ALGEBRAIC INVARIANTS, MUTATION, AND COMMENSURABILITY OF LINK COMPLEMENTS

ERIC CHESEBRO AND JASON DEBLOIS

Abstract. We construct a family of hyperbolic link complements, all with trace field $\mathbb{Q}(i, \sqrt{2})$, by gluing tangles along totally geodesic four-punctured spheres, and investigate the commensurability relation among its members. Those with different volume are incommensurable, distinguished by their scissors congruence classes. Mutation produces arbitrarily large finite subfamilies of nonisometric manifolds with the same volume and scissors commensurability class. Depending on the choice of mutation, these manifolds may be commensurable or incommensurable, distinguished in the latter case by cusp parameters. Examples with integral and nonintegral traces are also produced.

1. Introduction

Manifolds M and M' are *commensurable* if there is a manifold N which is a finite cover for both M and M'. The study of the commensurability relation among hyperbolic knot and link complements in S^3 was initiated by W. Thurston, who described examples of commensurable and incommensurable manifolds in Chapter 6 of his notes [31]. The commensurability relation has been further studied on the chain link complements [24], on two-bridge knot complements [29], and on certain pretzel knot complements [17].

The following algebraic invariants distinguish commensurability classes of link complements under many circumstances. The *trace field* of a hyperbolic manifold $M = \mathbb{H}^3/\Gamma$, $\Gamma < \mathrm{PSL}_2(\mathbb{C})$, is the smallest field containing the traces of elements of Γ. The *cusp parameters* of M, used in [31] and [24], are algebraic invariants of the Euclidean structures on horospherical cross sections of the cusps of a hyperbolic manifold. *Scissors congruence* (cf. [8], [25]) and its relative the *Bloch invariant* [26] are determined by a polyhedral decomposition of a link complement.

In this paper, we construct a family of hyperbolic link complements, upon which the invariants above may be easily computed, and explore the commensurability relation among members of this family.

Theorem 1. *For each $n \in \mathbb{N}$, there is a link $L_n \subset S^3$ such that the complement $M_n = S^3 - L_n$ is hyperbolic with trace field $\mathbb{Q}(i, \sqrt{2})$ and integral traces. If $m \neq n$ then M_m and M_n have distinct Bloch invariants and cusp parameters.*

The link L_2 is pictured in Figure 1. The grey lines in the figure are meant to indicate the presence of 2-spheres $S^{(0)}$, $S^{(1)}$ and $S^{(2)}$ in S^3, each meeting L_2 in 4 points, which separate it left-to-right into a tangle S in the three-ball B^3, two copies of a tangle $T \subset S^2 \times I$, and the mirror image $\overline{S}$ of S. For arbitrary $n \in \mathbb{N}$, the link L_n is constructed analogously, using S, $\overline{S}$, and n copies of T.

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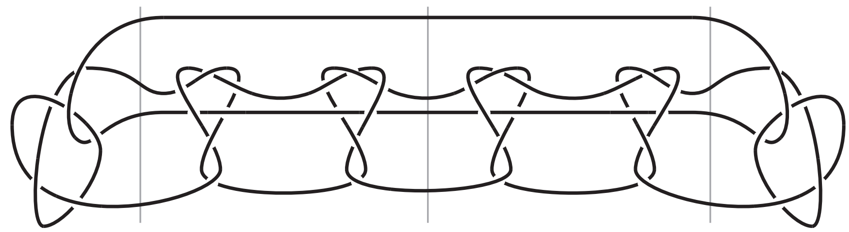


FIGURE 1. The link L_2

It is of fundamental importance here that the tangle complements $M_S = B^3 - S$ and $M_T = S^2 \times I - T$ each admit hyperbolic structures with totally geodesic boundary such that ∂M_S is isometric to a component of ∂M_T. In forming M_n, we have glued $\partial B^3 - S$ to $S^2 \times \{0\} - T$ by a map isotopic to an isometry, so that the separating four-punctured spheres $F^{(i)} = S^{(i)} - T$, $0 \leq i \leq n$, are totally geodesic in M_n, and each copy of M_S and M_T inherits its structure with totally geodesic boundary from the ambient hyperbolic structure. This is one of few examples in which the existence of totally geodesic surfaces in hyperbolic link complements has been proved to occur for geometric, as opposed to topological, reasons. For others, see [18] and [4], and for related results see [21], [16], [1], [2].

The hyperbolic structure with totally geodesic boundary on M_S, which was previously known, is obtained as an identification space of the regular ideal octahedron in $\mathbb{H}^3$. T is itself the double of $T_0 = T \cap (S^2 \times [0, 1/2])$, across $S^2 \times \{1/2\}$. We take doubles here as a convienience, it simplifies our gluing construction by ensuring that the totally geodesic boundary components of M_T are orientation-reversing isometric. We describe M_S and an identification space of the right-angled ideal cuboctahedron producing M_{T_0} in Section 2. In Section 3 we describe assembly of the tangle complements producing the link complements M_n, and in Section 4 we compute the algebraic commensurability invariants described above.

The construction above is a variant of those used by Adams [3] and Neumann-Reid (cf. [19, §5.6]), which produce families of hyperbolic 3-manifolds — in some cases, link complements — by gluing together manifolds bounded by 3-punctured spheres. Unlike the 4-punctured sphere, the 3-punctured sphere is *rigid*: up to isometry, it has a unique complete, finite-volume hyperbolic structure. This automatically implies the analogous total geodesicity property to that described above. A sample result which can be obtained from this construction, due to Neumann-Reid, is that for each imaginary biquadratic extension k of $\mathbb{Q}$, there are infinitely many commensurability classes of hyperbolic 3-manifolds with trace field k.

Our second concern is the commensurability relation among manifolds related to the M_n by *mutation* along the $F^{(i)}$: cutting along the surface, then regluing by a nontrivial homeomorphism. The mapping class group of a sphere with four marked points has a "mutation subgroup" isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, each member of which is determined by its action as an even permutation of the marked points. We will thus refer to a mutation homeomorphism of a sphere with four marked points by its permutation representation. With L_n projected as in Figure 1, and numbering the spheres separating the copies of S and T left-to-right by $S^{(i)}$, $0 \leq i \leq n$, for each i we mark the points of $S^{(i)} \cap L_n$ by 2, 3, 4, and 1, reading top to bottom.

Since homeomorphisms of S^2 extend over B^3, if $L \subset S^3$ is a link meeting a two-sphere S in four points, cutting (S^3, L) along $(S, S \cap L)$ and regluing by a mutation yields a link $L' \subset S^3$. We say L' is obtained from L by mutation along S.

In any fixed hyperbolic structure on the 4-punctured sphere, each mutation determines a homeomorphism which is properly isotopic to an isometry [30]. The hyperbolic structure on the $F^{(i)}$, which we describe in Lemma 2.4, corresponds to an arithmetic group $\Lambda < \mathrm{PSL}_2(\mathbb{Z})$ with the property that isometries realizing (13)(24) and (12)(34) lie in different maximal discrete groups containing Λ.

Theorem 2. *For fixed $n \in \mathbb{N}$, each manifold obtained from M_n by mutation along a subcollection of $\{F^{(i)}\}$, using isometries realizing* (13)(24)*, is commensurable to M_n. As $n \to \infty$, this produces arbitrarily large finite collections of pairwise nonisometric hyperbolic link complements with identical volumes.*

Theorem 2 follows from the fact that the isometry of ∂M_S which realizes (13)(24) extends over M_S but not over M_T; however the isometry of ∂M_T which realizes (13)(24) on each component lifts to an isometry that extends over a cover of M_T. We describe the isometries realizing (13)(24) and (12)(34) and prove Theorem 2 in Section 6.

That M_n is commensurable with its mutants by (13)(24) implies that it has a large collection of *hidden symmetries*: isometries of a cover which do not lift isometries of M_n. It is known that nonarithmetic manifolds have only finitely many hidden symmetries; however the nature of the collections which may occur remains mysterious.

Mutants of M_n by (12)(34) behave quite differently from mutants by (13)(24), as we will show in Theorem 3. Below, for $n \in \mathbb{N}$ and an element $I \in \{0,1\}^{n+1}$, we let L_I be the link obtained from L_n by the mutation (12)(34) along $S^{(i)}$, for each i such that the ith entry of I is nonzero. We let $\mathcal{L}_n = \{L_I \mid I \in \{0,1\}^{n+1}\}$ and write $M_I = S^3 - L_I$ for each I.

Theorem 3. *For every $n \in \mathbb{N}$ we have the following.*

1. *For each $I \in \{0,1\}^{n+1} - \{(0,\dots,0)\}$, M_I has the same volume, Bloch invariant, and trace field as M_n, but has a nonintegral trace.*
2. *There is a subfamily of $\mathcal{L}_n$ with at least $n/2$ mutually incommensurable members, distinguished by their cusp parameters.*
3. *There is a subfamily of $\mathcal{L}_n$ with n members which all share cusp parameters.*

Remarks.

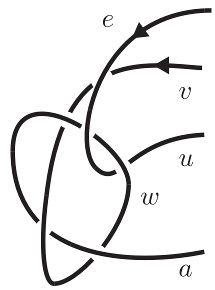
1. Mutation along 4-punctured spheres always preserves hyperbolic volume, by [30], and the trace field, by [23]. It almost certainly preserves the Bloch invariant as well, but we do not know a reference for this assertion.

2. $L_n = L_{(0,\dots,0)}$, which accounts for the gap in statement (1) of the theorem.

3. We do not know the commensurability relation among members of the subfamily described in (3) above.

We will compute algebraic invariants of the M_I in Section 7, yielding Theorem 3.

Our work is motivated by questions regarding knot complements. Reid-Walsh conjectured that the commensurability class of a hyperbolic knot complement in S^3 contains at most two others [29]. This would imply in particular that any hyperbolic knot complement is incommensurable with most of its mutants. There



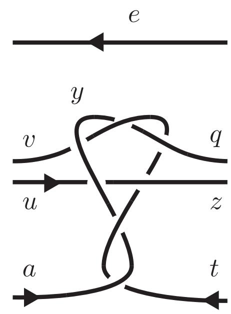


Figure 2. Tangles S and T_0, labeled with Wirtinger generators

seems to be no general method for deciding this issue, as mutants are notoriously difficult to distinguish. (Theorem 2 suggests that this may be for good reason!) Methods of Goodman-Heard-Hodgson [11] may be used to decide specific instances; for example, they show that the "Kinoshita-Terasaka knot," 11n42 in the knot tables, is incommensurable with its mutant, the "Conway knot" 11n34.

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2. A pair of tangles

This section is devoted to describing hyperbolic structures with totally geodesic boundary on the complements of the tangles S, in B^3, and T_0, in $S^2 \times I$, depicted in Figure 2. For a manifold M with boundary, we refer by a *tangle* in M to a pair (M, T), where T is the image of a disjoint union of circles and closed intervals, embedded in M by a map taking each circle into the interior of M and restricting on each interval to a proper embedding.

We will prove there is a homeomorphism taking $B^3 - S$ to a hyperbolic manifold with totally geodesic boundary which is an identification space of an ideal octahedron by pairing certain faces. This was previously known, and it follows from results in [27] upon taking a geometric limit, but we do not know a reference for a direct proof. We also prove there is a homeomorphism taking $S^2 \times I - T_0$ to a certain identification space of the right-angled ideal cuboctahedron. As far as we are aware, this description was not previously known.

We prove existence of these homeomorphisms by describing faithful representations from the fundamental groups of the tangle complements to Kleinian groups generated by the face pairings. Our main tools drawing connections between the geometric, algebraic, and topological objects involved are Lemma 2.1, which relates a hyperbolic 3-manifold with totally geodesic boundary produced by pairing some faces of a right-angled polyhedron to the Kleinian group generated by the face pairing isometries, and Lemma 2.6, which describes a homeomorphism between a pared manifold and the convex core of a Kleinian group to which its fundamental group represents.

In the remainder of the paper, we will let $\mathbb{H}^3 = \{(z,t)\,|\,z \in \mathbb{C}, t \in (0,\infty)\}$, the upper half space model of hyperbolic space, equipped with the complete Riemannian metric of constant sectional curvature -1. In this model, the group of orientation–preserving isometries, $\mathrm{PSL}_2(\mathbb{C})$, acts by extending its action by Möbius transformations on the *ideal boundary* or *sphere at infinity* $\mathbb{C} \cup \{\infty\}$.

The *horosphere of height t centered at* ∞ is $\mathbb{C} \times \{t\} \subset \mathbb{H}^3$. This inherits the Euclidean metric, scaled by $1/t$, from the ambient hyperbolic metric. For $v \in \mathbb{C} \times \{0\}$, a *horosphere centered at v* is a Euclidean sphere in $\mathbb{C} \times \mathbb{R}$ centered at a point in $\mathbb{H}^3$ and tangent to $\mathbb{C} \times \{0\}$ at $(v, 0)$. It is a standard fact that isometries of $\mathbb{H}^3$ take horospheres to horospheres.

A *hyperplane* of $\mathbb{H}^3$ is a totally geodesic subspace of the form $\ell \times \mathbb{R}^+$ for a line $\ell \subset \mathbb{C}$, or the intersection with $\mathbb{H}^3$ of a Euclidean sphere centered at a point in $\mathbb{C} \times \{0\}$. A *half space* is the closure of a component of the complement in $\mathbb{H}^3$ of a hyperplane, and a *polyhedron* is the nonempty intersection of half-spaces. A *face* of a polyhedron is its intersection with one of its defining hyperplanes. A polyhedron is *right-angled* if the defining hyperplanes meet at right angles whenever they meet at all. A polyhedron is *ideal* if any point at which more than two of its defining hyperplanes meet is on the sphere at infinity. Such points are *ideal vertices*.

If $\mathcal{P} \subset \mathbb{H}^3$ is a right-angled ideal polyhedron, we will say that a *checkering* of $\mathcal{P}$ is a partition of the set of faces of $\mathcal{P}$ into sets $\mathcal{S}_i$ and $\mathcal{S}_e$ of *internal* and *external* faces, respectively, so that each $f \in \mathcal{S}_i$ intersects only faces in $\mathcal{S}_e$ and vice–versa. If f is a face of $\mathcal{P}$, let $\mathcal{H}_f$ be the geodesic hyperplane in $\mathbb{H}^3$ containing f, and let U_f be the half-space determined by $\mathcal{H}_f$ which contains $\mathcal{P}$. We define the *expansion* of $\mathcal{P}$ as

$$E(\mathcal{P}) = \bigcap_{f \in \mathcal{S}_i} U_f.$$

Then $E(\mathcal{P})$ is a polyhedron of infinite volume containing $\mathcal{P}$, such that the components of the frontier of $\mathcal{P}$ in $E(\mathcal{P})$ are the external faces of $\mathcal{P}$.

An *internal face pairing* for a checkered polyhedron $\mathcal{P}$ is a collection $\{\phi_f \,|\, f \in \mathcal{S}_i\}$ of isometries with the properties that for each $f \in \mathcal{S}_i$, $\phi_f(f) = f'$, where $f' \neq f$ is an internal face of $\mathcal{P}$, $\phi_f(\mathcal{P}) \cap \mathcal{P} = f'$, and $\phi_{f'} = \phi_f^{-1}$. Given an internal face pairing and an edge e of $\mathcal{P}$, there are faces $g \in \mathcal{S}_e$ and $f \in \mathcal{S}_i$ such that $e \subset f \cap g$ and $f' = \phi_f(f)$ intersects an external face g' along $\phi_f(e)$. Let $M_{\mathcal{P}} \doteq \mathcal{P}/\{\phi_f\}$ be the identification space determined by the equivalence relation $x \sim \phi_f(x)$. Since $\mathcal{P}$ is right-angled, $M_{\mathcal{P}}$ has the structure of a hyperbolic manifold with totally geodesic boundary, where the boundary is the quotient of the disjoint union of the external faces by the edge pairing induced by $\{\phi_f\}$.

If Γ is a Kleinian group — that is, a discrete group of isometries of $\mathbb{H}^3$ — we will refer by $C(\Gamma)$ to the convex core of $\mathbb{H}^3/\Gamma$. This the convex submanifold of $\mathbb{H}^3/\Gamma$, minimal with respect to inclusion, with the property that the inclusion-induced homomorphism $\pi_1 C(\Gamma) \to \mathbb{H}^3/\Gamma$ is surjective. (See the beginning of [22, §6] for background on convex cores.)

Lemma 2.1. *Let $\mathcal{P} \subset \mathbb{H}^3$ be a finite-sided, checkered right-angled ideal polyhedron, with an internal face pairing $\{\phi_f \mid f \in \mathcal{S}_i\}$. Then $\Gamma \doteq \langle \phi_f \mid f \in \mathcal{S}_i \rangle$ is a free Kleinian group, and the inclusion $\mathcal{P} \hookrightarrow \mathbb{H}^3$ induces an isometry $p\colon M_{\mathcal{P}} \to C(\Gamma)$.*

Proof. The internal face pairing for $\mathcal{P}$ determines a genuine face pairing for $E(\mathcal{P})$, such that the inclusion $\mathcal{P} \to E(\mathcal{P})$ induces an isometric embedding $M_{\mathcal{P}} \to E(\mathcal{P})/\{\phi_f\}$. If $E(\mathcal{P})/\{\phi_f\}$ is a complete hyperbolic 3-manifold, then by Poincaré's polyhedron theorem (see eg. [28, Theorem 11.2.2]), $\Gamma = \langle \phi_f \mid f \in \mathcal{S}_i \rangle$ is discrete and $E(\mathcal{P})$ is a fundamental domain for Γ.

We will use the terminology and some results of [28]. By [28, Theorem 11.1.6], to show completeness it suffices to check that the link of any cusp is a complete Euclidean surface. Let $\lfloor v \rfloor = \{v_0, v_1, \ldots, v_{n-1}\}$ be an equivalence class of ideal vertices of $\mathcal{P}$ under the relation generated by $x \sim \phi_f(x)$, $f \in \mathcal{S}_i$, enumerated so that for each j there exists $f_j \in \mathcal{S}_i$ with $\phi_{f_j}(v_j) = v_{j+1}$ (taken modulo n). In particular, v_j is an ideal vertex of f_j and also of $f'_j \doteq \phi_{j-1}(f_{j-1})$.

For each j, let $\mathcal{B}_j$ be a horosphere centered at v_j, chosen small enough that $\mathcal{B}_j \cap \mathcal{B}_{j'} = \emptyset$ for $j \neq j'$. Since $\mathcal{P}$ is right-angled, $\mathcal{B}_j \cap \mathcal{P}$ is a Euclidean rectangle for each j. We may assume, by renumbering if necessary, that $\mathcal{B}_0 \cap f_0$ has shortest length of all the arcs $\mathcal{B}_j \cap f_j$. Then since $\phi_0(\mathcal{B}_0) \cap f'_1$ is parallel to $\phi_0(\mathcal{B}_0) \cap f_1$ in $\phi_0(\mathcal{B}_0) \cap \mathcal{P}$, they have the same length: that of $\mathcal{B}_0 \cap f_0$. Since this is less than the length of $\mathcal{B}_1 \cap f_1$, we have $\phi_0(\mathcal{B}_0) \subset \mathcal{B}_1$.

We may replace $\mathcal{B}_1$ by $\phi_0(\mathcal{B}_0)$, then replace $\mathcal{B}_2$ with $\phi_1(\mathcal{B}_1)$ and so on, yielding a new collection of horospheres which are pairwise disjoint and have the additional property that they are interchanged by the face pairings of $\mathcal{P}$. Equivalence classes of ideal vertices of $E(\mathcal{P})$ are the same as those of $\mathcal{P}$; thus this collection satisfies the hypotheses of [28, Theorem 11.1.4], and the link of $\lfloor v \rfloor$ is complete. It follows that $E(\mathcal{P})/\{\phi_f\}$ is a complete hyperbolic 3-manifold.

Now by the polyhedron theorem, Γ is discrete and $E(\mathcal{P})$ is a fundamental domain for Γ. It follows from a ping-pong argument that Γ is free, since the fact that $\mathcal{P}$ is right-angled implies that the hyperplanes containing its internal faces are mutually disjoint. The inclusion $E(\mathcal{P}) \to \mathbb{H}^3$ induces an isometry $E(\mathcal{P})/\{\phi_f\} \to \mathbb{H}^3/\Gamma$, so the inclusion $\mathcal{P} \to \mathbb{H}^3$ induces an isometric embedding $M_{\mathcal{P}} \to \mathbb{H}^3/\Gamma$.

For $g \in \mathcal{S}_e$, let V_g be the closure of $\mathbb{H}^3 - U_g$. There is a homeomorphism $V_g \to \mathcal{H}_g \times [0,\infty)$ taking x to $(r_g(x), d(x, r_g(x)))$, where $r_g\colon \mathbb{H}^3 \to \mathcal{H}_g$ is the nearest-point retraction and $d(x,y)$ is the distance in $\mathbb{H}^3$ between x and y. If e is an edge of g, there exists $f \in \mathcal{S}_i$ with $e \subset g \cap f$. The map ϕ_f takes $\mathcal{H}_g$ to $\mathcal{H}_{g'}$, where g' intersects $f' = \phi_f(f)$ along $e' = \phi_f(e)$. Furthermore, $r_g^{-1}(e) \subset V_g$ is taken to $r_{g'}^{-1}(e')$, equivariantly with respect to the homeomorphisms to $e \times [0,\infty)$ and $e' \times [0,\infty)$. It follows that $\mathbb{H}^3/\Gamma$ is homeomorphic to $M_{\mathcal{P}} \cup_\partial \partial M_{\mathcal{P}} \times [0,\infty)$, so $M_{\mathcal{P}}$ contains $C(\Gamma)$.

