M}_n$, together with a recurrence. Since $\mathcal{M}_0$ is the area under the curve for the probability density function of a standard normal random variable, we have $\mathcal{M}_0 = 1$; the proof of Lemma 8 below verifies this directly. Since the integrand of $\mathcal{M}_1$ is an odd function, we have $\mathcal{M}_1 = 0$.

Now fix $n \geq 0$; starting with $\mathcal{M}_n$ and integrating by parts yields our desired recurrence:

$$\mathcal{M}_{n+2} = (n+1)\mathcal{M}_n \quad \text{for } n \geq 0. \tag{10}$$

Applying (10) repeatedly until our initial values come into play leads to

$$\mathcal{M}_n = 0 \quad \text{whenever } n \text{ is odd} \tag{11}$$

and

$$\mathcal{M}_{2m} = (2m-1)\mathcal{M}_{2m-2} = \cdots = (2m-1)(2m-3)\cdots 5\cdot 3\cdot 1 \quad \text{for integers } m \geq 1. \tag{12}$$

Comparing (11) and (12) with (8) shows that Theorem 7 is proved. □

Just as Proposition 2 generalizes to Theorem 6, so too does Theorem 7 generalize. For a given graph G, to determine $\Xi(G)$, we need to replace the factor t^n in the integrand of Theorem 7 by the 'matchings polynomial' of the complementary graph of G. Details appear in [10].

4 Appendix

After establishing that the 0th moment $\mathcal{M}_0 = 1$ (which was needed in the proof of Theorem 7), we indicate how to obtain (4). Evaluating the integral in the definition of $\mathcal{M}_0$ is an enjoyable polar coordinates exercise.

Lemma 8 $\int_{-\infty}^{\infty} e^{-u^2/2}\, du = \sqrt{2\pi}.$

Proof. Denoting the integral by $\mathcal{J}$, we have

$$\begin{aligned}
\mathcal{J}^2 &= \left(\int_{-\infty}^{\infty} e^{-u^2/2}\, du\right)\left(\int_{-\infty}^{\infty} e^{-v^2/2}\, dv\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(u^2+v^2)/2}\, du\, dv \qquad (13) \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} r\, e^{-r^2/2}\, dr\, d\vartheta, \qquad (14)
\end{aligned}$$

where we used Tonelli's Theorem to obtain (13) (see, e.g., [16]) and a switch to polar coordinates to reach (14). Since the inner integral here is unity, the result follows. □

Now the relation (4) is almost immediate:

Corollary 9 $\Gamma(1/2) = \sqrt{\pi}$.

Proof. By definition, $\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t}\,dt$. On putting $t = u^2/2$, we find that $\Gamma(1/2) = \sqrt{2}\int_0^\infty e^{-u^2/2}\,du$, or, since the last integrand is an even function, $\Gamma(1/2) = \sqrt{2}\int_{-\infty}^{\infty} e^{-u^2/2}\,du/2$. Now Lemma 8 gives the value of this integral to confirm the assertion. □

Solution to Problem 1

How many perfect matchings are contained in the graph H of Figure 5?

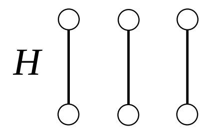


Figure 5: a graph induced by the edges of a perfect matching in $K_{3,3}$

The answer is obviously 1 because H is induced by the edges of a perfect matching: $\Xi(H) = 1$. On the other hand, since H is a spanning subgraph of $K_{3,3}$, Theorem 6 tells us that

$$\Xi(H) = \int_0^\infty R_{\widetilde{H}}(t) e^{-t}\,dt. \tag{15}$$

But here, $\widetilde{H}$ is the graph G in Figure 2(c) and Example 3, so that $R_{\widetilde{H}}(t) = t^3 - 6t^2 + 9t - 2$. Therefore, the integrals in (1) and (15) are the same. □

Concluding remarks

Proposition 2 and Theorem 7 present only two examples of combinatorially interesting sequences that can be expressed in the form $\int_\Omega t^n\,d\mu$ for some measure μ and space Ω. This topic is considered in detail in [10].

What is one to make of these connections between integrals and enumeration? We're not attempting to argue that integrals grind the preferred lens to view the counting problems herein. (For example, nobody would make the case that the integral in Theorem 7 is the 'right way' to determine $\Xi(K_n)$; the explicit formula (8) provides a far more direct route.) However, perhaps surprisingly, integrals do provide *one* lens. And this connection between the continuous and the discrete reveals just one of the myriad ways in which mathematics intimately links to itself. These links can benefit the mathematical branches at either of their ends. Section 2.2 illustrates how continuous methods can shed light on a discrete problem, while Problem 1 and its solution in the Appendix indicate how a discrete viewpoint might yield a fresh approach to an essentially continuous question. This symbiotic relationship between the different branches of mathematics should inspire students (and their teachers) not to overly specialize. As in life, it's better to keep one's mind as open as possible.

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