It now follows from work of Kojima ([14], see also [15]) that the convex hull of Γ_S has boundary consisting of translates of the hyperplanes containing external faces of $\mathcal{P}$. These cover $\partial M_{\mathcal{P}}$, and so $p(M_{\mathcal{P}}) = C(\Gamma_S)$. □

Corollary 2.2. *Let $\mathcal{P}_1$ be the regular ideal octahedron in $\mathbb{H}^3$, embedded as indicated in Figure 2, and checkered by declaring the face A to be external. The collection $\{\mathsf{s}^{\pm 1}, \mathsf{t}^{\pm 1}\}$ is an internal face pairing for $\mathcal{P}_1$, where*

$$\mathsf{s} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathsf{t} = \begin{pmatrix} 2i & 2-i \\ i & 1-i \end{pmatrix}.$$

Let $M_S = \mathcal{P}/\{\mathsf{s}^{\pm1}, \mathsf{t}^{\pm1}\}$, and let $\Gamma_S = \langle \mathsf{s}, \mathsf{t}\rangle$. Then the inclusion $\mathcal{P} \to \mathbb{H}^3$ induces an isometry $p_S\colon M_S \to C(\Gamma_S)$.

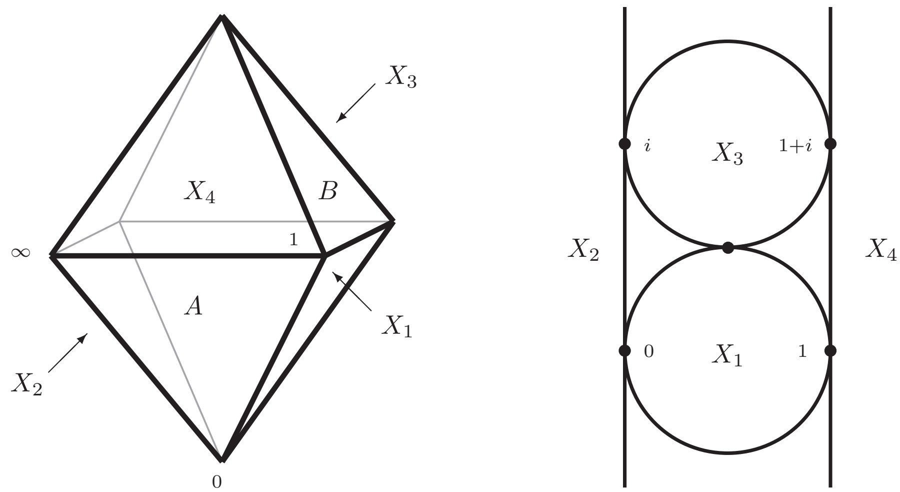

FIGURE 3. The regular ideal octahedron $\mathcal{P}_1$, and its expansion $E(\mathcal{P}_1)$

Proof. With the indicated embedding, $\mathcal{P}_1$ is a tile of the $\mathrm{PSL}_2(\mathcal{O}_1)$–invariant tesselation $\mathcal{T}_1$ constructed in [12]. Here $\mathcal{O}_1 = \mathbb{Z}[i]$ is the ring of integers of the field $\mathbb{Q}(i)$. In particular, the face A shown on the left in Figure 2 has ideal vertices 0, 1, and ∞, and all other ideal vertices of $\mathcal{P}_1$ have positive imaginary part.

Since A is external, the faces X_1, X_2, X_3, and X_4 of $\mathcal{P}_1$ indicated on the left in Figure 2 are internal. Direct computation reveals that s takes X_1 to X_2, fixing the ideal vertex they share, and t takes X_3 to X_4 so that the vertex they share goes to the vertex shared by X_4 and X_2. Hence $\{\mathsf{s}^{\pm1}, \mathsf{t}^{\pm1}\}$ is an internal face pairing for $\mathcal{P}_1$. The corollary now follows from Lemma 2.1. □

The external faces of $\mathcal{P}_1$ triangulate ∂M_S, and their images under p_S determine a triangulation of $\partial C(\Gamma_S)$, which we will denote by Δ_S.

Corollary 2.3. *Let $\mathcal{P}_2$ be the right-angled ideal cuboctahedron in $\mathbb{H}^3$, embedded as indicated in Figure 4, and checkered by declaring triangular faces external. The collection $\{\mathsf{f}^{\pm1}, \mathsf{g}^{\pm1}, \mathsf{h}^{\pm1}\}$ is an internal face pairing for $\mathcal{P}_2$, where*

$$\mathsf{f} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad \mathsf{g} = \begin{pmatrix} -1+i\sqrt{2} & 1-2i\sqrt{2} \\ -2 & 3-i\sqrt{2} \end{pmatrix} \qquad \mathsf{h} = \begin{pmatrix} 2i\sqrt{2} & -3-i\sqrt{2} \\ -3+i\sqrt{2} & -3i\sqrt{2} \end{pmatrix}.$$

Let $M_{T_0} = \mathcal{P}_2/\{\mathsf{f}^{\pm1}, \mathsf{g}^{\pm1}, \mathsf{h}^{\pm1}\}$, and let $\Gamma_{T_0} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h}\rangle$. The inclusion $\mathcal{P}_2 \to \mathbb{H}^3$ induces an isometry $p_{T_0}\colon M_{T_0} \to C(\Gamma_{T_0})$.

Proof. With the indicated embedding, $\mathcal{P}_2$ is a tile of the $\mathrm{PSL}_2(\mathcal{O}_2)$–invariant tesselation $\mathcal{T}_2$ of $\mathbb{H}^3$ defined in [12], where $\mathcal{O}_2 = \mathbb{Z}[i\sqrt{2}]$ is the ring of integers of $\mathbb{Q}(i\sqrt{2})$. In particular, the face C labeled in the figure has ideal vertices 0, 1, and ∞.

Label the internal faces Y_i as indicated on the left in Figure 4, and label the square face opposite Y_i as Y_i'. Direct computation reveals that f takes Y_2 to Y_1, fixing the ideal vertex they share, g takes Y_3 to Y_1', fixing the ideal vertex they

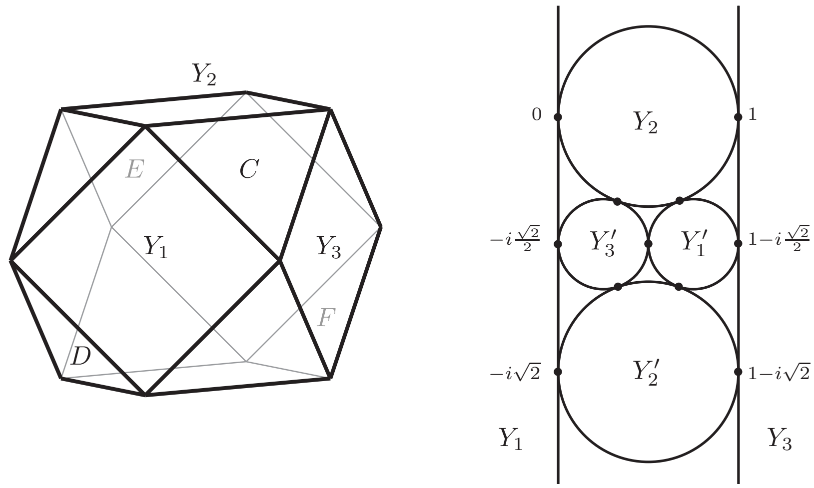

FIGURE 4. The right-angled ideal cuboctahedron $\mathcal{P}_2$, and $E(\mathcal{P}_2)$

share, and h takes Y_2' to Y_3', taking the vertex they share to the opposite vertex on Y_3'. Hence $\{\mathsf{f}^{\pm 1}, \mathsf{g}^{\pm 1}, \mathsf{h}^{\pm 1}\}$ is an internal face pairing for $\mathcal{P}_2$. The corollary now follows from Lemma 2.1. □

The external faces of $\mathcal{P}_2$ triangulate ∂M_{T_0}. We will call $\partial_- M_{T_0}$ the component which is triangulated by the faces labeled in Figure 4 with letters, and $\partial_+ M_{T_0}$ the other component. Let $\partial_\pm C(\Gamma_{T_0}) = p_{T_0}(\partial_\pm M_{T_0})$ and let $\Delta^\pm_{T_0}$ refer to the triangulation for $\partial_\pm C(\Gamma_{T_0})$ determined by the images of the external faces of $\mathcal{P}$ under p_{T_0}.

In the remainder of the paper, if g and h are elements of a group and G is a subgroup, we let g^{h} denote the conjugate of g by h, hgh^{-1} and $G^{\mathsf{h}} = \mathsf{h}G\mathsf{h}^{-1}$. Below we describe parabolic isometries p_1, p_2 and p_3 which lie in $\Gamma_S \cap \Gamma_{T_0}$.

$$\begin{aligned}
\mathsf{p}_1 &= \mathsf{s}^{-1} &&= \mathsf{f}^{-1} &&= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\
\mathsf{p}_2 &= \mathsf{stst}^{-2} &&= \mathsf{fg}^{-1}\mathsf{f}^{-1}\mathsf{h}^{-1}\mathsf{g} &&= \begin{pmatrix} -1 & 5 \\ 0 & -1 \end{pmatrix} \\
\mathsf{p}_3 &= (\mathsf{s}^{-1})^{\mathsf{tst}} &&= (\mathsf{g}^{-1})^{\mathsf{g}^{-1}\mathsf{f}^{-1}\mathsf{h}} &&= \begin{pmatrix} -14 & 25 \\ -9 & 16 \end{pmatrix}.
\end{aligned}$$

Let $\mathcal{H} = \mathcal{H}_A$. Since p_1, p_2, and p_3 are in $\mathrm{PSL}_2(\mathbb{R})$, they stabilize $\mathcal{H}$.

Lemma 2.4. *Let $\Lambda = \langle \mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3 \rangle < \mathrm{PSL}_2(\mathbb{R})$. Then the polyhedron $\mathcal{F}$ of Figure 5 is a fundamental domain for Λ, and $F^{(0)} = \mathcal{H}/\Lambda$ is a 4-punctured sphere. Furthermore:*

(1) $\Lambda = \mathrm{Stab}_{\Gamma_S}(\mathcal{H}) = \mathrm{Stab}_{\Gamma_{T_0}}(\mathcal{H})$,
(2) *the inclusion $\mathcal{H} \hookrightarrow \mathbb{H}^3$ induces an isometry $\iota_-^{(0)} \colon F^{(0)} \to \partial C(\Gamma_S)$ and an isometry $\iota_+^{(0)} \colon F^{(0)} \to \partial_- C(\Gamma_{T_0})$, and*
(3) *the triangulation of $\mathcal{F}$ pictured in Figure 5 projects to a triangulation Δ_F of $F^{(0)}$ taken by $\iota_-^{(0)}$ and $\iota_+^{(0)}$, respectively, to Δ_S and $\Delta^-_{T_0}$.*

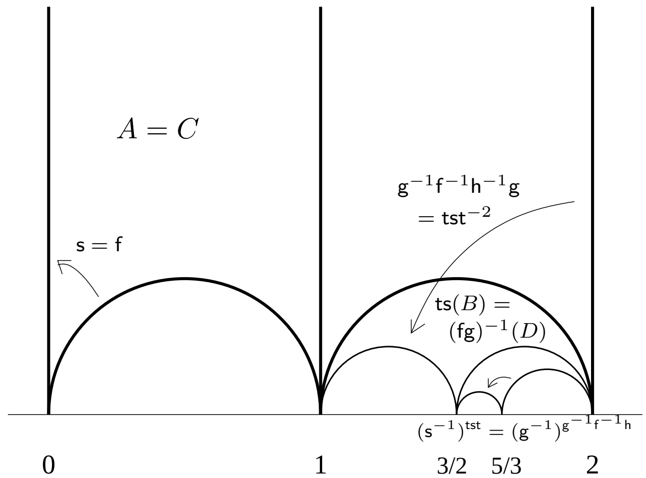

FIGURE 5. A triangulated fundamental domain $\mathcal{F}$ for the action of Λ on $\mathcal{H}$, and side pairings.

Proof. Let A and B be the external faces of $\mathcal{P}_1$ labeled in Figure 2, let A' be the external face which shares the ideal vertex 0 with A, and let B' be the other external face, which shares the vertex ∞ with A and $1+i$ with B. Let C, D, E, and F be the external faces of $\mathcal{P}_2$ indicated in Figure 4. With the prescribed embeddings of $\mathcal{P}_1$ and $\mathcal{P}_2$, their faces A and C coincide and lie in $\mathcal{H}$.

Each triangle of the domain $\mathcal{F}$ pictured in Figure 5 is a Γ_S-translate of a face of $\mathcal{P}_1$, and a Γ_{T_0}-translate of a face of $\mathcal{P}_2$. For example, since t takes X_3 to X_4, with the edge $X_3 \cap B'$ taken to $X_4 \cap A$, it follows that $\mathsf{t}(B')$ lies in $\mathcal{H}$, abutting A along the geodesic between 1 and ∞. Similarly, $\mathsf{g}^{-1}(E)$ abuts $C = A$ along same edge.

Direct calculation verifies that $\mathsf{t}(B') = \mathsf{g}^{-1}(E)$ is the triangle whose other vertex is at 2. This also follows from the fact noted in the proofs of Corollaries 2.2 and 2.3, that Γ_S-translates of $\mathcal{P}_1$ lie in the tesselation $\mathcal{T}_1$ described in [12], and Γ_{T_0}-translates of $\mathcal{P}_2$ lie in $\mathcal{T}_2$, and each of these intersects $\mathcal{H}$ in the Farey tesselation. One further finds that $\mathsf{ts}(B) = (\mathsf{fg})^{-1}(D)$, as indicated in Figure 5, and $\mathsf{tst}(A') = \mathsf{g}^{-1}\mathsf{f}^{-1}\mathsf{h}(F)$ has vertices at $3/2$, 2, and $5/3$.

Combinatorial considerations or direct calculation establish that $\mathsf{s} = \mathsf{f}$, $\mathsf{tst}^{-2} = \mathsf{g}^{-1}\mathsf{f}^{-1}\mathsf{h}^{-1}\mathsf{g}$, and $(\mathsf{s}^{-1})^{\mathsf{tst}} = (\mathsf{g}^{-1})^{\mathsf{g}^{-1}\mathsf{f}^{-1}\mathsf{h}}$, and that each stabilizes $\mathcal{H}$ and pairs edges of $\mathcal{F}$ as indicated in Figure 5. Inspection reveals that the quotient of $\mathcal{F}$ by these edge pairings is a 4-punctured sphere. By the polyhedron theorem, $\mathcal{F}$ is a fundamental domain for the group that they generate, and its quotient by the edge pairings is the quotient of $\mathcal{H}$ by this group. Since p_1, p_2, and p_3 are easily obtained from the edge pairings above and vice-versa, it follows that

$$\Lambda = \left\langle \mathsf{s}, \mathsf{tst}^{-2}, (\mathsf{s}^{-1})^{\mathsf{tst}} \right\rangle = \left\langle \mathsf{f}, \mathsf{g}^{-1}\mathsf{f}^{-1}\mathsf{h}^{-1}\mathsf{g}, (\mathsf{g}^{-1})^{\mathsf{g}^{-1}\mathsf{f}^{-1}\mathsf{h}} \right\rangle.$$

Therefore $\mathcal{F}$ is a fundamental fundamental domain for Λ, and $\mathcal{H}/\Lambda$ is a four-punctured sphere $F^{(0)}$.

It is clear from the descriptions of its generators that Λ is contained in $\mathrm{Stab}_{\Gamma_S}(\mathcal{H})$ and $\mathrm{Stab}_{\Gamma_{T_0}}(\mathcal{H})$, so the inclusion $\mathcal{H} \to \mathbb{H}^3$ induces immersions $\iota_-^{(0)} \colon F^{(0)} \to \mathbb{H}^3/\Gamma_S$

and $\iota_+^{(0)}\colon F^{(0)} \to \mathbb{H}^3/\Gamma_{T_0}$ which factor through covering maps to $\mathcal{H}/\mathrm{Stab}_{\Gamma_S}(\mathcal{H})$ and $\mathcal{H}/\mathrm{Stab}_{\Gamma_{T_0}}(\mathcal{H})$, respectively.

Since $\mathcal{H}$ contains the external face A of $\mathcal{P}_1$, the projection to $\mathbb{H}^3/\Gamma_S$ sends it to a component of $\partial C(\Gamma_S)$. Since it contains C, the projection to $\mathbb{H}^3/\Gamma_{T_0}$ sends it to a component of $\partial C(\Gamma_{T_0})$. Hence $\iota_-^{(0)}$ and $\iota_+^{(0)}$ are covering maps to components of $\partial C(\Gamma_S)$ and $\partial C(\Gamma_{T_0})$, respectively. But by Lemma 2.1, these components are triangulated by the external faces of $\mathcal{P}_1$ and $\mathcal{P}_2$ described above, whence conclusions (1), (2), and (3) follow immediately. □

Remarks.

1. The parabolic elements of Λ which fix the ideal vertices 0, ∞, and 5/3 of $\mathcal{F}$ are p_1, p_2, and p_3. The final conjugacy class of parabolic elements in Λ is represented by

$$\mathsf{p}_4 \;=\; \mathsf{p}_1\mathsf{p}_2\mathsf{p}_3^{-1} \;=\; (\mathsf{stst}^{-2})^{\mathsf{tst}^{-1}} = \begin{pmatrix} 29 & -45 \\ 20 & -31 \end{pmatrix}.$$

Evidently p_1 and p_3 are conjugate in Γ_S, as are p_2 and p_4. The combinatorial considerations of Lemma 4.10 will show that $C(\Gamma_S)$ has exactly two cusps, each of rank one, so every parabolic element of Γ_S is conjugate to one of p_1 or p_2.

2. There exists $\mathsf{k} \in \mathrm{PSL}_2(\mathbb{C})$, with order 2, which normalizes Γ_{T_0}:

$$\mathsf{k} \;=\; \begin{pmatrix} i & i-\sqrt{2} \\ 0 & -i \end{pmatrix}. \tag{1}$$

The action of k on the generators f, g, and h is given by

$$\mathsf{f}^{\mathsf{k}} = \mathsf{g}^{\mathsf{fg}^{-1}}, \qquad \mathsf{g}^{\mathsf{k}} = \mathsf{f}^{\mathsf{fg}^{-1}}, \qquad \text{and} \qquad \mathsf{h}^{\mathsf{k}} = (\mathsf{h}^{-1})^{\mathsf{fg}^{-1}}.$$

If Γ is a Kleinian group and $\mathsf{u} \in \mathrm{Isom}(\mathbb{H}^3)$, we write $\phi_{\mathsf{u}}\colon C(\Gamma) \to C(\Gamma^{\mathsf{u}})$ for the induced isometry. Note that $\phi_{\mathsf{u}}\phi_{\mathsf{v}} = \phi_{\mathsf{uv}}\colon C(\Gamma) \to C(\Gamma^{\mathsf{uv}})$.

Since k normalizes Γ_{T_0}, $\phi_{\mathsf{k}}\colon C(\Gamma_{T_0}) \to C(\Gamma_{T_0})$ acts as an orientation-preserving involution. The elements $\mathsf{p}_i^{\mathsf{k}}$, $i \in \{1,2,3\}$, all preserve the geodesic hyperplane $\mathsf{k}(\mathcal{H})$ which lies over the line $\mathbb{R} - i\sqrt{2}$. This hyperplane contains an external face of $\mathcal{P}_2$ which projects to $\partial_+ C(\Gamma_{T_0})$. The lemma below follows and, together with Lemma 2.4, completely describes $\partial C(\Gamma_{T_0})$.

Lemma 2.5.
$$\Lambda^{\mathsf{k}} = \mathrm{Stab}_{\Gamma_T}(\mathsf{k}(\mathcal{H})).$$
Furthermore, the inclusion $\mathsf{k}(\mathcal{H}) \to \mathbb{H}^3$ *induces an isometry* $F' = \mathsf{k}(\mathcal{H})/\Lambda^{\mathsf{k}} \to \partial_+ C(\Gamma_{T_0})$.

It is easy to see that $\mathsf{p}_1^{\mathsf{k}}$ is conjugate in Γ_{T_0} to p_3 and that $\mathsf{p}_2^{\mathsf{k}} = \mathsf{p}_2^{-1}$. The combinatorial considerations of Lemma 4.11 will imply that M_{T_0} has four cusps. Hence by Lemma 2.5, each of the cusps of $C(\Gamma_{T_0})$ joins $\partial_- C(\Gamma_{T_0})$ to $\partial_+ C(\Gamma_{T_0})$, and each parabolic in Γ_{T_0} is conjugate to exactly one p_i, $i \in \{1,2,3,4\}$.

Our second main tool in this section is Lemma 2.6 below. We refer to [22, Definition 4.8] for the definition of a pared manifold.

Lemma 2.6. *Let (M,P) be a pared manifold, and suppose that $\rho : \pi_1 M \to \Gamma < \mathrm{PSL}_2(\mathbb{C})$ is a faithful representation onto a non-Fuchsian geometrically finite Kleinian group where $C(\Gamma)$ has totally geodesic boundary. If there is a one–to–one* *correspondence under ρ between conjugacy classes of $\pi_1(M)$ corresponding to elements of P and conjugacy classes of maximal parabolic subgroups of Γ, then ρ is induced by a homeomorphism of $M - P$ to $C(\Gamma)$.*

This is well known to experts in Kleinian groups, but we do not know of a reference for a written proof. It seems worth writing down as it may fail if $C(\Gamma)$ does not have totally geodesic boundary (see [7] for a thorough exploration of this phenomenon). The proof follows easily from results in [7] for example, but requires introduction of the characteristic submanifold machinery. Since this is somewhat outside the scope of the rest of the paper, we defer the proof to Appendix A.

Let (B^3, S) be the tangle pictured on the left in Figure 2. Take a base point for $\pi_1(B^3 - S)$ on $\partial(B^3 - S)$ high above the projection plane, and let Wirtinger generators correspond in the usual way to labeled arcs of the diagram.

Proposition 2.7. *There is a homeomorphism $f_S\colon B^3 - S \to C(\Gamma_S)$, such that*

$$f_{S*}\colon \pi_1(B^3 - S) \to \Gamma_S$$

is given by $f_{S*}(a) = \mathsf{p}_1^{-1}$, $f_{S*}(e) = \mathsf{p}_2$, and $f_{S*}(v) = \mathsf{p}_3^{-1}$.

Proof. Reducing a standard Wirtinger presentation for $\pi_1(B^3 - S)$, we obtain

$$\left\langle a, w, e \,\middle|\, ewe^{-1}a = awaw^{-1} \right\rangle = \left\langle a, w, e \,\middle|\, w(e^{-1}aw) = (e^{-1}aw)a \right\rangle$$

Thus taking $b = e^{-1}aw$, one finds that $\pi_1(B^3 - S)$ is freely generated by a and b.

By Lemma 2.1 and Corollary 2.2, Γ_S is free on the generators s and t. Hence, the map $f_{S*}\colon \pi_1(B^3 - S) \longrightarrow \Gamma_S$ given by $a \mapsto \mathsf{s}$ and $b \mapsto \mathsf{t}$ is an isomorphism. Notice that the subgroup of $\pi_1(B^3 - S)$ corresponding to the 4-punctured sphere $\partial B^3 - \partial S$ is freely generated by a, v, and e. It is easily checked that

$$f_{S*}(v) \quad = \quad \begin{pmatrix} 16 & -25 \\ 9 & -14 \end{pmatrix} \quad = \quad \mathsf{p}_3^{-1}$$

and

$$f_{S*}(e) \quad = \quad \begin{pmatrix} -1 & 5 \\ 0 & -1 \end{pmatrix} \quad = \quad \mathsf{p}_2.$$

f_{S*} takes $\pi_1(\partial B^3 - S)$ isomorphically to Λ, since a, v, and e generate $\pi_1(\partial B^3 - S)$ and their images in Γ_S generate Λ. Since any meridian of S is conjugate in $\pi_1(B^3 - S)$ to either a or e, and these are taken to p_1 and p_2 respectively, meridians are taken to parabolic elements of Γ_S.

Now let $N(S)$ be a small open tubular neighborhood of S in B^3. Then $B^3 - N(S)$ is a compact manifold with genus two boundary, and the pair $(B^3 - N(S), \partial N(S))$ is a pared manifold. The proposition follows from Lemma 2.6, after noting that $(B^3 - N(S)) - \partial N(S)$ is homeomorphic to $B^3 - S$. □

Let $(S^2 \times I, T_0)$ be the tangle pictured on the right side of Figure 2, where I is oriented so that $\partial_- T_0 \doteq T_0 \cap S^2 \times \{0\}$ contains the endpoints labeled a, u, and v. Take a base point for $\pi_1(S^2 \times I - T_0)$ on $S^2 \times \{0\}$ high above the projection plane and let Wirtinger generators correspond to the labeled arcs of Figure 2.

The proposition below is the analog for T_0 of Proposition 2.7.

Proposition 2.8. *There is a homeomorphism* $f_{T_0}\colon\ S^2\times I - T_0 \longrightarrow C(\Gamma_{T_0})$ *such that*

$$f_{T_0*}\colon\ \pi_1(S^2\times I - T_0)\longrightarrow \Gamma_{T_0}$$

is given by $f_{T_0*}(a)=\mathsf{p}_1^{-1}$, $f_{T_0*}(e)=\mathsf{p}_2$, *and* $f_{T_0*}(v)=\mathsf{p}_3^{-1}$.

Proof. $(S^2\times I - N(T_0))$ may be isotoped in S^3 to a standard embedding of a genus-3 handlebody. Thus $\pi_1(S^2\times I - T_0)$ is free on three generators. We claim that the group is generated by a, e, and t. This follows after noticing that $v=y^{-1}xy$ where $y=(ta)^{-1}a(ta)$ and $x=(azq)^{-1}t(azq)=(ate)^{-1}t(ate)$. (The relation $zq=te$ used in the last equality comes from the relationship between four peripheral elements in a 4-punctured sphere group.) So far, we have established that $v,y\in\langle a,e,t\rangle$. Now using the other punctured sphere relation, we have $u=a^{-1}ev\in\langle a,e,t\rangle$. Finally, $z=yuy^{-1}$ and $q=z^{-1}te$. Therefore a, e, and t generate the group as claimed.

By Lemma 2.1 and Corollary 2.3, Γ_{T_0} is freely generated by f, g, and h. For our purposes, a more convenient free generating set for Γ_{T_0} is $\{\mathsf{f},\ \mathsf{fgf}^{-1},\ \mathsf{p}_2\}$. Note that all of these generators are parabolic and peripheral, and conjugation by k interchanges the first two and takes the third to its inverse. The representation of $\pi_1(S^2\times[0,1/2]-T_0)$ given by

$$a\mapsto \mathsf{f} \qquad\qquad t\mapsto \mathsf{fgf}^{-1} \qquad\qquad e\mapsto \mathsf{p}_2$$

is clearly faithful, and it is easily checked that v maps to p_3^{-1}. Because $u=a^{-1}ev$ is mapped to $\mathsf{p}_1\mathsf{p}_2\mathsf{p}_3^{-1}=\mathsf{p}_4$, we conclude that meridians are mapped to parabolic elements and that $\pi_1(S^2\times\{0\}-\partial_-T_0)$ is taken to Λ. The result now follows from Lemma 2.6 as previously. □

There is a visible involution of $S^2\times I - T_0$ which is a rotation by π around a circle in $S^2\times\{1/2\}$. This involution exchanges the two boundary components. With a proper choice of path between our basepoint and its image under this involution, the corresponding action on $\pi_1(S^2\times I - T_0)$ is given by

$$a\leftrightarrow t \qquad\qquad e\leftrightarrow e^{-1}$$

This commutes with the action of the element k defined in (1) on Γ_{T_0}, under the representation f_{T_0*}. Hence this involution is isotopic to the pullback of ϕ_{k} by f_{T_0}.

Recall from Lemma 2.4 that $\Lambda=\mathrm{Stab}_{\Gamma_{T_0}}(\mathcal{H})$, and from Lemma 2.5 that $\Lambda^{\mathsf{k}}=\mathrm{Stab}_{\Gamma_{T_0}}(\mathsf{k}(\mathcal{H}))$. By its definition in Proposition 2.8, it is clear that f_{T_0*} maps $\pi_1\left(S^2\times\{0\}-\partial_-T_0\right)$ isomorphically to Λ. Since $\mathcal{H}$ projects to $\partial_-C(\Gamma_{T_0})$, using the involution equivariance of f_{T_0} we obtain the corollary below.

Corollary 2.9. *Let* $\partial_+T_0=T_0\cap S^2\times\{1\}$. *Then* $f_{T_0}(S^2\times\{0\}-\partial_-T_0)=\partial_-C(\Gamma_{T_0})$, *and* $f_{T_0}(S^2\times\{1\}-\partial_+T_0)=\partial_+C(\Gamma_{T_0})$.

3. Combination

In this section, we will describe how to join copies of the tangles S and T_0 to construct links in S^3 whose complements are uniformized by combinations of Γ_S and Γ_{T_0}. The main tool in this section is a corollary of Maskit's combination theorem for free products with amalgamation [20]. Denote the convex hull of the limit set for a Kleinian group Γ by $\mathrm{Hull}(\Gamma)$.

Definition 3.1. Kleinian groups Γ_0 and Γ_1 *meet cute* along a hyperplane $\mathcal{K}\subset\mathbb{H}^3$ if $\mathcal{K}=\mathrm{Hull}(\Gamma_0)\cap\mathrm{Hull}(\Gamma_1)$ and $\mathrm{Stab}_{\Gamma_0}(\mathcal{K})=\mathrm{Stab}_{\Gamma_1}(\mathcal{K})$.

The utility of this definition is due to the following fact, an easy consequence of the definitions.

Fact. *If* Γ_0 *and* Γ_1 *meet cute along* $\mathcal{K}$, $\mathcal{K}$ *divides* $\mathbb{H}^3$ *into* $\mathcal{B}_0$ *and* $\mathcal{B}_1$, *such that:*

(1) $\Gamma_0 \cap \Gamma_1$ *stabilizes* $\mathcal{K}$.
(2) *For* $i \in \{0,1\}$, *if* $\mathsf{g}_i \in \Gamma_i$ *satisfies* $\mathsf{g}_i(\mathcal{B}_{1-i}) \cap \mathcal{B}_{1-i} \neq \emptyset$, *then* $\mathsf{g}_i \in \Gamma_0 \cap \Gamma_1$.

In general, if Θ is a subgroup of Γ, the limit set of Θ is contained in that of Γ, and so the covering map $\mathbb{H}^3/\Theta \to \mathbb{H}^3/\Gamma$ maps $C(\Theta)$ into $C(\Gamma)$ — we will call this restriction the *natural map* $C(\Theta) \to C(\Gamma)$. When Γ_0 and Γ_1 meet cute along $\mathcal{K}$ then the natural map $C(\Gamma_0 \cap \Gamma_1) \to C(\Gamma_i)$ restricts to an embedding of the 2-orbifold $\mathcal{K}/(\Gamma_0 \cap \Gamma_1)$.

The lemma below is a geometric combination theorem for Kleinian groups which meet cute along a hyperplane. It follows from Maskit's combination theorem and an observation due to Morgan.

Lemma 3.2. *Suppose* Γ_0 *and* Γ_1 *meet cute along a plane* $\mathcal{K}$. *Let* $E = \mathcal{K}/\Theta$, *where* $\Theta = \Gamma_0 \cap \Gamma_1$, *and for* $i = 0$, 1 *let* $\iota_i\colon E \to C(\Gamma_i)$ *be the natural embedding. Then* $\langle \Gamma_0, \Gamma_1 \rangle$ *is a Kleinian group, and the inclusions* $\Gamma_i \to \langle \Gamma_0, \Gamma_1 \rangle$ *determine an isomorphism* $\Gamma_0 *_\Theta \Gamma_1 \to \langle \Gamma_0, \Gamma_1 \rangle$ *as abstract groups. The natural maps* $C(\Gamma_i) \to C(\langle \Gamma_0, \Gamma_1 \rangle)$ *determine an isometry* $C(\Gamma_0) \cup_{\iota_1 \iota_0^{-1}} C(\Gamma_1) \to C(\langle \Gamma_0, \Gamma_1 \rangle)$.

In using Lemma 3.2, we often write $C(\Gamma_0) \cup_E C(\Gamma_1)$ when the maps ι_i are clear.

Proof. We will use the version of Maskit's combination theorem recorded in [22, Theorem 8.2]). The fact above implies that Γ_0 and Γ_1 which meet cute along a hyperplane $\mathcal{K}$ satisfy the hypotheses of [22, Theorem 8.2], whence the group-theoretic conclusions of the lemma follow. The existence of the required isometry follows from the remarks in [22] below Theorem 8.2, noting that in this case the surface $X = f^{-1}(\frac{1}{2})$ *is* the totally geodesic surface uniformized by Θ. □

If M is an oriented manifold with a boundary component F, the *double* of M across F is $M \cup_F \overline{M}$, where $\overline{M}$ is a copy of M with orientation reversed, and the gluing map $F \to \overline{F} \subset \overline{M}$ is the identity map.

Our first application of Lemma 3.2 describes a Kleinian group with totally geodesic convex core boundary isometric to the double of $C(\Gamma_{T_0})$ across $\partial_+ C(\Gamma_{T_0})$.

Recall from above Lemma 2.4 that we have defined $\mathcal{H}$ to be the geodesic hyperplane of $\mathbb{H}^3$ with ideal boundary $\mathbb{R} \cup \{\infty\}$. Let $\mathsf{r} \in \mathrm{Isom}(\mathbb{H}^3)$ be the reflection through $\mathcal{H}$. This acts on $\mathbb{C} \cup \{\infty\}$ by complex conjugation; thus if $\mathsf{q} \in \Gamma < \mathrm{PSL}_2(\mathbb{C})$, then $\mathsf{q}^{\mathsf{r}} = \bar{\mathsf{q}}$, where $\bar{\mathsf{q}} \in \mathrm{PSL}_2(\mathbb{C})$ is the element whose entries are the conjugates of the entries of q. Hence, we let $\overline{\Gamma}$ denote Γ^{r}.

Lemma 3.3. *Define* $\mathsf{c} = \begin{pmatrix} 1 & i\sqrt{2} \\ 0 & 1 \end{pmatrix}$, *and let* $\Gamma_T = \langle \Gamma_{T_0}, \overline{\Gamma}_{T_0}^{\mathsf{c}^{-2}} \rangle$. *There is an inclusion-induced isomorphism*

$$\Gamma_{T_0} *_{\Lambda^{\mathsf{k}}} \overline{\Gamma}_{T_0}^{\mathsf{c}^{-2}} \to \Gamma_T,$$

and an isometry $C(\Gamma_{T_0}) \cup_{F'} C(\overline{\Gamma}_{T_0}^{\mathsf{c}^{-2}}) \to C(\Gamma_T)$ *determined by the natural maps. Furthermore,* $\mathsf{c}^{-2}\mathsf{r}$ *normalizes* Γ_T, *and* $\phi_{\mathsf{c}^{-2}\mathsf{r}}\colon C(\Gamma_T) \to C(\Gamma_T)$ *exchanges complementary components of* F'.

Proof. Recall from Lemmas 2.4 and 2.5 that Λ and Λ^{k} are the stabilizers in Γ_{T_0} of the geodesic planes $\mathcal{H}$ and $\mathsf{k}(\mathcal{H})$, respectively, and that these planes project to the components of $\partial C(\Gamma_{T_0})$. It follows that $\mathcal{H}$ and $\mathsf{k}(\mathcal{H})$ are components of the boundary of $\mathrm{Hull}(\Gamma_{T_0})$, hence that $\mathrm{Hull}(\Gamma_{T_0})$ is contained in the region between them.

With c as defined in the statement of the lemma, note that $\mathsf{c}(\mathcal{H}) = (\mathbb{R}+i\sqrt{2})\times\mathbb{R}^+$ and that $\mathsf{ck}(\mathcal{H}) = \mathcal{H}$. Since $\mathrm{Hull}(\Gamma^{\mathsf{c}}_{T_0})$ has boundary components $\mathsf{c}(\mathcal{H})$ and $\mathsf{ck}(\mathcal{H}) = \mathcal{H}$, and $\Lambda^{\mathsf{ck}} = \mathrm{Stab}_{\Gamma^{\mathsf{c}}_{T_0}}(\mathcal{H})$ is invariant under conjugation by r, $\Gamma^{\mathsf{c}}_{T_0}$ and $\overline{\Gamma^{\mathsf{c}}_{T_0}}$ meet cute along $\mathcal{H}$.

Applying Lemma 3.2, we obtain an isomorphism $\Gamma^{\mathsf{c}}_{T_0} *_{\Lambda^{\mathsf{ck}}} \overline{\Gamma^{\mathsf{c}}_{T_0}} \to \langle \Gamma^{\mathsf{c}}_{T_0}, \overline{\Gamma^{\mathsf{c}}_{T_0}}\rangle$ and an isometry

$$C(\Gamma^{\mathsf{c}}_{T_0}) \cup_{\phi_{\mathsf{c}}(F')} C(\overline{\Gamma^{\mathsf{c}}_{T_0}}) \to C(\langle \Gamma^{\mathsf{c}}_{T_0}, \overline{\Gamma^{\mathsf{c}}_{T_0}}\rangle)$$

induced by the natural maps. It is clear that r normalizes $\langle \Gamma^{\mathsf{c}}_{T_0}, \overline{\Gamma^{\mathsf{c}}_{T_0}}\rangle$, exchanging amalgamands, hence ϕ_{r} acts as an orientation-reversing involution of $C(\langle \Gamma^{\mathsf{c}}_{T_0}, \overline{\Gamma^{\mathsf{c}}_{T_0}}\rangle)$, fixing F' and exchanging $C(\Gamma^{\mathsf{c}}_{T_0})$ with $C(\overline{\Gamma^{\mathsf{c}}_{T_0}})$.

Observe that $\bar{\mathsf{c}} = \mathsf{c}^{-1}$. It follows that $\overline{\Gamma^{\mathsf{c}}_{T_0}} = \overline{\Gamma}^{c^{-1}}_{T_0}$, and hence that $\Gamma_T = \langle \Gamma^{\mathsf{c}}_{T_0}, \overline{\Gamma^{\mathsf{c}}_{T_0}}\rangle^{\mathsf{c}^{-1}}$. Conjugating the groups of the paragraph above by c^{-1}, we obtain an inclusion-induced isomorphism $\Gamma_{T_0} *_{\Lambda^k} \overline{\Gamma}^{\mathsf{c}^{-2}}_{T_0} \to \Gamma_T$, and an isometry

$$C(\Gamma_{T_0}) \cup_{F'} C(\overline{\Gamma}^{\mathsf{c}^{-2}}_{T_0}) \to C(\Gamma_T)$$

induced by the natural maps. Furthermore, $\mathsf{c}^{-1}\mathsf{rc} = \mathsf{c}^{-2}\mathsf{r}$ normalizes Γ_T and induces an orientation-reversing involution $\phi_{\mathsf{c}^{-2}\mathsf{r}}$ fixing F' and exchanging its sides. □

The advantage that Γ_T has over Γ_{T_0} for our purposes is that the components $\partial C(\Gamma_T)$ are exchanged by the "doubling involution" $\phi_{\mathsf{c}^{-2}\mathsf{r}}$. In particular, they are naturally orientation-reversing isometric. Recall from Lemma 2.4 that $\Lambda = \mathrm{Stab}_{\Gamma_{T_0}}(\mathcal{H})$; thus by Lemma 3.3, $\Lambda = \mathrm{Stab}_{\Gamma_T}(\mathcal{H})$. We will again refer by $i^{(0)}_+$ to the natural map $F^{(0)} \to C(\Gamma_T)$. Then the lemma below follows from Lemma 3.3.

Lemma 3.4. *Let $F^{(1)} = \Lambda^{\mathsf{c}^{-2}}$, and let $\phi_{\mathsf{c}^{-2}}\colon F^{(0)} \to F^{(1)}$ and $\iota^{(1)}_-\colon F^{(1)} \to C(\Gamma_T)$ be the natural maps. Then $\partial C(\Gamma_T) = \partial_- C(\Gamma_T) \sqcup \partial_+ C(\Gamma_T)$, where $\partial_- C(\Gamma_T) \doteq \iota^{(0)}_+(F^{(0)})$ and $\partial_+ C(\Gamma_T) \doteq \iota^{(1)}_-(F^{(1)})$, and $\iota^{(1)}_-\phi_{\mathsf{c}^{-2}} = \phi_{\mathsf{c}^{-2}\mathsf{r}}\iota^{(0)}_+$.*

$C(\Gamma_T)$ is a geometric model for the double of $(S^2\times I, T_0)$ across $(S^2\times\{1\}, \partial_+T_0)$. Note that the double of $S^2\times I$ across $S^2\times\{1\}$ is again homeomorphic to $S^2\times I$, by a map taking $(p,t)\in S^2\times I$ to $(p,t/2)$ and $(p,t)\in\overline{S^2\times I}$ to $(p,1-t/2)$.

Definition 3.5. Let $(S^2\times I, T)$ be the double of $(S^2\times I, T_0)$ across $(S^2\times\{1\}, \partial_+T_0)$. We will identify $(S^2\times I, T_0)\subset(S^2\times I, T)$ with its image under the map discussed above, so that $T_0 = T\cap S^2\times[0,1/2]$. In particular, we have $\partial_-T = \partial_-T_0 = T\cap S^2\times\{0\}$ and $\partial_+T_0 = T\cap S^2\times\{1/2\}$, and we will take $\partial_+T = T\cap S^2\times\{1\}$.

The tangle $(S^2\times I, T)$ is pictured in Figure 6, with $T_0\subset T$ visible to the left of the grey vertical line representing $S^2\times\{1/2\}$. The mirror symmetry fixing $(S^2\times\{1/2\}, \partial_+T_0)$,

$$r_T\colon (S^2\times I, T)\to(S^2\times I, T),$$

given by $r_T(p,x) = (p,1-x)$, is visible in the figure as reflection through the grey vertical line.

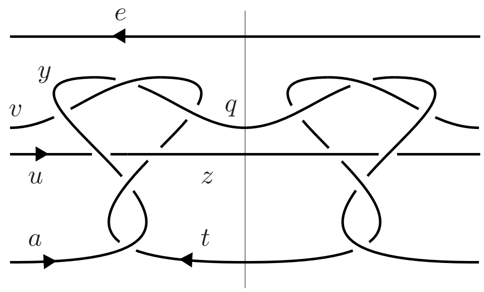

FIGURE 6. The tangle $T \subset S^2 \times I$ with labeled Wirtinger generators for T_0

Proposition 3.6. *There is a homeomorphism $f_T\colon\ S^2 \times I - T \to C(\Gamma_T)$, which restricts on $S^2 \times [0, 1/2] - T_0$ to f_{T_0} followed by the natural map, such that the diagram below commutes.*

$$\begin{array}{ccc} S^2 \times I - T & \xrightarrow{f_T} & C(\Gamma_T) \\ \downarrow r_T & & \downarrow \phi_{\mathsf{c}^{-2}\mathsf{r}} \\ S^2 \times I - T & \xrightarrow{f_T} & C(\Gamma_T) \end{array}$$

Furthermore, f_T takes $S^2 \times \{0\} - \partial_- T$ to $\partial_- C(\Gamma_T)$ and $S^2 \times \{1\} - \partial_+ T$ to $\partial_+ C(\Gamma_T)$.

Proof. We define f_T using the properties described in the statement of the lemma. Namely, we first require f_T to restrict on $S^2 \times [0, 1/2] - T_0$ to the homeomorphism f_{T_0} defined in Proposition 2.8, followed by the natural map $C(\Gamma_{T_0}) \to C(\Gamma_T)$. For $x \in S^2 \times [1/2, 1] - T$, we define $f_T(x) = \phi_{\mathsf{c}^{-2}\mathsf{r}} f_T\, r_T(x)$. The resulting map is well-defined, since r_T fixes $S^2 \times \{1/2\} - \partial_+ T_0$ and $\phi_{\mathsf{c}^{-2}\mathsf{r}}$ fixes F'. It is a homeomorphism, since r_T, f_{T_0}, and $\phi_{\mathsf{c}^{-2}\mathsf{r}}$ are. By Corollary 2.9, f_{T_0} takes $S^2 \times \{0\} - \partial_- T_0$ to $\partial_- C(\Gamma_{T0})$; it therefore follows from the definitions and Lemma 3.4 that $f_T(S^2 \times \{0\} - \partial_- T) = \partial_- C(\Gamma_T)$. The conclusion thus follows from the reflection equivariance of f_T. □

Definitions 3.7.

(1) Let $j\colon\ (\partial B^3, \partial S) \to (S^2 \times \{0\}, \partial_- T)$ be the homeomorphism such that $(B^3, S) \cup_j (S^2 \times I, T)$ is the tangle pictured in Figure 7.

(2) For each $i \in \mathbb{Z}$, define $h_i\colon\ S^2 \times \mathbb{R} \to S^2 \times \mathbb{R}$ by $h_i(p, x) = (p, x + (i - 1))$, and with $T \subset S^2 \times I \subset S^2 \times \mathbb{R}$, let $T^{(i)} = h_i(T)$. Now for $n \in \mathbb{N}$, define

$$(S^3, L_n)\ =\ (B^3, S) \cup_j \left(S^2 \times [0, n], \cup_{i=1}^n T^{(i)}\right) \cup_{j_n} (\overline{B}^3, \overline{S}).$$

For $i \in \{0, 1, \ldots, n\}$, let $S^{(i)}$ be the image in (S^3, L_n) of $S^2 \times \{i\}$. Above, $(\overline{B}^3, \overline{S}) = r_S(B^3, S)$, where r_S is the reflective involution of S^3 fixing the boundary of an embedding of B^3 and exchanging its sides, and $j_n = r_S j^{-1} r_T h_n^{-1}\colon\ \left(S^{(n)}, \partial_+ T^{(n)}\right) \longrightarrow \left(\partial \overline{B}^3, \partial \overline{S}\right)$.

(3) Using Figure 6 and taking $T \subset S^2 \times I \subset S^2 \times \mathbb{R}$, label the points of $S^{(0)} \cap L_n = S^2 \times \{0\} \cap T$ by 2, 3, 4, and 1 top-to-bottom, so that for example 2 is the terminal point of the tangle string labeled e and 1 is the

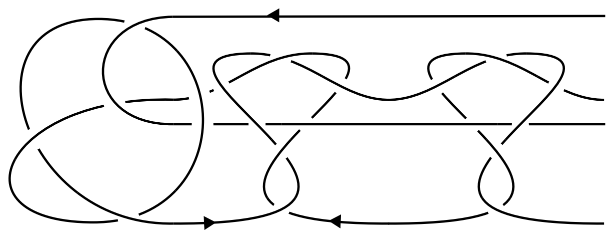

FIGURE 7. $S \cup T$

initial point of the string labeled a. Label each point of $S^{(i)} \cap L_n$ by its image under h_{i+1}^{-1}.

Remarks.

1. With Wirtinger generators for $\pi_1(B^3 - S)$ and $\pi_1(S^2 \times I - T)$ as labeled in Figures 2 and 6, we have $j_*(a) = a$, $j_*(u) = u$, and $j_*(v) = v$.
2. Under the natural embedding of $S^2 \times I \subset S^2 \times \mathbb{R}$ used above, we have $T = T^{(1)}$.

We now go about constructing a geometric model of $S^3 - L_n$.

Definitions 3.8.

(1) For $i \geq 0$, let $\Lambda^{(i)} = \Lambda^{\mathsf{c}^{-2i}}$ and $F^{(i)} = \mathsf{c}^{-2i}(\mathcal{H})/\Lambda^{(i)}$
(2) For $i \geq 1$, let $\Gamma_T^{(i)} = \Gamma_T^{\mathsf{c}^{-2(i-1)}}$, and define $\phi_i = \phi_{\mathsf{c}^{-2(i-1)}} \colon C(\Gamma_T) \to C(\Gamma_T^{(i)})$.

The definitions above of $F^{(0)}$ and $F^{(1)}$ above agree with our previous definitions. Also, $\Gamma_T^{(1)} = \Gamma_T$, and since $\Gamma_T^{(i)} = \mathsf{c}^{-2(i-1)}\Gamma_T \mathsf{c}^{2(i-1)}$, Lemma 3.4 implies that

$$\Lambda^{(i-1)} = \mathrm{Stab}_{\Gamma_T^{(i)}}(\mathsf{c}^{-2(i-1)}(\mathcal{H})) \quad \text{and} \quad \Lambda^{(i)} = \mathrm{Stab}_{\Gamma_T^{(i)}}(\mathsf{c}^{-2i}(\mathcal{H})),$$

and the resulting natural maps $\iota_+^{(i-1)} \colon F^{(i-1)} \to C(\Gamma_T^{(i)})$ and $\iota_-^{(i)} \colon F^{(i)} \to C(\Gamma_T^{(i)})$ map to the components of its totally geodesic boundary.

Proposition 3.9. *For $n \in \mathbb{N}$, define $M_n = C(\Gamma_S) \cup C(\Gamma_T^{(1)}) \cup \ldots \cup C(\Gamma_T^{(n)}) \cup C(\overline{\Gamma}_S)$, using gluing maps defined as follows:*

$$\iota_+^{(0)}(\iota_-^{(0)})^{-1} \colon \partial C(\Gamma_S) \to \partial_- C(\Gamma_T^{(1)})$$
$$\iota_+^{(i)}(\iota_-^{(i)})^{-1} \colon \partial_+ C(\Gamma_T^{(i)}) \to \partial_- C(\Gamma_T^{(i+1)}) \text{ for } 1 \leq i < n$$
$$\phi_{\mathsf{r}}\iota_-^{(0)}\phi_{n+1}^{-1}(\iota_-^{(n)})^{-1} \colon \partial_+ C(\Gamma_T^{(n)}) \to \partial C(\overline{\Gamma}_S)$$

There is a homeomorphism $f_n \colon S^3 - L_n \to M_n$ which restricts on $B^3 - S$ to f_S, on $S^2 \times [i-1, i] - T^{(i)}$ to $\phi_i f_T h_i^{-1}$ for $1 \leq i \leq n$, and on $\overline{B}^3 - \overline{S}$ to $\phi_{\mathsf{r}} f_S r_S$.

Proof. We will use the description of f_n above as its definition. Then by Proposition 2.7 and the definitions, f_n restricts on $B^3 - S$ and $\overline{B}^3 - \overline{S}$ to homeomorphisms to $C(\Gamma_S)$ and $C(\overline{\Gamma}_S)$, respectively. By Proposition 3.6 and the definitions, for each i between 1 and n it restricts on $S^2 \times [i-1, i] - T^{(i)}$ to a homeomorphism to $C(\Gamma_T^{(i)})$. Thus in order to show that f_n is a homeomorphism, we must only show that it is well defined on the spheres $S^{(i)} - \{1, 2, 3, 4\}$ that separate these tangle complements.

We will show first that f_n is well defined on $S^{(0)} - \{1,2,3,4\}$. When $i=1$, $T^{(i)} = T$, $\Gamma_T^{(i)} = \Gamma_T$, and h_i and ϕ_i are each the identity map. Checking well-definedness in this case therefore reduces to showing that on $\partial B^3 - \partial S$, $f_T \circ j = \iota_+^{(0)}(\iota_-^{(0)})^{-1} \circ f_S$.

By their definitions above, f_S and $f_T \circ j$ induce the same isomorphism from $\pi_1(\partial B^3 - \partial S)$ to $\Lambda = \Gamma_S \cap \Gamma_T$. Recall from Lemma 2.4 and the remarks above Lemma 3.4 that the $\iota_\pm^{(0)}$ are induced by the inclusions of Λ into Γ_S and Γ_T. Therefore at the level of fundamental group, $\left(\iota_+^{(0)}(\iota_-^{(0)})^{-1} \circ f_S\right)_* = (f_T \circ j)_*$. Since any two homeomorphisms between 4-punctured spheres that induce the same map on fundamental groups are properly isotopic, we may isotope j so that f_S and $f_T j$ agree on $S^{(0)}$. The conclusion thus follows in this case.

For $1 \leq i < n$, we may use the fact that $\Gamma_T^{(i)}$ and $\Gamma_T^{(i+1)}$ are conjugates of Γ_T to obtain the following model descriptions for $\iota_+^{(i)}$ and $\iota_-^{(i)}$:

$$\iota_+^{(i)} = \phi_{i+1}\iota_+^{(0)}\phi_{i+1}^{-1} \qquad\qquad \iota_-^{(i)} = \phi_i \iota_-^{(1)} \phi_i^{-1} \tag{2}$$

Here $\iota_+^{(0)} \colon F^{(0)} \to \partial_- C(\Gamma_T)$ and $\iota_-^{(1)} \colon F^{(1)} \to \partial_+ C(\Gamma_T)$ are the natural maps of Lemma 3.4. Using the reflection-invariance property described there, we thus obtain

$$\iota_+^{(i)}(\iota_-^{(i)})^{-1} = \phi_{i+1}\iota_+^{(0)}\left(\iota_-^{(1)}\phi_2\right)^{-1}\phi_i^{-1} = \phi_{i+1}\phi_{\mathsf{c}^{-2}\mathsf{r}}^{-1}\phi_i^{-1}. \tag{3}$$

Then by the reflection-equivariance property of f_T from Proposition 3.6, we have

$$\iota_+^{(i)}(\iota_-^{(i)})^{-1} \circ \phi_i f_T h_i^{-1} = \phi_{i+1}\phi_{\mathsf{c}^{-2}\mathsf{r}}^{-1} f_T h_i^{-1} = \phi_{i+1} f_T r_T h_i^{-1}.$$

It follows directly from the definitions that $r_T h_i^{-1} = h_{i+1}^{-1}$ on $S^{(i)}$, whence f_n is well defined on $S^{(i)} - \{1,2,3,4\}$ for $1 \leq i < n$.

To show f_n is well defined on $S^{(n)}$ requires another definition-chase, this time to check

$$\phi_{\mathsf{r}} f_S r_S \circ j_n = \phi_{\mathsf{r}} \iota_-^{(0)} \phi_{n+1}^{-1} (\iota_-^{(n)})^{-1} \circ \phi_n f_T h_n^{-1}.$$

By Definition 3.7(2), $j_n = r_S j^{-1} r_T h_n^{-1}$; therefore simplifying the left-hand side above yields $\phi_{\mathsf{r}} f_S j^{-1} r_T h_n^{-1}$. On the other hand, using the model description of $\iota_-^{(n)}$ from (2), the right-hand side above simplifies to $\phi_{\mathsf{r}}\iota_-^{(0)}\phi_2^{-1}(\iota_-^{(1)})^{-1} f_T h_n^{-1}$. The reflection-invariance property of Lemma 3.4 and an appeal to the case $i=0$ establish the desired equation. □

Corollary 3.10. *For $0 \leq i < n$, refer again by $F^{(i)}$ to the image of $\iota_+^{(i)}(F^{(i)}) \subset C(\Gamma_T^{(i)})$ under inclusion into M_n, and refer by $F^{(n)}$ to the included image of $\iota_-^{(n)}(F^{(n)})$. For each i, $F^{(i)}$ is totally geodesic in M_n and $f_n(S^{(i)} - \{1,2,3,4\}) = F^{(i)}$.*

This follows immediately from Proposition 3.9 and the fact that the maps $\iota_\pm^{(i)}$ are isometric embeddings. The proposition below describes an algebraic model for M_n.

Proposition 3.11. *For $n \in \mathbb{N}$, define $\Gamma_n = \langle \Gamma_S, \Gamma_T^{(1)}, \ldots, \Gamma_T^{(n)}, \overline{\Gamma}_S^{\mathsf{c}^{-2n}} \rangle$. There is an isometry $M_n \to \mathbb{H}^3/\Gamma_n$ which restricts on $C(\Gamma_S)$ and each $C(\Gamma_T^{(i)})$ to the natural map, and on $C(\overline{\Gamma}_S)$ to ϕ_{n+1} followed by the natural map.*

Proof. We first recall from Lemma 2.4 that the plane $\mathcal{H}$ with ideal boundary $\mathbb{R} \cup \{\infty\}$ projects $\partial C(\Gamma_S)$ under the quotient map $\mathbb{H}^3 \to \mathbb{H}^3/\Gamma_S$. From this and the fact that all ideal vertices of the octahedron $\mathcal{P}_1$ have non-negative imaginary part, it follows that

$$\mathrm{Hull}(\Gamma_S) \subset \{z \in \mathbb{C} \,|\, \Im z \geq 0\} \cup \{\infty\}.$$

Similarly, from Lemma 3.4 and the positioning of $\mathcal{P}_2$ we find that

$$\mathrm{Hull}(\Gamma_T) \subset \{z \in \mathbb{C} \,|\, 0 \geq \Im z \geq -2\sqrt{2}\} \cup \{\infty\}.$$

Then inspecting the action of c on $\mathbb{C} \cup \{\infty\}$, we find that any point of $\mathrm{Hull}(\Gamma_T^{(i)})$ have imaginary part between $-2(i-1)\sqrt{2}$ and $-2i\sqrt{2}$ for $i \in \mathbb{N}$ (unless the point is at ∞).

The claim below builds an inductive picture of a family of isometrically embedded, codimension-0 submanifolds of M_n with totally geodesic boundary.

Claim. For $1 \leq i \leq n$, define $\Gamma_-^{(i)} = \langle \Gamma_S, \Gamma_T^{(1)}, \ldots, \Gamma_T^{(i)} \rangle$. There is an isometry $C(\Gamma_S) \cup C(\Gamma_T^{(1)}) \cup \ldots \cup C(\Gamma_T^{(i)}) \to C(\Gamma_-^{(i)})$, where the gluing maps for the domain are as in Proposition 3.9, which restricts on $C(\Gamma_S)$ and each $C(\Gamma^{(j)})$, $j < i$, to the natural map. Furthermore:

(1) $\Lambda^{(i)} = \mathrm{Stab}_{\Gamma_-^{(i)}}(\mathsf{c}^{-2n}(\mathcal{H}))$, and the induced natural map is an isometry $F^{(i)} \to \partial C(\Gamma_-^{(i)})$ which factors as $\iota_-^{(i)}$ followed by the natural map $C(\Gamma_T^{(i)}) \to C(\Gamma_-^{(i)})$.
(2) $\mathrm{Hull}(\Gamma_-^{(i)}) \subset \{z \in \mathbb{C} \,|\, \Im\, z \geq -i\sqrt{2}\} \cup \{\infty\}$.

Proof of claim. We will prove the claim by induction. If it holds for some $i < n$, then (1) and (2) above, together with the observations above the claim imply that $\Gamma_-^{(i)}$ and $\Gamma_T^{(i+1)}$ meet cute along $\mathsf{c}^{-2i}(\mathcal{H})$. Then by Lemma 3.2, the natural maps determine an isometry $C(\Gamma_-^{(i)}) \cup C(\Gamma_T^{(i+1)}) \to C(\Gamma_-^{(i+1)})$, where by the inductive hypothesis and the observation above Proposition 3.9, the gluing map for the domain is $\iota_+^{(i)}(\iota_-^{(i)})^{-1}$ following the inverse of the natural map.

Furthermore, since $C(\Gamma_-^{(i)})$ has a unique totally geodesic boundary component, which is isometrically identified with $\partial_- C(\Gamma_T^{(i+1)})$ in the isometry to $C(\Gamma_-^{(i+1)})$ described above, the unique totally geodesic boundary component of $C(\Gamma_-^{(i+1)})$ is the isometric image of $\partial_+ C(\Gamma_T^{(i+1)})$. Therefore the observations above Proposition 3.9 imply that this boundary component is the image of $\iota_-^{(i+1)}(F^{(i+1)})$ under the natural map. Assertion (1) of the claim thus follows for $\Gamma_-^{(i+1)}$. It follows that $\mathrm{Hull}(\Gamma_-^{(i+1)})$ is entirely on one side or the other of the boundary at infinity of $\mathsf{c}^{-2(i+1)}(\mathcal{H})$. Since $\Gamma_T^{(i+1)} < \Gamma_-^{(i+1)}$, assertion (2) now follows.

By our definition of "natural map" above Lemma 3.2, the composition of the natural map $C(\Gamma_T^{(j)}) \to C(\Gamma_-^{(i)})$, with the natural map $C(\Gamma_-^{(i)}) \to C(\Gamma_-^{(i+1)})$ is itself natural, for $j \leq i$. Hence if the claim holds for $\Gamma_-^{(i)}$, $i < n$, it holds for $\Gamma_-^{(i+1)}$. The claim will therefore hold by induction if it is true in the base case $i = 1$. But this follows from the fact that Γ_S and $\Gamma_T^{(1)}$ meet cute along $\mathcal{H}$. This follows in turn from Lemmas 2.4 and 3.4, which establish that $\Lambda^{(0)} = \mathrm{Stab}_{\Gamma_S}(\mathcal{H}) = \mathrm{Stab}_{\Gamma_T}(\mathcal{H})$, and the first paragraph of the proof. □

Using the claim, it now follows that $\Gamma_-^{(n)}$ and $\overline{\Gamma}_S^{\mathsf{c}^{-2n}}$ meet cute along $\mathsf{c}^{-2n}(\mathcal{H})$; hence a final application of Lemma 3.2 implies that the natural maps determine an isometry $C(\Gamma_-^{(n)}) \cup C(\overline{\Gamma}_S^{\mathsf{c}^{-2n}}) \to C(\Gamma_n)$. Since each of $C(\Gamma_-^{(n)})$ and $C(\overline{\Gamma}_S^{\mathsf{c}^{-2n}})$ has a unique boundary component, $C(\Gamma_n)$ is boundaryless and hence equal to $\mathbb{H}^3/\Gamma_n$. The conclusion of the proposition follows. □

The result below follows Proposition 3.11, or really, its proof.

Corollary 3.12. *For fixed n and $0 \leq i \leq n$, define*

$$\Gamma_-^{(i)} = \left\langle \Gamma_S, \Gamma_T^{(1)}, \dots, \Gamma_T^{(i)} \right\rangle \qquad \Gamma_+^{(i)} = \left\langle \Gamma_T^{(i+1)}, \dots, \Gamma_T^{(n)}, \overline{\Gamma}_S^{\mathsf{c}^{-2n}} \right\rangle,$$

Then $\Gamma_+^{(i)}$ and $\Gamma_-^{(i)}$ meet cute along $\mathsf{c}^{-2i}(\mathcal{H})$ and the natural maps determine an isometry $C(\Gamma_-^{(i)}) \cup C(\Gamma_+^{(i)}) \to \mathbb{H}^3/\Gamma_n$. The isometry of Proposition 3.11 factors through this map, so that the component of $M_n - F^{(i)}$ containing $C(\Gamma_S)$ is taken isometrically to its image in $C(\Gamma_-^{(i)})$.

In the remainder of the paper, we will frequently take the isometry above for granted and refer to the components obtained by splitting M_n along $F^{(i)}$ by $C(\Gamma_\pm^{(i)})$.

4. Invariants

4.1. **Traces.** If $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ is a discrete group, its trace field $\mathbb{Q}(\mathrm{tr}\,\Gamma)$ is obtained by adjoining to $\mathbb{Q}$ the traces of elements of Γ. If the hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ has finite volume, Mostow rigidity implies that this is a topological invariant of M. It follows from the local rigidity theorems of Garland-Prasad that in this case the trace field is a number field; ie, a finite extension of $\mathbb{Q}$. The trace field is not generally an invariant of the commensurability class of M, however, and to obtain one we pass to the *invariant trace field* $k\Gamma$. This is obtained by adjoining to $\mathbb{Q}$ the traces of squares of elements of Γ. When M is the complement of a link in a $\mathbb{Z}_2$–homology sphere, its trace field and invariant trace field coincide (cf. [19]).

Proposition 4.1. *$k(\Gamma_S) = \mathbb{Q}(i)$, $k(\Gamma_T) = \mathrm{Q}(i\sqrt{2})$, and $k(M_n) = \mathbb{Q}(i, i\sqrt{2})$ for all $n \in \mathbb{N}$.*

Proof. Its definition in Corollary 2.2 immediately implies $\Gamma_S < \mathrm{PSL}_2(\mathbb{Q}(i))$. The description in Corollary 2.3, of Γ_{T_0}, and Lemma 3.3 imply that $\Gamma_T < \mathrm{PSL}_2(\mathbb{Q}(i\sqrt{2}))$. Thus $k\Gamma_S \subseteq \mathbb{Q}(i)$, and $k\Gamma_T \subseteq \mathbb{Q}(i\sqrt{2})$. That the containments are equalities is clear by noting that $\mathrm{Tr}(\mathsf{h}) = \pm i\sqrt{2}$ and $\mathrm{Tr}(\mathsf{t}) = \pm(1+i)$. To establish the claim regarding Γ_n, we note that the element c defined in Lemma 3.3 lies in $\mathrm{PSL}_2(\mathbb{Q}(i\sqrt{2}))$, and each Γ_n is contained in the group generated by Γ_S, Γ_T, and c. □

We say $M = \mathbb{H}^3/\Gamma$ has *integral traces* if for each $\gamma \in \Gamma$, tr γ is an algebraic integer. Otherwise we say M_n has a nonintegral trace. M has integral traces if and only if all manifolds commensurable to M do as well (cf. [19]).

Proposition 4.2. *For each n, M_n has integral traces.*

Proof. As in the proposition above, this follows from the fact that each Γ_n is contained in the group generated by Γ_S, Γ_T, and c. It is easy to see that the *entries* of the generators for Γ_S and Γ_{T_0} are algebraic integers. Since c has integral entries as well, all elements of Γ_n have integral entries, hence integral traces. □

Remark. Bass showed that if $M = \mathbb{H}^3/\Gamma$ where Γ has an element with a nonintegral trace, there are closed essential surfaces in M associated to this trace [5]. We say that such surfaces are *detected by the trace ring*. For fixed n and $1 \le i \le n$, closed essential surfaces in M_n can be obtained by "tubing" $S^{(i)}$ through $B^3 - L_-^{(i)}$. More precisely, let $\mathcal{N}_i$ be a regular neighborhood of $L_-^{(i)}$ in $(B^3, L_-^{(i)}) \subset (S^3, L_n)$, let $A_i = \mathcal{N}_i \cap \overline{B^3 - \mathcal{N}_i}$, and let

$$\hat{S}_i = \overline{S^{(i)} - (S^{(i)} \cap \mathcal{N}_i)} \cup A_i.$$

Then $\hat{S}_i$ is a closed surface of genus two which is incompressible in M_n. We will show below that certain mutants have nonintegral traces, and one easily finds surfaces analogous to $\hat{S}_i$ in the mutants. It is interesting to note that although these surfaces are present in all of these link complements, the trace ring does not detect any closed surfaces in the M_n.

4.2. **Scissors congruence and the Bloch invariant.** We will prove in Proposition 4.5 below that *scissors congruence* distinguishes the commensurability class of M_m from that of M_n for $m \neq n$. Manifolds are scissors congruent if they can be cut into identical collections of hyperbolic polyhedra. A fundamental tool for proving results about the scissors congruence relation is its connection to the Bloch invariant, which allows access to deep theorems from algebraic k-theory. We focus on this connection below, referring the reader to the thorough and very readable survey [25] for a broader exposition.

Definition 4.3. Let $k \subset \mathbb{C}$ be a field. Define the *pre-Bloch group* $\mathcal{P}(k)$ to be the quotient of the free $\mathbb{Z}$–module on $k - \{0,1\}$ by all instances of the following relations.

$$(4) \quad [x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] = 0, \quad x \neq y \in k - \{0,1\}$$

$$(5) \quad [z] = \left[1 - \frac{1}{z}\right] = \left[\frac{1}{1-z}\right] = -\left[\frac{1}{z}\right] = -\left[\frac{z}{z-1}\right] = -[1-z], \quad z \in k - \{0,1\}$$

There is a map $\delta\colon \mathcal{P}(k) \to k^* \wedge k^*$ given by $[z] \mapsto 2(z \wedge (1-z))$. (Here k^* is considered a $\mathbb{Z}$–module with multiplication as the group operation and $\mathbb{Z}$–action given by $a.x = x^a$, $a \in \mathbb{Z}$.) The *Bloch group* is $\mathcal{B}(k) = \ker \delta$.

Remark. If k is algebraically closed, the relation (4) above, called the *five term relation*, implies the relation (5). For instance, taking $\sqrt{z}$ and $\sqrt{z^{-1}}$ as x and y, respectively, in (4), then interchanging their roles and summing the results yields $[z] + [1/z] = 0$.

An ideal tetrahedron T in $\mathbb{H}^3$ determines an element $[z] \in \mathcal{P}(\mathbb{C})$ as follows. Let z_1, z_2, z_3, and z_4 be the ideal vertices of T on $\widehat{\mathbb{C}} = \partial_\infty \mathbb{H}^3$, ordered as follows. The intersection of T with a sufficiently small horosphere H centered at z_4 determines a Euclidean triangle in T, which we give the boundary orientation from the component of $\mathbb{H}^3 - H$ containing the remaining ideal vertices. We order them so that traversing the boundary of the triangle $H \cap T$ in the direction determined by its orientation visits the vertices determined by z_1, z_2, and z_3 in that order (up to cyclic permutation). We define the *cross ratio parameter* of T as

$$z = [z_1 : z_2 : z_3 : z_4] = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_3 - z_1)(z_4 - z_2)}.$$

The choices made above only determine the ordering on the ideal vertices of T up to even permutation. It is easily verified that changing the order by an even permutation replaces z by one of z, $\frac{1}{1-z}$, or $1-\frac{1}{z}$. Thus by the relation (5), $[z]$ is a well-defined element of $\mathcal{P}(\mathbb{C})$. Since the cross ratio is invariant under Möbius transformations, $[z]$ is an invariant of the orientation–preserving isometry class of T. Changing the ordering of the ideal vertices by an odd permutation replaces z by one of $\frac{1}{z}$, $\frac{z}{z-1}$, or $1-z$.

Definition 4.4. Let $M=\mathbb{H}^3/\Gamma$ be a complete, orientable hyperbolic 3-manifold of finite volume with a specified triangulation; that is, $M=T_1\cup\ldots\cup T_n$, each T_i isometric to an ideal hyperbolic tetrahedron, with $T_i\cap T_j$ either empty, an edge of each, or a face of each, for $i\neq j$. For each i in $\{1,\ldots,n\}$, let z_i be the cross ratio parameter of T_i, and define the *Bloch invariant* $\beta(M)$ as follows.

$$\beta(M)=[z_1]+[z_2]+\ldots+[z_n]\in\mathcal{B}(\mathbb{C})$$

If M is a complete, orientable hyperbolic 3-manifold of finite volume with totally geodesic boundary, a triangulation of M determines one of its double DM. Define $\beta(M)=\frac{1}{2}\beta(DM)$.

Remark. It is not clear *a priori* that $\delta(\beta(M))=0$, but this follows from a geometric interpretation of the Bloch invariant. Neumann-Yang [26] showed that $\beta(M)$ does not depend on the choice of triangulation, and furthermore that $\beta(M)\in\mathcal{B}(k)$ when M is not compact, where k is the trace field of M.

Recall that the cross ratio $z=[z_1:z_2:z_3:z_4]$ is characterized by the property that z is the destination of z_1 under the Möbius transformation sending z_2, z_3, and z_4 to 1, ∞, and 0, respectively. For an ideal tetrahedron T with its ideal vertices ordered as described above, it follows that the cross ratio parameter of T lies in the upper half plane.

The *Bloch-Wigner dilogarithm function* $D_2\colon\ \mathbb{C}-\{0,1\}\to\mathbb{R}$ describes the volume of an ideal tetrahedron in terms of its cross ratio parameter. We record the formula below.

$$D_2(z)=\Im\psi(z)+\log|z|\arg(1-z),\ \text{where}\ \psi(z)=\sum_{i=1}^{\infty}\frac{z^n}{n^2}.$$

For z in the upper half plane, $D_2(z)$ is the volume of the ideal tetrahedron with ideal vertices z, 1, ∞, and 0. Note that $D_2(\bar{z})=-D_2(z)$. The theorem below uses D_2 to define a map known as the *Borel regulator.*

Theorem ([6])**.** *Suppose k is a number field with complex embeddings $\sigma_1,\ldots,\sigma_{r_2}$, one representing each complex–conjugate pair. There is a map $\mathcal{B}(k)\to\mathbb{R}^{r_2}$ extending $[z]\mapsto(D_2(\sigma_1(z)),\ldots,D_2(\sigma_{r_2}(z)))$, which takes $\mathcal{B}(k)$ onto a lattice in $\mathbb{R}^{r_2}$, with kernel consisting entirely of torsion elements.*

The octahedron $\mathcal{P}_1$ may be divided into a collection of 4 tetrahedra by the addition of a single edge, joining the ideal vertices $(1+i)/2$ and ∞, and four triangular faces. Each new face has vertices $(1+i)/2$, ∞, and one of the remaining vertices of $\mathcal{P}_1$. The resulting triangulation projects to a triangulation of M_S. We leave it as an exercise to the reader to triangulate the cuboctahedron $\mathcal{P}_2$, with the additional requirement that the induced triangulation of its internal faces is preserved by the face pairings. Such a triangulation projects to one of M_{T_0}.

The map p_S of Corollary 2.2 takes the triangulation of M_S described above to one of $C(\Gamma_S)$, and the map p_{T_0} of Corollary 2.3 does the same for $C(\Gamma_{T_0})$. Since the external faces of $\mathcal{P}_1$ and $\mathcal{P}_2$ are ideal triangles, they are faces of any triangulation obtained by subdivision; hence the triangulations inherited by $C(\Gamma_S)$ and $C(\Gamma_{T_0})$ intersect their boundaries in Δ_S and $\Delta_{T_0}^{\pm}$, respectively (as defined below Corollaries 2.2 and 2.3). Then Lemma 3.3 implies that $C(\Gamma_T)$ is triangulated by the union of the triangulation of $C(\Gamma_{T_0})$ and its image under $\phi_{\mathsf{c}^{-2}\mathsf{r}}$.

Now fix $n \in \mathbb{N}$. For each i between 1 and n, $C(\Gamma_T^{(i)})$ is triangulated by the image under ϕ_i of the triangulation of $C(\Gamma_T)$. Also, $C(\overline{\Gamma}_S^{c^{-2n}})$ is triangulated by the image under $\phi_{\mathsf{c}^{-2n}\mathsf{r}}$ of that of $C(\Gamma_S)$. By the construction in Section 3, the union of these triangulates M_n if the gluing maps $\iota_+^{(i)}(\iota^{(i)})_-)^{-1}$ preserve their boundaries. But this follows from assertion (3) of Lemma 2.4 and from Lemma 3.4. Hence M_n is triangulated in this fashion. Below, the details of the triangulation of M_n are not so important as this fact.

Proposition 4.5. *For $m \neq n$, M_m is not commensurable with M_n.*

Proof. Let β_1 be the Bloch invariant of $C(\Gamma_S)$, and let β_2 be the Bloch invariant of $C(\Gamma_{T_0})$. Then $2 \cdot \beta_1 \in \mathcal{B}(\mathbb{Q}(i))$, and applying the Borel regulator map to $2 \cdot \beta_1$ yields the volume of the double of $C(\Gamma_S)$, which is twice its volume v_1. Since $C(\Gamma_T)$ is itself a double, the Borel regulator takes $\beta_2 \in \mathcal{B}(\mathbb{Q}(i\sqrt{2}))$ to the volume v_2 of $C(\Gamma_T)$. It follows that β_1 and β_2 have infinite order in the pre-Bloch group.

The inclusion of $\mathbb{Q}(i)$ into $k = \mathbb{Q}(i, i\sqrt{2})$ induces a natural map $\mathcal{B}(\mathbb{Q}(i)) \to \mathcal{B}(k)$, and similarly for $\mathcal{Q}(i\sqrt{2}) \subset k$. It is clear that k has a pair of complex conjugate embeddings, each determined by its action on i and $i\sqrt{2}$. We will take $\sigma_1 = id_k$, and $\sigma_2(i) = i$, $\sigma_2(i\sqrt{2}) = -i\sqrt{2}$, in defining the Borel regulator on k. Since each σ_i restricts on $\mathbb{Q}(i)$ to the identity, the Borel regulator on k takes β_1 to (v_1, v_1). On the other hand, the Borel regulator on k takes β_2 to $(v_2, -v_2)$. Thus β_1 and β_2 are linearly independent in $\mathcal{B}(k)$, since their images are linearly independent in $\mathbb{R}^2$.

For any n, the triangulation which M_n inherits, from the two copies of $C(\Gamma_S)$ and n copies of $C(\Gamma_T)$ which comprise it, has Bloch invariant $2\beta_1 + 2n\beta_2 \in \mathcal{B}(k)$. If $\widetilde{M} \to M_n$ is a covering space of degree k, the preimage in $\widetilde{M}$ of each tetrahedron T in the triangulation of M_n is a disjoint union of k isometric copies of T. Thus $\beta(\widetilde{M}) = k\beta(M_n)$. If M_m were commensurable with M_n, it would follow that there is a rational number q such that

$$q\,[2\beta_1 + 2m\beta_2] = 2\beta_1 + 2n\beta_2.$$

But since β_1 and β_2 are linearly independent, this forces $q = 1$ and $m = n$. □

4.3. **Cusp parameters.**

Definition 4.6. Let $T = \mathbb{C}/\Lambda$ be a Euclidean torus, where $\Lambda \subset \mathbb{C}$ is a lattice. Define the *complex modulus* of T as $m(T) = \alpha/\beta$, where $\Lambda = \langle \alpha, \beta \rangle$.

Remark. This is well defined only up to the action of $\mathrm{PGL}_2(\mathbb{Z})$ by Möbius transformations: a different generating pair $\gamma = p\alpha + q\beta$, $\delta = r\alpha + s\beta$ determines an invertible matrix $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$. Computing the modulus with γ and δ yields

$$m(T) = \frac{p\alpha + q\beta}{r\alpha + s\beta} = \frac{p(\alpha/\beta) + q}{r(\alpha/\beta) + s}.$$

It will prove useful here to understand the complex modulus of a torus by decomposing it into annuli using a family of parallel geodesics.

Definition 4.7. For a Euclidean annulus A with core of length ℓ and distance d between geodesic boundary components, define the *real modulus* of A by $m(A) = d/\ell$.

If $T = \mathbb{C}/\Lambda$, and $\Lambda = \langle \alpha, \beta \rangle$, then α and β determine isotopy classes of simple closed geodesics on T with representatives which intersect once. These are the projections to T of the line segments in $\mathbb{C}$ joining 0 to α and β, respectively. Below let A_β denote the Euclidean annulus with geodesic boundary obtained as the path completion of the metric on $T - \beta$ inherited from T.

Lemma 4.8. *Let $T = C/\Lambda$ be a Euclidean torus, and suppose α, β is a generating pair for Λ. Decompose $m(T)$ into real and imaginary parts:*

$$m(T) = \tau_\beta + i \cdot \mu_\beta,$$

where $\tau_\beta = \Re(\alpha/\beta)$ and $\mu_\beta = \Im(\alpha/\beta) \in \mathbb{R}$. Then $\tau_\beta = \frac{\|\alpha\|}{\|\beta\|}\cos\theta$, where θ is the angle between the geodesics α and β on T, and $|\mu_\beta| = m(A_\beta)$.

Proof. Write $\alpha = \|\alpha\|e^{i\theta_1}$ and $\beta = \|\beta\|e^{i\theta_2}$. Then $\theta = \theta_1 - \theta_2$ is the angle between the geodesics corresponding to α and β, and $\frac{\alpha}{\beta} = \frac{\|\alpha\|}{\|\beta\|}e^{i\theta}$. Writing $e^{i\theta} = \cos\theta + i\sin\theta$ yields the first assertion immediately.

To establish the second, consider the strip $\widetilde{A}_\beta$ in $\mathbb{C}$ bounded by the line containing 0 and β and its translate by α, containing α and $\alpha+\beta$. The quotient of $\widetilde{A}_\beta$ induced by the action of β is the universal covering $\widetilde{A}_\beta \to A_\beta$. The distance between boundary components of $\widetilde{A}_\beta$ is $\|\alpha\||\sin\theta|$, and the length of the core of A_β is the translation length of β, which is $\|\beta\|$. □

Lemma 4.8 provides a convenient means for understanding the modulus of a Euclidean torus in terms of "Fenchel-Nielsen" coordinates (μ_β, τ_β) associated to a simple closed geodesic β. We regard μ_β as a length parameter for the annulus A_β, and τ_β as a twist parameter.

Lemma 4.9. *Suppose T is a Euclidean torus decomposed into annuli $A_1, \ldots, A_n$ by simple closed geodesics parallel to β. Then*

$$|\mu_\beta| = m(A_1) + m(A_2) + \ldots + m(A_n).$$

Proof. By isotoping β if necessary, we may assume that it is one of the geodesics determining the A_i; hence $A_\beta = A_1 \cup A_2 \ldots \cup A_n$. Then if α_0 is an arc perpendicular to ∂A_β, joining one component to the other, for each i, $\alpha_0 \cap A_i$ is an arc perpendicular to ∂A_i joining one component to the other. This is because ∂A_i is parallel to β. Since $\ell(\alpha_0) = \sum_i \ell(\alpha_0 \cap A_i)$ and the core of each A_i has length $\ell(\beta)$, the result follows. □

The annuli we are concerned with arise as horospherical cross sections of the cusps of M_S and M_T. Recall from Lemma 2.4 that $\mathrm{Stab}_{\Gamma_S}(\mathcal{H})$ is a group Λ generated by parabolic isometries p_1, p_2 and p_3. Furthermore, as pointed out in Remark 1 below the lemma, p_1 and p_3 are conjugate in Γ_S, as are p_2 and $\mathsf{p}_4 = \mathsf{p}_1\mathsf{p}_2\mathsf{p}_3^{-1}$. We asserted there that $C(\Gamma_S)$ has two cusps, one corresponding to p_1 and one to p_2. This follows from the lemma below.

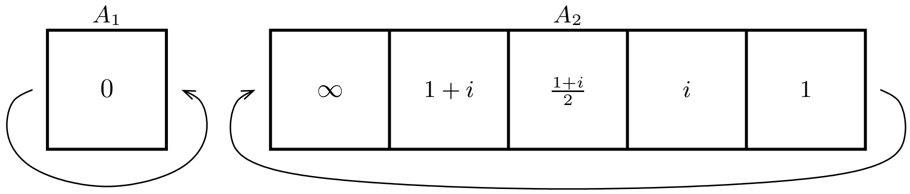

FIGURE 8. Cross sections of the cusps of M_S

In what follows, we let $\mathcal{V}_1 = \{\infty, 0, 1, i, 1+i, (1+i)/2\}$, the set of ideal vertices of the ideal octahedron $\mathcal{P}_1$. Let $\{h_v \,|\, v \in \mathcal{V}_1\}$ be a collection of horospheres invariant under the action of the symmetry group of $\mathcal{P}_1$, such that h_v is centered at v for each $v \in \mathcal{V}$ and h_∞ is at height 2.

Lemma 4.10. *The projection to M_S of $\bigcup (h_v \cap \mathcal{P}_1)$ is a disjoint union of Euclidean annuli A_1 and A_2 with geodesic boundary, such that $p_S(A_1)$ is a horospherical cross section of the cusp of $C(\Gamma_S)$ corresponding to p_1, $p_S(A_2)$ is a cross section of the cusp corresponding to p_2, and $m(A_1) = 1$, $m(A_2) = 1/5$.*

Proof. Since h_∞ is at height 2 and our embedding of $\mathcal{P}_1$ is as in Figure 2, $h_\infty \cap \mathcal{P}_1$ is a square with sides of length $1/2$. Since the symmetry group of $\mathcal{P}_1$ acts transitively on vertices, this holds for all $h_v \cap \mathcal{P}_1$, $v \in \mathcal{V}$. We will call a side of $h_v \cap \mathcal{P}_1$ *internal* if it is contained in an internal face of $\mathcal{P}_1$ and *external* otherwise. The face–pairing s has the property that if v and v' are ideal vertices of $\mathcal{P}_1$ and $\mathsf{s}(v) = v'$, then $\mathsf{s}(h_v) = h_{v'}$, and $\mathsf{s}(h_v \cap \mathcal{P}_1)$ abuts $h_{v'} \cap \mathcal{P}_1$ along an internal side. The analogous property holds for t.

Each of s and t identifies a pair of internal faces of $\mathcal{P}_1$, yielding M_S. The isometry p_S of Corollary 2.2 is induced by the inclusion $\mathcal{P}_1 \to \mathbb{H}^3$. Since $\mathsf{p}_1 = \mathsf{s}^{-1}$ fixes the ideal vertex of $\mathcal{P}_1$ at 0, it identifies the opposite internal sides of $h_0 \cap \mathcal{P}_1$. This square thus projects to a cross section of the cusp of M_S mapping under p_S to the cusp of $C(\Gamma_S)$ corresponding to p_1; that is, A_1. This is depicted on the left side of Figure 8.

The other cusp cross section of M_S, the annulus A_2, is the identification space of the collection

$$\{h_v \cap \mathcal{P}_1 \,|\, v \in \mathcal{V} - \{0\}\}$$

shown on the right side of Figure 8. In the figure, each square is the projection to M_S of $h_v \cap \mathcal{P}_1$ for the ideal vertex v by which it is labeled. The combinatorics can be verified by considering the action of s and t on $\mathcal{V}$.

By assumption each square in Figure 8 has side length $1/2$, and so the cores of A_1 and A_2 have lengths $1/2$ and $5/2$, respectively. For any square in Figure 8, a vertical side projects to an arc joining the distinct boundary components of the corresponding A_i, hence the distance between them is $1/2$. Thus it follows directly from the definition that $m(A_1) = 1$ and $m(A_2) = 1/5$. □

The lemma below describes the moduli of the cusps of $C(\Gamma_{T_0})$. We asserted below Lemma 2.5 that $C(\Gamma_{T_0})$ has four cusps, one corresponding to each p_i, $i \in \{1, 2, 3, 4\}$, and each joining $\partial_- C(\Gamma_{T_0})$ to $\partial_+ C(\Gamma_{T_0})$. This follows from Lemma 4.11 below. Let $\mathcal{V}_2$ be the set of ideal vertices of $\mathcal{P}_2$, and consider a collection of horospheres

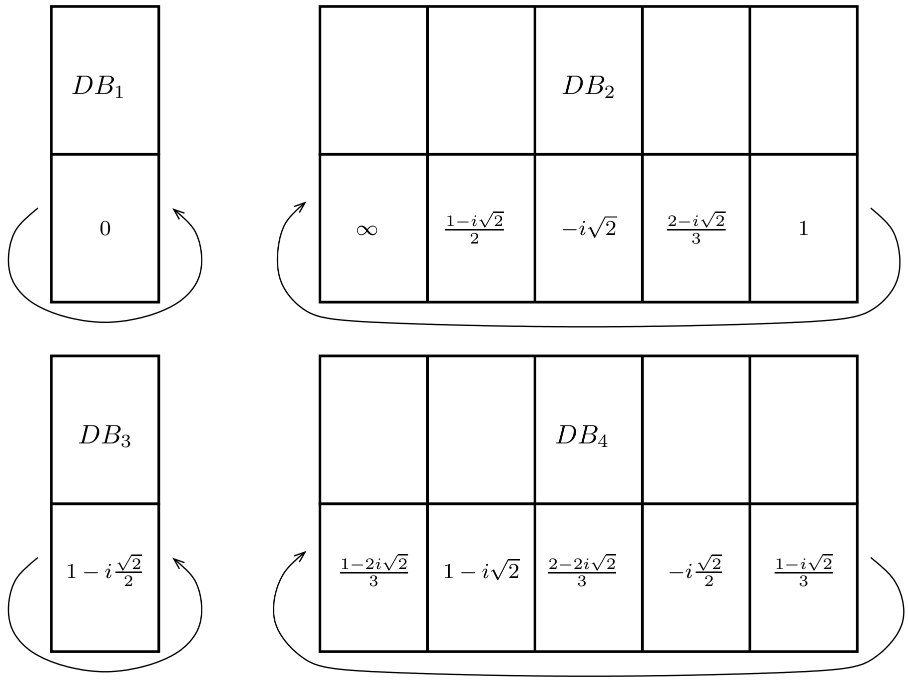

FIGURE 9. Cross sections of the cusps of M_T

$\{h_v \mid v \in \mathcal{V}_2\}$, invariant under the symmetry group of $\mathcal{P}_2$, such that h_v is centered at v for each $v \in \mathcal{V}$ and h_∞ is at height 2.

Lemma 4.11. *The projection of $\bigcup(h_v \cap \mathcal{P}_2)$ to M_{T_0} is a collection of disjoint Euclidean annuli B_j with geodesic boundary, $j \in \{1,2,3,4\}$, such that $p_{T_0}(B_j)$ is a cross section of the cusp of $C(\Gamma_{T_0})$ corresponding to $\mathsf{p}_j \in \Lambda$, and $m(B_1) = m(B_3) = \sqrt{2}$, $m(B_2) = m(B_4) = \sqrt{2}/5$.*

Proof. For $v \in \mathcal{V}_2$, we again call a side of $h_v \cap \mathcal{P}_2$ external if it is contained in an external face of $\mathcal{P}_2$ and internal otherwise. Each cusp cross section of M_{T_0} is the projection of a subcollection of the $h_v \cap \mathcal{P}_2$, identified along their internal faces. From Figure 4, we find that $h_\infty \cap \mathcal{P}_2$ is a Euclidean rectangle with two opposite internal sides and two external. Since the symmetry group of $\mathcal{P}_2$ is transitive on its set of ideal vertices, this holds for the other h_v as well. It follows that each cusp cross section of M_{T_0} is a Euclidean annulus with geodesic boundary.

In Figure 9, the lower rectangles of each annulus DB_j are labeled by vertices v such that $h_v \cap \mathcal{P}_2$ projects to a subrectangle of cross section of the cusp of M_{T_0} whose image under p_{T_0} corresponds p_j. Then B_j is the lower half of DB_j. We will justify this below.

The isometries f, g, and h defined in Corollary 2.3 identify the internal sides of $\mathcal{P}_2$ in pairs, yielding the manifold M_{T_0} with totally geodesic boundary. The parabolic $\mathsf{p}_1 = \mathsf{f}^{-1}$ fixes 0, identifying the internal sides of $\mathcal{P}_2$ sharing this ideal vertex. Thus in M_{T_0}, B_1 consists of $h_0 \cap \mathcal{P}_2$ with its internal sides identified. The description of the p_i in terms of f, g, and h above Lemma 2.4 shows that p_3 is a conjugate of g^{-1}.

Since g^{-1} fixes $1-i\frac{\sqrt{2}}{2}$, identifying the internal edges of $\mathcal{P}_2$ which abut it, $h_{1-i\sqrt{2}/2}$ projects to B_3 in M_{T_0}. This justifies the depictions of B_1 and B_3 in Figure 9.

Since p_2 fixes ∞, $h_\infty \cap \mathcal{P}_2$ projects to a subrectangle of B_2. Since g takes the internal side Y_3 to Y_1' and ∞ to $(1-i\sqrt{2})/2$, in B_2 the projection of $h_\infty \cap \mathcal{P}_2$ meets the projection of $h_{(1-i\sqrt{2})/2} \cap \mathcal{P}_2$ along a side contained in the projection of Y_3 to M_{T_0}. Since the internal face of $\mathcal{P}_2$ meeting Y_1' at $(1-i\sqrt{2})/2$ is Y_3', and this is taken to Y_2' by h^{-1}, the rectangle meeting the projection of $h_{(1-i\sqrt{2})/2}$ in B_2 on the internal side opposite its intersection with h_∞ is $h_{-i\sqrt{2}}$. Further consideration of the combinatorics of the action of f, g, and h yields the depictions of B_2 and B_4 in the figure.

From Figure 4, we find that the internal sides of $h_\infty \cap \mathcal{P}_2$ have length $\sqrt{2}/2$ and the external sides length $1/2$. Since the symmetry group of $\mathcal{P}_2$ is transitive on its ideal vertices, the same holds for each rectangle $h_v \cap \mathcal{P}_2$. Thus the cores of B_1 and B_3 have length $1/2$, and the cores of B_2 and B_4 have length $5/2$. For any square $h_v \cap \mathcal{P}_2$, an internal side projects to a perpendicular arc joining opposite sides of the cusp cross section in M_{T_0} containing $h_v \cap \mathcal{P}_2$. The moduli are thus as described. □

Recall that Lemma 3.3 describes $C(\Gamma_T)$ as the double of $C(\Gamma_{T_0})$ across the image of $F' = \mathsf{k}(\mathcal{H})/\Lambda^{\mathsf{k}}$ under the natural map described in Lemma 2.5. Therefore p_{T_0} determines a reflection-invariant map from the double M_T of M_{T_0} across $\partial_+ M_{T_0}$ to $C(\Gamma_T)$. Furthermore, as we remarked below Lemma 2.5, each cusp of $C(\Gamma_{T_0})$ joins one component of $\partial C(\Gamma_{T_0})$ to the other. Therefore taking $DB_j \subset M_T$, $j \in \{1,2,3,4\}$, to be the double of B_j across its component of intersection with $\partial_+ M_{T_0}$, we have:

Lemma 4.12. *For each $j \in \{1,2,3,4\}$, the image in $C(\Gamma_T)$ of DB_j is a cross section of the cusp corresponding to p_j, and $m(DB_1) = m(DB_3) = 2\sqrt{2}$, $m(DB_2) = m(DB_4) = 2\sqrt{2}/5$.*

The complex modulus is clearly a similarity invariant of a Euclidean torus. We will say T and T' are *commensurable* if T has a finite cover which is similar to a cover of T'. The following lemma can be verified by comparing the modulus of a lattice in $\mathbb{C}^2$ to moduli of finite–index sublattices.

Lemma 4.13. *If T and T' are commensurable, their moduli differ by the action of an element of* $\mathrm{PGL}_2(\mathbb{Q})$.

It is a well known consequence of Margulis' lemma that each cusp C of a hyperbolic manifold $M = \mathbb{H}^3/\Gamma$ of finite volume is foliated by similar Euclidean tori, the projections to M of horospheres in $\mathbb{H}^3$ centered at the fixed point of a parabolic subgroup of Γ corresponding to C.

Definition 4.14. The *parameter* of a cusp C of a finite–volume complete hyperbolic manifold is the complex modulus of a horospherical cross–section of C.

Below we describe the cusp parameters of the M_n up to the action of $\mathrm{PGL}_2(\mathbb{Q})$. For $j = 1,2$, define $A_j' = p_S(A_j) \subset C(\Gamma_S)$ and $\overline{A}_j' = \phi_{\mathsf{r}} \circ p_S(A_j) \subset C(\overline{\Gamma_S})$. For $j = 1,\ldots,4$ and $i \in \mathbb{N}$, define $DB_j^{(i)} = \phi_i \circ p_T(DB_j) \subset C(\Gamma_T^{(i)})$. We will also refer by A_j', $\overline{A}_j'$, and $DB_j^{(i)}$ to their images in $C(\Gamma_n)$ under the natural map and to their images in M_n under the isometry $C(\Gamma_n) \to M_n$.

Proposition 4.15. *For $j = 1, 2$, let T_j be a cusp cross section of M_n such that $T_j \supset A'_j$. Then $m(T_1) = i(2 + 4n\sqrt{2})$, and $m(T_2) = i(2 + 4n\sqrt{2})/5$.*

Remark. These parameters are identical up to the action of $\mathrm{PGL}_2(\mathbb{Q})$, but distinct under the action of $\mathrm{PGL}_2(\mathbb{Z})$.

Proof. Recall from Proposition 3.9 that $M_n = C(\Gamma_S) \cup C(\Gamma_T^{(1)}) \cup \ldots \cup C(\Gamma_T^{(n)}) \cup C(\overline{\Gamma}_S)$, where the gluing maps factor through the inclusion induced isometries $\iota_\pm^{(i)}$ defined on $F^{(i)}$, $0 \le i \le n$.

Let $\partial_\pm DB_j^{(i)} = DB_j^{(i)} \cap \partial_\pm C(\Gamma_T^{(i)})$. By Lemma 4.10, A'_1 is a cross section of the cusp of $C(\Gamma_S)$ corresponding to p_1, and by Lemma 4.12, $DB_1^{(1)}$ is a cross section of the cusp of $C(\Gamma_T)$ corresponding to p_1. Lemma 2.4 implies that the gluing map $\iota_+^{(0)}(\iota_-^{(0)})^{-1}$ takes a component of $\partial A'_1$ to $\partial_- DB_1^{(0)}$. In Remark 1 below that lemma, we note that p_1 and p_3 are conjugate in Γ_S. Hence the other component of ∂A_1, must be a cross section of the cusp of $\partial C(\Gamma_S)$ corresponding to p_3. So $\iota_+^{(0)}(\iota_-^{(0)})^{-1}$ identifies this component of $\partial A'_1$ to $\partial_- DB_3^{(1)}$.

The definitions and Lemma 3.4 imply that for each i between 1 and $n-1$, $\iota_+^{(i)}(\iota_-^{(i)})^{-1}$ takes $\partial_+ DB_j^{(i)}$ to $\partial_- DB_j^{(i+1)}$. This uses the fact that DB_j is by its construction invariant under the doubling involution $\phi_{\mathsf{c}^{-2}\mathsf{r}}$, and the characterization (3) from the proof of Proposition 3.9:

$$\iota_+^{(i)}(\iota_-^{(i)})^{-1} = \phi_{i+1}\iota_+^{(0)}\phi_2^{-1}\left(\iota_-^{(1)}\right)^{-1}\phi_i^{-1} = \phi_{i+1}\phi_{\mathsf{c}^{-2}\mathsf{r}}\phi_i^{-1}.$$

The above implies that T_1 is decomposed by its intersection in M_n with the separating spheres $F^{(i)}$ into the following collection of Euclidean annuli with geodesic boundary.

$$A'_1 \cup DB_1^{(1)} \cup \ldots \cup DB_1^{(n)} \cup \overline{A}'_1 \cup DB_3^{(n)} \cup \ldots DB_3^{(1)}$$

Similarly, we find that T_2 decomposes into the union of A'_2, $\overline{A}'_2$, and $DB_j^{(i)}$ for $1 \le i \le n$ and $j = 2, 4$. We may take β_1 to be the geodesic $\partial_- DB_1^{(1)}$ on T_1 and $\beta_2 = \partial_- DB_2^{(1)} \subset T_2$. Then we obtain the following from Lemma 4.9, applying lemmas 4.10 and 4.12.

$$\Im(m(T_1)) = \pm(2 + 4n\sqrt{2}) \qquad\qquad \Im(m(T_2)) = \pm\frac{2 + 4n\sqrt{2}}{5}$$

We will show $m(T_1)$ and $m(T_2)$ have real part equal to 0 by describing geodesics α_j, $j = 1, 2$, which meet the β_j once, perpendicularly. Let a_1 be the arc in A'_1 which is the image of the projection of the internal edges of $h_0 \cap \mathcal{P}_1$ (the vertical arcs on the left-hand square in Figure 8) in $C(\Gamma_S)$. Recall that the internal edges of $h_0 \cap \mathcal{P}$ are its intersection with internal faces of $\mathcal{P}_1$. In particular, ∂a_1 is the intersection of A'_1 with the one-skeleton of the triangulation Δ_S defined below Corollary 2.2.

Let $b_1 \subset B_1$ and $b_3 \subset B_3$ similarly be projections of internal edges of $h_0 \cap \mathcal{P}_2$ and $h_{1-i\sqrt{2}/2} \cap \mathcal{P}_2$, respectively (see Figure 9), and let db_1 and db_3 be the geodesic arcs of DB_1 and DB_3 containing them. Let $db_j^{(i)} = \phi_i \circ p_T(db_j)$, and let $\partial_\pm db_j^{(i)} = db_j^{(i)} \cap \partial_\pm DB_j^{(i)}$, $j = 1, 3$ and $i \in \mathbb{N}$. Then $\partial_\pm db_j^{(i)}$ is the intersection of $\partial DB_j^{(i)}$ with the one-skeleton of $\phi_i(\Delta_T^\pm)$, where Δ_T^- is the image of the triangulation $\Delta_{T_0}^-$ defined below Corollary 2.3 under the inclusion $M_{T_0} \to M_T$, and Δ_T^+ is its reflected image.

By Lemma 2.4, $\iota_+^{(0)}(\iota_-^{(0)})^{-1}$ preserves triangulations, and the discussion above implies that the other gluing maps do as well. From Figure 5 it is apparent that the cusps of $F^{(0)}$ corresponding to p_1 and p_3 each contain only one end of an edge of the triangulation $\mathcal{T}$ which it inherits from $\mathcal{F}$. Therefore $\iota_0(\partial a_1) = \partial_- db_1^{(1)} \cup \partial_- db_3^{(1)}$. It then follows from (3) that

$$\alpha_1 = a_1 \cup db_1^{(1)} \cup \ldots \cup db_1^{(n)} \cup \bar{a}_1 \cup db_3^{(n)} \cup \ldots \cup db_3^{(1)}$$

is a closed geodesic on T_1 which meets β_1 once, at right angles. Therefore by Lemma 4.8, $\Re(m(T_1)) = 0$, and up to the action of $\mathrm{PGL}_2(\mathbb{Z})$, $m(T_1) = i(2 + 4n\sqrt{2})$.

A similar argument will give $m(T_2)$. Let $\mathcal{A}_2$ be the collection of arcs in A_2' which are the projections of internal edges of the squares comprising it. From Figure 8, one finds that $\mathcal{A}_2$ consists of five arcs evenly spaced around A_2', each joining one component of $\partial A_2'$ to the other and perpendicular to $\partial A_2'$ at each endpoint. For $j = 2, 4$, we define a collection of arcs $D\mathcal{B}_j \subset DB_j$ analogously, and take $D\mathcal{B}_j^{(i)} = \phi_i \circ p_T(D\mathcal{B}_j)$. Let $\partial_\pm D\mathcal{B}_j^{(i)} = D\mathcal{B}_j^{(i)} \cap \partial_\pm DB_j^{(i)}$, and note that the points of $\partial_\pm D\mathcal{B}_j^{(i)}$ are the points of intersection of $\partial DB_j^{(i)}$ with the one-skeleton of $\phi_i(\Delta_T^\pm)$.

For the same reasons as above, $i_+^{(0)}(i_-^{(0)})^{-1}$ takes $\partial\mathcal{A}_2$ to $\partial_- D\mathcal{B}_2^{(1)} \cup \partial_- D\mathcal{B}_4^{(1)}$, and the other gluing maps take the $\partial_+ D\mathcal{B}_j^{(i)}$ to $\partial_- D\mathcal{B}_j^{(i+1)}$ for the appropriate i and j. Then the collection

$$\mathcal{A}_2 \cup D\mathcal{B}_2^{(1)} \cup \ldots \cup D\mathcal{B}_2^{(n)} \cup \overline{\mathcal{A}}_2 \cup D\mathcal{B}_4^{(n)} \cup \ldots \cup D\mathcal{B}_4^{(1)}$$

consists of a disjoint union of up to five closed geodesics, each of which meets β_2 perpendicularly in at most 5 points.

Fix a component α_2 of the collection above, let k be the intersection number of α_2 with β_2, and let $\widetilde{T}_2$ be the k-fold cover of T_2 dual to α_2. Then β_2 lifts to $\widetilde{T}_2$, and any lift intersects the preimage $\widetilde{\alpha}_2$ of α once, perpendicularly. Computing the modulus of $\widetilde{T}_2$ using this pair, we obtain $\pm k \cdot i(2+4n\sqrt{2})/5$. This is $\mathrm{PGL}_2(\mathbb{Q})$–equivalent to $i(2 + 4n\sqrt{2})/5$ and $m(T_2)$, as claimed. □

Since commensurable hyperbolic manifolds have commensurable cusps, the collection of $\mathrm{PGL}_2(\mathbb{Q})$–orbits of the cusp parameters of M is a commensurability invariant. This fact and the lemma below provide another means for verifying Proposition 4.5.

Lemma 4.16. *Suppose $z = i(m+n\sqrt{2})$ is $\mathrm{PGL}_2(\mathbb{Q})$–equivalent to $z' = i(m+n'\sqrt{2})$, where $m, n, n' \in \mathbb{Q}$ and $m \neq 0$. Then $n' = \pm n$.*

Proof. Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{Q})$ takes z to z'. After clearing denominators (which does not change the action by Möbius transformations), we may assume that $a, b, c, d \in \mathbb{Z}$. We have

$$\frac{ai(m + n\sqrt{2}) + b}{ci(m + n\sqrt{2}) + d} = i(m + n'\sqrt{2}).$$

Multiplying by the denominator on the left, and collecting the real and imaginary parts, we find

$$m(a - d) + (an - dn')\sqrt{2} = 0 \qquad b + c(m^2 + 2nn') + cm(n' + n)\sqrt{2} = 0$$

Since 1 and $\sqrt{2}$ are linearly independent over $\mathbb{Q}$, the left-hand equation above implies that $m(a-d) = 0$ and $an - dn' = 0$. Since $m \neq 0$, the first equation implies

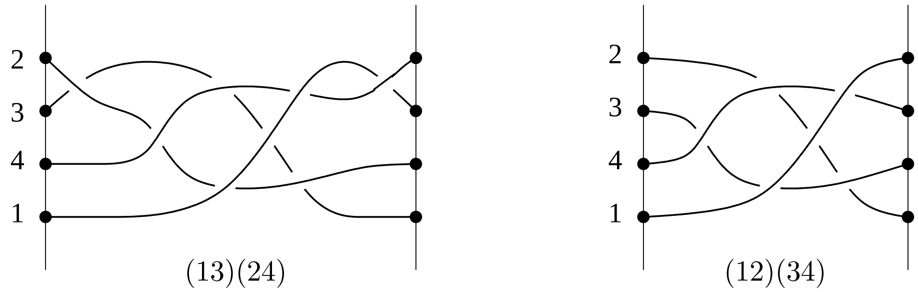

FIGURE 10. The mutations as braids.

$a = d$. Then the second equation implies $n = n'$ unless $a = d = 0$. But in this case, $c \neq 0$ since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{Q})$. Hence, using the coefficient of $\sqrt{2}$ in the right-hand equation above, we find $n' = -n$. □

5. MUTANTS

In the remaining sections, we will consider the relationship between L_n and links obtained from it by mutation along the separating spheres $S^{(i)}$, $0 \leq i \leq n$, from Definition 3.7(3). As remarked in the introduction, a *mutation* of a sphere with four marked points is a mapping class of order 2 which acts on the set of marked points by an even permutation. Such a map is uniquely determined up to isotopy by its image in the symmetric group on the marked points — this is Proposition B.2 of Appendix B.

Let S be the sphere obtained by compactifying each cusp of $F^{(0)} = \mathcal{H}/\Lambda$, defined in Lemma 2.4, with a single point. Label each additional point by a number between 1 and 4, according to the parabolic p_i corresponding to the cusp it compactifies. We will refer to the mutations of $(S, \{1, 2, 3, 4\})$ by their permutation representations in the alternating group on 4 letters.

Corollary 3.10 and the definition of $F^{(i)}$ together imply that for each i between 0 and n, $\phi_{i+1}^{-1} \circ f_n$ extends to a homeomorphism $S^{(i)} \to S$ taking marked points to marked points, which preserves the numbering of $S^{(i)} \cap L_n$ from Definition 3.7(3).

Lemma 5.1. *With L_n projected as in Figure 1, the link obtained from it by the mutation* $(13)(24)$ *(respectively,* $(12)(34)$*) along $S^{(i)}$ is obtained by cutting L_n along $S^{(i)}$ and inserting the braid on the left (resp. right) of Figure 10.*

Proof. As we asserted in the Introduction, with L_n projected in the plane and $S^{(i)}$ represented by a vertical line as in Figure 1, the points of $S^{(i)} \cap L_n$ read off as 2, 3, 4, 1 from top to bottom. This follows from Definition 3.7(3) and the projections of S and T_0 pictured in Figure 2.

If we regard a horizontal axis as running midway between points 3 and 4 so that points 2 and 1 are also equidistant from it, then the 180-degree rotation in this axis restricts on $S^{(i)}$ to an involution acting on the marked points as the permutation $(12)(34)$. There is a homeomorphism $R\colon S^{(i)} \times I \to S^{(i)} \times I$, which preserves slices $S^2 \times \{t\}$ and restricts on each to rotation by $-180 \cdot t$ degrees in the horizontal axis. This interpolates between the identity, on $S^2 \times \{0\}$, and the inverse of $(12)(34)$ on $S^2 \times \{1\}$, although it does not preserve marked points for $0 < t < 1$.

The 2-sphere $S^{(i)}$ divides S^3 into balls B^- and B^+ that contain the tangles S and $\overline{S}$ respectively. We will denote by $L_-^{(i)}$ the tangle $L_n \cap B^-$, so that the link L' obtained by the mutation $(12)(34)$ along $S^{(i)}$ is $L_-^{(i)} \cup_{(12)(34)} (L_n \cap B^+)$. Let C be a collar of $S^{(i)}$ in B^-, small enough that it intersects $L_-^{(i)}$ in the collection of horizontal arcs $\{\{j\} \times I \,|\, j \in \{1,2,3,4\}\}$. A homeomorphism $B^- \cup_{\mathsf{m}_2} B^+ \to S^3$ may be defined as the identity on B^+ and the complement of C in B^-, and as R on C. By the definition of R, the image of $L_-^{(i)} \cap C$ under R is as pictured on the right-hand side of Figure 10, thus the image of L' in S^3 under the homeomorphism described above is as stated in the lemma.

The reader should note that the braid on the left-hand side of Figure 10 is the conjugate of the braid on the right-hand side by a left-handed half-twist exchanging the points 2 and 3. This reflects the fact that the conjugate of $(12)(34)$, by any homeomorphism of $(S, \{1,2,3,4\})$ which exchanges 2 and 3 and fixes 1 and 4, is a mapping class of order 2 acting on the marked points as $(13)(24)$; hence such a conjugate is $(13)(24)$. The conjugating braid in Figure 10 tracks the marked points under an isotopy $S \times I \to S$ taking the simplest such conjugator to the identity. The conclusion for $(13)(24)$ thus follows as it did above for $(12)(34)$. □

After adjoining the braid realizing $(13)(24)$ to the tangle S of Figure 2, there is an obvious isotopy removing two crossings between the arcs labeled e and v there. The diagram which results is symmetric under the rotation of B^3 given by 180-degree rotation about the horizontal axis described in the proof of Lemma 5.1. An attribute which we ascribed to $(13)(24)$ below Theorem 2 thus follows:

Corollary 5.2. *The mutation* $(13)(24)$ *extends over* (B^3, S).

By [30, Theorem 2.2], each mutation of $(S, \{1,2,3,4\})$ is *realized* by an isometry of $F^{(0)}$. (That is, there exists an isometry of $F^{(0)}$ whose extension to $(S, \{1,2,3,4\})$ represents the mutation mapping class.) The lemma below identifies lifts to $\mathrm{PSL}_2(\mathbb{R})$ of the isometries realizing $(13)(24)$ and $(12)(34)$.

Lemma 5.3. *Define*

$$\mathsf{m}_1 = \begin{pmatrix} -3 & 5 \\ -2 & 3 \end{pmatrix} \qquad\qquad \mathsf{m}_2 = \begin{pmatrix} 0 & \sqrt{5} \\ \frac{-1}{\sqrt{5}} & 0 \end{pmatrix}$$

Each of m_1 *and* m_2 *normalizes* Λ, *and the induced isometries* ϕ_{m_1} *and* ϕ_{m_2} *of* $F^{(0)}$ *realize* $(13)(24)$ *and* $(12)(34)$, *respectively.*

Proof. Since each of m_1 and m_2 has trace equal to zero, it has order 2 in $\mathrm{PSL}_2(\mathbb{C})$. Their actions by conjugation described below, on the generators p_1, p_2, and p_3 for Λ defined above Lemma 2.4, may be verified by direct computation.

$$\mathsf{p}_1^{\mathsf{m}_1} = \mathsf{p}_3^{-1} \qquad\qquad \mathsf{p}_2^{\mathsf{m}_1} = \mathsf{p}_4^{-1}$$

$$\mathsf{p}_1^{\mathsf{m}_2} = \mathsf{p}_2 \qquad\qquad \mathsf{p}_3^{\mathsf{m}_2} = \mathsf{p}_4^{\mathsf{p}_1^{-1}}$$

Here $\mathsf{p}_4 = \mathsf{p}_1\mathsf{p}_2\mathsf{p}_3^{-1}$ is the parabolic described in Remark 1 below Lemma 2.4. It follows that m_1 and m_2 normalize Λ and induce isometries ϕ_{m_1} and ϕ_{m_2}, respectively, of $F^{(0)} = C(\Lambda)$.

Each of ϕ_{m_1} and ϕ_{m_2} has order 2, since m_1 and m_2 have order 2, and their extensions to S act on the set of marked points as described in the statement of the lemma. Its conclusion therefore follows from Proposition B.2. □

Corollary 5.4. *For $j = 1, 2$ and $i \in \mathbb{Z}$, let $\mathsf{m}_j^{(i)} = \mathsf{c}^{-2i}\mathsf{m}_j\mathsf{c}^{2i}$. Each of $\mathsf{m}_1^{(i)}$ and $\mathsf{m}_2^{(i)}$ normalizes $\Lambda^{(i)}$, and the induced isometries of $F^{(i)}$ realize $(13)(24)$ and $(12)(34)$, respectively.*

Lemma 5.1 gives a prescription for describing the links obtained from L_n by the mutations $(13)(24)$ and $(12)(34)$. The proposition below is the analog of Proposition 3.9 for the mutants, describing hyperbolic manifolds to which their complements are homeomorphic.

Proposition 5.5. *For $I = (a_0, a_1, \ldots, a_n) \in \{0, 1, 2\}^{n+1}$, let L_I be the link obtained from L_n by the following prescription: for $0 \leq i \leq n$, if $a_i = 0$, do not mutate along $S^{(i)}$; if $a_i = 1$, mutate by $(13)(24)$; and if $a_i = 2$, mutate by $(12)(34)$. Let $M_I = C(\Gamma_S) \cup C(\Gamma_T^{(1)}) \cup \ldots \cup C(\Gamma_T^{(n)}) \cup C(\overline{\Gamma}_S)$, where for each i such that $a_i = 0$ the gluing is as in Proposition 3.9, and otherwise is given by*

$$\iota_+^{(i)}\phi_{\mathsf{m}_j^{(i)}}(\iota_-^{(i)})^{-1} \text{ for } 0 \leq i < n, \text{ where } a_i = j \in \{1, 2\}; \text{ and}$$

$$\phi_{\mathsf{r}}\iota_-^{(0)}\phi_{\mathsf{c}^{2n}}\phi_{\mathsf{m}_j^{(n)}}(\iota_-^{(n)})^{-1} \text{ if } a_n = j \in \{1, 2\}.$$

Then there is a homeomorphism $f_I \colon S^3 - L_I \to M_I$ whose restriction to each complementary component of the collection $\{S^{(i)}\}$ agrees with that of f_n.

Proposition 5.5 follows immediately from Proposition 3.9 and Corollary 5.4. We note the following "obvious" isometry relation on the set of mutants $\{M_I\}$ that follows from this description.

Corollary 5.6. *For $I = (a_0, a_1, \ldots, a_n) \in \{0, 1, 2\}^{n+1}$, let $\bar{I} = (a_n, a_{n-1}, \ldots, a_0)$. There is an orientation-reversing isometry $M_I \to M_{\bar{I}}$ that, for each $i \in \{1, \ldots, n\}$, takes the image of $C(\Gamma_T^{(i)})$ in M_I to the image of $C(\Gamma_T^{(n-i)})$ in $M_{\bar{I}}$.*

Proof. It is straightforward to check that the braids in Figure 10 are isotopic (in $S^2 \times I$) to their mirror images. Therefore, there is an orientation reversing homeomorphism $L_I \to L_{\bar{I}}$. By composing this homeomorphism with f_I^{-1} and $f_{\bar{I}}$ we get a homeomorphism $M_I \to M_{\bar{I}}$. The result follows by Mostow rigidity. □

The lemma below describes the effect at the level of Kleinian groups, of cutting a hyperbolic manifold along a separating totally geodesic surface and regluing by an isometry.

Lemma 5.7. *Suppose Γ_0 and Γ_1 meet cute along a plane $\mathcal{K}$, and let Θ, $E = \mathcal{K}/\Theta$, and ι_0 and ι_1 be as in Lemma 3.2. If n normalizes Θ, then $\langle \Gamma_0, \Gamma_1^{\mathsf{n}} \rangle$ is a Kleinian group, and there is an isometry*

$$C(\Gamma_0) \cup_{\iota_1\phi_{\mathsf{n}}\iota_0^{-1}} C(\Gamma_1) \to C(\langle \Gamma_0, \Gamma_1^{\mathsf{n}} \rangle)$$

which restricts on $C(\Gamma_0)$ to the natural map, and on $C(\Gamma_1)$ to $\phi_{\mathsf{n}} \colon C(\Gamma_1) \to C(\Gamma_1^{\mathsf{n}})$ followed by the natural map.

Proof. Since n normalizes Θ, it preserves $\mathcal{K}$; hence Γ_0 and Γ_1^{n} meet cute along $\mathcal{K}$, and Lemma 3.2 applies. Thus $\langle \Gamma_0, \Gamma_1^{\mathsf{n}} \rangle$ is a Kleinian group, and in particular, the natural maps $C(\Gamma_0) \to C(\langle \Gamma_0, \Gamma_1^{\mathsf{n}} \rangle)$ and $C(\Gamma_1^{\mathsf{n}}) \to C(\langle \Gamma_0, \Gamma_1^{\mathsf{n}} \rangle)$ determine an isometry

$$C(\Gamma_0) \cup_{n\iota_1\iota_0^{-1}} C(\Gamma_1^{\mathsf{n}}) \to C(\langle \Gamma_0, \Gamma_1^{\mathsf{n}} \rangle).$$

Here we are using $n\iota_1\colon E \to C(\Gamma_1^{\mathsf{n}})$ to refer to the natural map. It is now an exercise in definition-chasing to show that $n\iota_1 \circ \phi_{\mathsf{n}} = \phi_{\mathsf{n}} \circ \iota_1$, whence the map

$$C(\Gamma_0) \cup_{\iota_1 \phi_{\mathsf{n}} \iota_0^{-1}} C(\Gamma_1) \to C(\Gamma_0) \cup_{n\iota_1 \iota_0^{-1}} C(\Gamma_1^{\mathsf{n}}),$$

defined as the identity on $\mathbb{C}(\Gamma_0)$ and ϕ_{n} on $C(\Gamma_1)$, is well-defined. The lemma follows. □

Lemma 5.7 yields the result below, which describes how the algebraic model for M_n from Proposition 3.11 changes under mutation.

Proposition 5.8. *For $I = (a_0, a_1, \dots, a_n) \in \{0,1,2\}^{n+1}$ and $0 \le i \le n$, let $\mathsf{q}_{i+1} = \mathsf{m}_{a_0}^{(0)} \cdots \mathsf{m}_{a_i}^{(i)}$, where $\mathsf{m}_0^{(j)} := id$ for every j. Define*

$$\Gamma_I \;=\; \left\langle \Gamma_S, \left(\Gamma_T^{(1)}\right)^{\mathsf{q}_1}, \dots, \left(\Gamma_T^{(n)}\right)^{\mathsf{q}_n}, \left(\overline{\Gamma}_S^{\mathsf{c}^{-2n}}\right)^{\mathsf{q}_{n+1}} \right\rangle.$$

There is an isometry $M_I \to C(\Gamma_I)$ that restricts on $C(\Gamma_S)$ to the natural map, and on $C\left(\Gamma_T^{(i)}\right)$ to $\phi_{\mathsf{q}_i} \circ \phi_i$ followed by the natural map, for $1 \le i \le n$.

Proof. □

6. Commensurable mutants

The main goal of this section is to prove that Γ_n is commensurable with the Kleinian groups associated to each of its mutants by (13)(24). There is a group that contains them all, generated by reflections in a certain family of hyperplanes.

Below, let $\mathcal{B}_0$ be the open half-ball in $\mathbb{H}^3$ bounded by the Euclidean hemisphere of unit radius centered at $0 \in \mathbb{C}$, and let $\mathcal{B}_j = \mathsf{c}^{-j}(\mathcal{B}_0)$, where c is as defined in Lemma 3.3. Recall that we have defined $\mathcal{H}$ as the geodesic hyperplane of $\mathbb{H}^3$ with ideal boundary $\mathbb{R} \cup \{\infty\}$. If w and z are complex numbers, we will take $w\mathcal{H} + z$ to be the hyperplane with ideal boundary $(w\mathbb{R} + z) \cup \{\infty\}$.

Definitions 6.1.

(1) Let f_0 be obtained by first reflecting in $i\mathcal{H}$ and then in $i\mathcal{H} + 1/2$.
(2) Let b_0 be obtained by first reflecting in $\mathcal{H} + i/2$ and then in $\partial\mathcal{B}_0$.
(3) For $i \ge 0$, let a_i be obtained by reflecting in $i\mathcal{H} + 1/2$ and then in $\partial\mathcal{B}_i$.

Since $i\mathcal{H}$ and $i\mathcal{H}+1/2$ are parallel and share the ideal point ∞, f_0 is a parabolic isometry fixing ∞. $\mathcal{H}+i/2$ meets $\partial\mathcal{B}_0$ at an angle of $\pi/3$, so b_0 is an elliptic isometry of order 3 rotating around the geodesic of intersection. For the same reason, a_i is an elliptic element of order 3 rotating around the geodesic $i\mathcal{H} + 1/2 \cap \partial\mathcal{B}_i$ for each $i \ge 0$.

Lemma 6.2. *Let G_n be the group generated by reflections in the face of P_n, where*

$$P_n = \left\{(z,t) \in \mathbb{H}^3 \mid 0 \le \Re(z) \le 1/2, -n\sqrt{2} \le \Im(z) \le 1/2\right\} - \left(\bigcup_{k=0}^{n} \mathcal{B}_k\right).$$

Then G_n contains a_i for $0 \le i \le 2n$, as well as f_0 and b_0.

Proof. By its definition, P_n is cut out by $\mathcal{H}+i/2$, $i\mathcal{H}$, $i\mathcal{H}+1/2$, $\mathcal{H}-n\cdot i\sqrt{2}$, and the $\partial\mathcal{B}_k$, $0 \le k \le n$. It therefore follows from the definitions above that G_n contains f_0, b_0, and the a_i for $0 \le i \le n$. It remains to establish that G_n contains a_i for $n < i \le 2n$.

Note that the bounding hyperplane $\mathcal{H} - n \cdot i\sqrt{2}$ for P_n is the image of $\mathcal{H}$ under c^{-n}, so that reflection in $\mathcal{H} - n \cdot i\sqrt{2}$ is given by $\mathsf{c}^{-n}\mathsf{r}\mathsf{c}^{n}$, where r is the reflection through $\mathcal{H}$. Using the property of r observed above Lemma 3.3, the conjugate of any element $\mathsf{x} \in \mathrm{PSL}_2(\mathbb{C})$ by the reflection in $\mathcal{H} - n \cdot i\sqrt{2}$ is:

$$\mathsf{c}^{-n}\mathsf{r}\mathsf{c}^{n}\mathsf{x}\mathsf{c}^{-n}\mathsf{r}\mathsf{c}^{n} = \mathsf{c}^{-2n}\bar{\mathsf{x}}\mathsf{c}^{2n} \tag{6}$$

We further observe that c conjugates a_i to a_{i-1} for $i \geq 1$, since $\mathsf{c}(i\mathcal{H} + 1/2) = i\mathcal{H} + 1/2$ and $\mathsf{c}(\mathcal{B}_i) = \mathcal{B}_{i-1}$, and we note that $\bar{\mathsf{a}}_0 = \mathsf{a}_0$. Thus:

$$\overline{\mathsf{c}^i\mathsf{a}_i\mathsf{c}^{-i}} = \bar{\mathsf{a}}_0 = \mathsf{a}_0 = \mathsf{c}^i\mathsf{a}_i\mathsf{c}^{-i} \quad \Rightarrow \quad \mathsf{c}^{-2i}\bar{\mathsf{a}}_i\mathsf{c}^{2i} = \mathsf{a}_i \tag{7}$$

For $0 \leq i \leq n$, it follows that the conjugate of a_i by reflection in $\mathcal{H} - n \cdot i\sqrt{2}$ is:

$$\mathsf{c}^{-2n}\bar{\mathsf{a}}_i\mathsf{c}^{2n} = \mathsf{c}^{-2(n-i)}\mathsf{a}_i\mathsf{c}^{2(n-i)} = \mathsf{a}_{2n-i} \in G_n.$$

Therefore G_n contains a_i for $n \leq i \leq 2n$ as well, and the lemma is proved. □

Since $\mathcal{H}$ meets both $\partial\mathcal{B}_0$ and $i\mathcal{H} + 1/2$ at right angles, it does the same for the fixed geodesic of a_0 and is therefore preserved by a_0. In fact, the following description of $\mathsf{a}_0 \in \mathrm{PSL}_2(\mathbb{C})$ is easily obtained from its definition:

$$\mathsf{a}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

In particular, a_0 acts on the ideal points of $\mathcal{P}_1 \cap \mathcal{P}_2$ by $0 \mapsto 1 \mapsto \infty \mapsto 0$. Similarly, it is easy to see that $f_0(z,t) = (z+1,t)$

Then the face pairings f (defined in Lemma 2.3) and s (defined in Lemma 2.2), which are equal, are obtained from f_0 by conjugating by a_0:

$$\mathsf{s} = \mathsf{f} = \mathsf{a}_0\,\mathsf{f}_0\,\mathsf{a}_0^{-1}. \tag{8}$$

One may use similar analyses to establish the following.

$$\mathsf{t} = (\mathsf{b}_0\mathsf{a}_0)^{-1}\mathsf{f}_0(\mathsf{b}_0\mathsf{a}_0)\,\mathsf{a}_0 \qquad \mathsf{g} = (\mathsf{a}_0^{-1}\mathsf{a}_1)\mathsf{f}_0^{-1}(\mathsf{a}_0^{-1}\mathsf{a}_1)^{-1} \qquad \mathsf{h} = \mathsf{a}_1\mathsf{a}_0\mathsf{f}_0^{-1}\mathsf{a}_1 \tag{9}$$

The main group-theoretic fact of this section extends these observations.

Proposition 6.3. *For each $n \in \mathbb{N}$, G_n contains Γ_n and $\mathsf{m}_1^{(i)}$ for $0 \leq i \leq n$.*

Proof. We recall from Proposition 3.11 that $\Gamma_n = \langle \Gamma_S, \Gamma_T^{(1)}, \ldots, \Gamma_T^{(n)}, \overline{\Gamma}_S^{\mathsf{c}^{-2n}} \rangle$, where by Definition 3.8(2), $\Gamma_T^{(i)} \doteq \Gamma_T^{\mathsf{c}^{-2(i-1)}}$ for each i between 1 and n. Furthermore, by Lemma 3.3, $\Gamma_T = \langle \Gamma_{T_0}, \overline{\Gamma}_{T_0}^{\mathsf{c}^{-2}} \rangle$.

It is a direct consequence of the descriptions (8) and (9) above that $\Gamma_S < G_n$ and $\Gamma_{T_0} < G_n$. Furthermore, since f_0 commutes with c and $\bar{\mathsf{f}}_0 = \mathsf{f}_0$, (7) implies for instance that

$$\mathsf{c}^{-2}\bar{\mathsf{f}}\mathsf{c}^2 = \mathsf{c}^{-2}(\bar{\mathsf{a}}_0\bar{\mathsf{f}}_0\bar{\mathsf{a}}_0^{-1})c^2 = \mathsf{a}_2\mathsf{f}_0\mathsf{a}_2^{-1} \in \Gamma_n,$$

since $\bar{\mathsf{a}}_0 = \mathsf{a}_0$ and $\mathsf{c}^{-2}\mathsf{a}_0c^2 = \mathsf{a}_2$. Using the same strategy, we find:

$$\mathsf{c}^{-2}\bar{\mathsf{g}}c^2 = (\mathsf{a}_2^{-1}\mathsf{a}_1)\mathsf{f}_0^{-1}(\mathsf{a}_2^{-1}\mathsf{a}_1)^{-1} \in G_n \text{ and } \mathsf{c}^{-2}\bar{\mathsf{h}}\mathsf{c}^2 = \mathsf{a}_1\mathsf{a}_2\mathsf{f}_0^{-1}\mathsf{a}_1 \in G_n$$

Thus G_n contains $\Gamma_T = \Gamma_T^{(1)}$. Since conjugation by c^{-1} takes a_i to a_{i+1}, and $\mathsf{a}_i \in \Gamma_n$ for each i between 0 and $2n$, it follows from the descriptions above and in (8) and (9) that each $\Gamma_T^{(i)}$, $1 \leq i \leq n$. Finally the relation (6) immediately implies that $\overline{\Gamma}_S^{\mathsf{c}^{-2n}} < G_n$, and we have established that $\Gamma_n < G_n$.

To show that G_n contains the elements $\mathsf{m}_1^{(j)}$ for each j between 0 and n, we observe that the element obtained by reflecting first in $\partial\mathcal{B}_0$ and then in $i\mathcal{H}$ is the rotation of order 2 described by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This is well known to generate $\mathrm{PSL}_2(\mathbb{Z})$, along with a_0. Since $\mathsf{m}_1 \in \mathrm{PSL}_2(\mathbb{Z})$, it follows that $\mathsf{m}_1 \in G$.

We note that c^{-2j} preserves $i\mathcal{H}$ and takes $\mathcal{B}_0$ to $\mathcal{B}_{2j}$, and that $\mathcal{B}_{2j}$ intersects P_n, for $j \leq n/2$, and intersects its image under reflection in $\mathcal{H} - n \cdot i\sqrt{2}$ for $n/2 \leq j \leq n$. Thus for each j between 0 and n, the rotation obtained by reflecting first in $\partial\mathcal{B}_{2j}$ and then in $i\mathcal{H}$ is contained in G_n. If m_1 is expressed as a word in the two elements described in the paragraph above, then $\mathsf{c}^{-2j}\mathsf{m}_1\mathsf{c}^{2j}$ is expressed as the same word in a_{2j} and the rotation obtained from $\partial\mathcal{B}_{2j}$ as above. The lemma follows. $\square$

It is now easy to prove the first part of Theorem 2, that for each link L_I obtained from L_n using only the mutation (13)(24), its complement M_I is commensurable to M_n.

Proposition 6.4. *For any $I \in \{0,1\}^{n+1}$, M_I is commensurable to M_n.*

Proof. Since G_n is a discrete reflection group, it is enough to show that $\Gamma_I \subset G_n$. This is immediate from Propositions 5.8 and 6.3. $\square$

To finish the proof of Theorem 2 we need an isometry classification of link complements that fall under the purview of Proposition 6.4. Our first step is to show that G_n is the commensurator of Γ_n.

Given a Kleinian group Γ, the group

$$\mathrm{Comm}(\Gamma) \ = \ \left\{\mathsf{g} \in \mathrm{Isom}(\mathbb{H}^3) \,|\, [\Gamma : \mathsf{g}\Gamma\mathsf{g}^{-1}] < \infty\right\}$$

is the *commensurator* of Γ. It is an easy consequence of the definition that if Γ' is commensurable to Γ then $\Gamma' < \mathrm{Comm}(\Gamma)$. So, by Proposition 6.4, G_n and Γ_I are subgroups of $\mathrm{Comm}(\Gamma_n)$ when $I \in \{0,1\}^{n+1}$. Margulis proves in [**?**, (1) Theorem] that $\mathrm{Comm}(\Gamma)$ is discrete if and only if Γ is non-arithmetic. Recall that Γ_n is non-arithmetic, hence $\mathrm{Comm}(\Gamma_n)$ is discrete.

Fix $I \in \{0,1\}^{n+1}$ and let O_n be the hyperbolic orbifold $\mathbb{H}^3/\mathrm{Comm}(\Gamma_n)$. We claim that O_n has exactly one cusp. Since O_n has non-zero volume and M_I has finite volume, we have that $|\mathrm{Comm}(\Gamma_n) : \Gamma_I| < \infty$ for every I. M_n has two cusps so there are exactly two Γ_n-orbits of parabolic fixed points for Γ_n. Also, Γ_n has finite index in G_n so they have the same parabolic fixed point set, yet there are elements of G_n which mix the Γ_n-orbits. Therefore, O_n has a single cusp.

We use the strategy of [11] to compute $\mathrm{Comm}(\Gamma_n)$. Recall the *hyperboloid* model for $\mathbb{H}^3$. The *Lorentz inner product* on $\mathbb{R}^4$ is the degenerate bilinear pairing

$$\langle \mathbf{v}, \mathbf{w} \rangle \ = \ v_1w_1 + v_2w_2 + v_3w_3 - v_4w_4.$$

$\mathbb{H}^3$ is the set $\{\mathbf{v} \,|\, \langle \mathbf{v}, \mathbf{v} \rangle = -1,\, v_4 > 0\}$ equipped with the Riemannian metric on tangent spaces determined by the Lorentz inner product. The *positive light cone* is the set $\{\mathbf{v} \,|\, \langle \mathbf{v}, \mathbf{v} \rangle = 0,\, v_4 \geq 0\}$. The group $\mathrm{Isom}(\mathbb{H}^3)$ is the set of matrices in $\mathrm{GL}_4(\mathbb{R})$, acting by matrix multiplication, which preserve the Lorentz inner product and the sign of the last coordinate of vectors in $\mathbb{R}^4$.

Given a vector $\mathbf{v} \in L^+$ the set $H_{\mathbf{v}} = \{\mathbf{w} \in \mathbb{H}^3 \,|\, \langle \mathbf{v}, \mathbf{w} \rangle = -1\}$ is a horosphere. If $\alpha \in \mathbb{R}^+$ the horosphere $H_{\alpha\mathbf{v}}$ is a horosphere centered at the same ideal point as $H_{\mathbf{v}}$ and if $\alpha \leq 1$ then $H_{\mathbf{v}}$ is contained in the horoball determined by $\alpha\mathbf{v}$. This correspondence between vectors in L^+ and horospheres in $\mathbb{H}^3$ is a bijection. Hence, we call the vectors in L^+ *horospherical vectors*.

We use the hyperboloid model to construct certain canonical tilings of $\mathbb{H}^3$ associated to M_n as in [9]. First, choose a horospherical vector $\mathbf{v} \in L^+$ so that under the covering map $\mathbb{H}^3 \to O_n$ the horosphere $H_{\mathbf{v}}$ projects to a cross section of the cusp. (Take any $\mathbf{v}$ which is fixed by some peripheral element of Γ_n.) Then $V_n = \mathrm{Comm}(\Gamma_n) \cdot \mathbf{v}$ is $\mathrm{Comm}(\Gamma_n)$-invariant and determines a $\mathrm{Comm}(\Gamma_n)$-invariant set of horospheres. The convex hull of V_n in $\mathbb{R}^4$ is called the *Epstein–Penner convex hull*, we denote it as C_n. Epstein and Penner show that ∂C_n consists of a countable set of 3-dimensional faces F_i, where each F_i is a finite sided Euclidean polyhedron in $\mathbb{R}^4$. Furthermore, this decomposition of ∂C_n projects to a $\mathrm{Comm}(\Gamma_n)$-invariant tiling $\mathcal{T}_n$ of $\mathbb{H}^3$ [9, Prop. 3.5 and Thm. 3.6]. We refer to a tiling obtained in this manner as a *canonical tiling*. (It is easy to see that if we make a different choice for the vector $\mathbf{v}$ then we obtain a convex hull which differs from C_n by multiplication by a positive scalar. Therefore, the canonical tiling obtained will be identical to that obtained from $\mathbf{v}$.)

Consider the group of symmetries $\mathrm{Sym}(\mathcal{T}_n) < \mathrm{Isom}(\mathbb{H}^3)$. Since $\mathcal{T}_n$ is $\mathrm{Comm}(\Gamma_n)$-invariant we have that $\mathrm{Comm}(\mathcal{T}_n) < \mathrm{Sym}(\mathcal{T}_n)$. On the other hand, $\mathrm{Sym}(\mathcal{T}_n)$ is discrete [11, Lemma 3.1] and since Γ_n is non-arithmetic $\mathrm{Comm}(\Gamma_n)$ is the maximal discrete group containing Γ_n. Therefore $\mathrm{Sym}(\mathcal{T}_n) = \mathrm{Comm}(\Gamma_n)$. We will show that $\mathcal{T}_n$ is the tiling whose tiles are the Γ_n-orbits of the polyhedra in the set $\mathcal{S} = \{\mathcal{P}_1, \mathsf{c}^{-2n}\overline{\mathcal{P}}_1, \mathcal{P}_2, \mathsf{c}^{-2}\mathcal{P}_2, \dots, \mathsf{c}^{-2n}\mathcal{P}_2\}$ and that the symmetry group of this tiling is G_n.

Lemma 6.5. *The tiles of $\mathcal{T}_n$ are the Γ_n–orbits of the polyhedra in $\mathcal{S}$.*

Proof. If X is a $4 \times n$ matrix we denote the i^{th} column of X as x_i. When the columns of X lie in L^+ and the convex hull of the corresponding ideal points is an ideal polyhedron we call the polyhedron $\mathcal{P}_X$. Consider the matrices

$$M = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & -1 & -2 & -1 & 1 & -1 & -1 & 1 \\ 0 & 1 & 2 & 1 & -2 & -1 & 0 & -1 & -1 & 1 & 1 & -1 \\ 0 & \sqrt{2} & 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

$$N = \begin{pmatrix} \sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix}$$

The columns of M and N are horospherical vectors and represent horospheres centered about the ideal vertices of a regular ideal cuboctahedron and octahedron respectively. These matrices are chosen so that, for $X = M, N$, the isometries in $\mathrm{Isom}(\mathcal{P}_X)$ all fix $(0,0,0,1)^T \in \mathbb{H}^3$ and the columns of X are $\mathrm{Isom}(\mathcal{P}_X)$–invariant. Furthermore, if h is the orientation preserving hyperbolic isometry that takes the triangular face (n_1, n_2, n_3) of $\mathcal{P}_N$ to the triangular face (m_1, m_9, m_4) of $\mathcal{P}_M$ so that $\mathsf{h}(\mathcal{P}_N) \cap \mathcal{P}_M$ is exactly this face, then our choice of horospheres agree on this intersection. That is, $\mathsf{h}(n_1, n_2, n_3) = (m_1, m_9, m_4)$. This arrangement corresponds to our embeddings of $\mathcal{P}_1$ and $\mathcal{P}_2$ in the upper half space model. In particular, we let $\mathcal{P}_1 = \mathsf{h}(\mathcal{P}_N)$ and $\mathcal{P}_2 = \mathcal{P}_M$ and note that there is a isomorphism between the models which takes $\mathcal{P}_i$ to $\mathcal{P}_i$ and takes ∞ to the center of the horosphere H_{m_1}. This isomorphism identifies the polyhedra in $\mathcal{S}$ with a set $\mathcal{S}$ of ideal octahedra and cuboctahedra in the hyperboloid model. Each of these polyhedra has, as an ideal vertex, the center of the horosphere H_{m_1}. Furthermore, we have well-defined matrices in $\mathrm{GL}_4(\mathbb{R})$ which represent each of the isometries we've used thus far in our constructions.

We have already chosen horospherical vectors for $\mathcal{P}_i$, $i = 1, 2$. Our choices are G_n-invariant, since the stabilizer of $\mathcal{P}_i$ in G_n is a subgroup of $\mathrm{Isom}(\mathcal{P}_i)$ which acts transitively on the set of ideal vertices of $\mathcal{P}_i$ and because our choices agree on the ideal vertices of $\mathcal{P}_1 \cap \mathcal{P}_2$. Three of the horospherical vectors for $\mathcal{P}_2$ are shared by $\mathsf{c}^{-2}\mathcal{P}_2$ and so G_n-invariance determines our choice for the remainder of the vertices of $\mathsf{c}^{-2}\mathcal{P}_2$. The horospherical vectors for the remainder of the elements of $\mathcal{S}$ are determined similarly. Finally, the union of all elements in $\mathcal{S}$ is a fundamental domain for Γ_n, so we let Γ_n act on our set of horospherical vectors to get a full set V_n of G_n-invariant horospherical vectors. Since $\mathbb{H}^3/G_n$ and O_n both have one cusp, it follows that V_n is $\mathrm{Comm}(\Gamma_n)$-invariant.

Since Γ_n and $\mathcal{S}$ together give a tiling of $\mathbb{H}^3$ we need only show that $\mathcal{S}$ is a subset of the tiles in $\mathcal{T}_n$. According to [11, Lemma 3.1], this is true if, for each element of $\mathcal{S}$, the horospherical vectors determining its vertices are coplanar in $\mathbb{R}^4$ and the angle between this plane and the planes for each of the neighboring tiles is convex. In other words, if $\{v_1, \dots v_k\}$ are the horospherical vectors for an element of $\mathcal{S}$ and w is a horospherical vector for a neighboring tile which is not shared by the first, then there exists a vector $\mathbf{n} \in \mathbb{R}^4$ such that

(1) $\mathbf{n} \cdot v_i = 1$ for every $i = 1, \dots k$, and
(2) $\mathbf{n} \cdot w > 1$.

Observe that these conditions are invariant under $\mathrm{Isom}(\mathbb{H}^3)$, for if $\mathbf{n} \cdot v = \alpha$ and $A \in \mathrm{Isom}(\mathbb{H}^3)$ then $(\mathbf{n}A^{-1}) \cdot Av = \alpha$.

Take $\mathbf{n} = (0, 0, 0, 1/2)^T$. Then $\mathbf{n} \cdot m_i = 1$ for $i = 1, \dots 12$ and $\sqrt{2}\mathbf{n} \cdot n_i = 1$ for $i = 1, \dots, 6$. So, our horospherical vectors for $\mathcal{P}_N$ and $\mathcal{P}_M$ are coplanar. Using the observation, if $\mathcal{P} \in \mathcal{S}$ then the horospherical vertices of $\mathcal{P}$ are coplanar. Again, by the observation, it remains only to show condition (2) holds for adjacent pair of cuboctahedra that meet along a triangular face, an adjacent pair of cuboctahedra that meet along a square face, and an octahedron adjacent to a cuboctahedron.

If $\mathcal{Q}$ is a cuboctahedron adjacent to $\mathcal{P}_M$ sharing the triangular face (m_1, m_9, m_4) with $\mathrm{Isom}(\mathcal{Q})$–invariant horospherical vectors which agree with (m_1, m_9, m_4) then $w = (7, 1, -5\sqrt{2}, 10)^T$ is a horospherical vector for $\mathcal{Q}$ which is not shared by $\mathcal{P}_M$. We have $\mathbf{n} \cdot w = 5 > 1$. If $\mathcal{Q}$ is a cuboctahedron adjacent to $\mathcal{P}_M$ sharing the square face (m_1, m_2, m_3, m_4) with $\mathrm{Isom}(\mathcal{Q})$-invariant horospherical vectors which agree with (m_1, m_2, m_3, m_4) then $w = (3, 5, -\sqrt{2}, 6)^T$ is a horospherical vector for $\mathcal{Q}$ which is not shared by $\mathcal{P}_M$. We have $\mathbf{n} \cdot w = 3 > 1$. The octahedron $\mathsf{h}(\mathcal{P}_N)$ is adjacent to $\mathcal{P}_M$ sharing the face (m_1, m_9, m_4). Its vectors are invariant under the isometry group of $\mathsf{h}(\mathcal{P}_N)$ and they agree with those of $\mathcal{P}_M$ along the shared face. The vector $w = (2 + 2\sqrt{2}, 0, -2 - 2\sqrt{2}, 4 + 4\sqrt{2})^T$ is a horospherical vector for $\mathsf{h}(\mathcal{P}_N)$ which is not shared by $\mathcal{P}_M$. We have $\mathbf{n} \cdot w = 2 + \sqrt{2} > 1$. □

By construction, we have that G_n is a subgroup of the symmetry group for $\mathcal{T}_n$. It remains to show that G_n is the full symmetry group for this tiling.

Proposition 6.6. *For any $I \in \{0, 1\}^{n+1}$, G_n is the commensurator of Γ_I.*

Proof. From Propositition 6.4 we have $\mathrm{Comm}(\Gamma_I) = \mathrm{Comm}(\Gamma_n)$. Take $\mathsf{x} \in \mathrm{Comm}(\Gamma_n)$. We want to show that $\mathsf{x} \in G_n$. Recall that $\mathsf{c}^{-n}\mathsf{r}\mathsf{c}^n \in G_n$. Since this element exchanges $\mathcal{P}_1$ and $\mathsf{c}^{-2n}\mathcal{P}_1$ we see that the octahedral tiles of $\mathcal{T}_n$ lie in a single G_n-orbit. Hence, we may assume that x fixes $\mathcal{P}_1$.

Recall, for instance from Corollary 2.2, that $\mathcal{P}_1$ is checkered and its face A spanned by the vertices 0, 1, and ∞ is external. We have that $\mathsf{a}_0, \mathsf{b}_0 \in \mathrm{Isom}(\mathcal{P}_1) \cap$

G_n. It follows immediately from the definitions of a_0 and b_0 that $\langle \mathsf{a}_0, \mathsf{b}_0\rangle$ acts transitively on the external faces of $\mathcal{P}_1$. The internal faces of $\mathcal{P}_1$ are paired by elements of Γ_S, so every internal face of $\mathcal{P}_1$ meets an octahedron in $\mathcal{T}_n$. In particular, $\mathsf{x}(A)$ must be an external face of $\mathcal{P}_1$. Hence, we may multiply by an element of $\langle \mathsf{a}_0, \mathsf{b}_0\rangle < G_n$ to assume that $\mathsf{x}(A) = A$. Now, from the definition of G_n it is clear that $\mathrm{Isom}(A) \cap \mathrm{Isom}(\mathcal{P}_1) < G_n$ and so we have $\mathsf{x} \in G_n$ as desired. □

The second half of Theorem 2 follows from the classification below.

Proposition 6.7. *Suppose* $I = (0, a_1 \dots, a_{n-1}, 0)$ *and* $J = (0, b_1, \dots, b_{n-1}, 0)$ *are elements of* $\{0,1\}^{n+1}$. M_I *is isometric to* M_J *if and only if* $J = I$ *or* $J = \bar{I}$.

Remark. We have assumed that the first and last entries of I and J are all zero to make the proposition easier to state. By Corollary 5.2, changing the first or last entry of either I or J to a one yields another isometric manifold.

Proof. Suppose $I = (0, a_1 \dots, a_{n-1}, 0)$ and $J = (0, b_1, \dots, b_{n-1}, 0)$ are elements of $\{0,1\}^{n+1}$ and that $\mathbb{H}^3/\Gamma_I$ and $\mathbb{H}^3/\Gamma_J$ are homeomorphic. By Mostow rigidity, $\mathbb{H}^3/\Gamma_I$ and $\mathbb{H}^3/\Gamma_J$ are isometric and there is some $\mathsf{x} \in \mathrm{Isom}(\mathbb{H}^3)$ such that $\Gamma_I^{\mathsf{x}} = \Gamma_J$. We may assume that this isometry is orientation preserving by, if necessary, replacing J with $\overline{J}$ and composing with the isometry from Corollary 5.6. Hence, the isometry x is also orientation preserving. We have

$$\begin{aligned} G_n^{\mathsf{x}} &= \mathrm{Comm}(\Gamma_I)^{\mathsf{x}} \\ &= \mathrm{Comm}(\Gamma_I^{\mathsf{x}}) \\ &= \mathrm{Comm}(\Gamma_J) \\ &= G_n. \end{aligned}$$

Therefore, $\mathsf{x} \in N(G_n) = G_n = \mathrm{Sym}(\mathcal{T}_n)$.

Consider the homomorphism $\phi\colon G_n \to \mathbb{Z}/2\mathbb{Z}$ whose kernel is the set of orientation preserving elements in G_n. Let $\widehat{G}_n = \ker\phi$ and let $\mathsf{u} \in G_n$ be the reflection through the plane $\mathcal{H} - n \cdot i\sqrt{2}$. So $\mathsf{u} \notin \widehat{G}_n$ and $[G_n : \widehat{G}_n] = 2$ which implies that $P_n \cup \mathsf{u}P_n$ is a fundamental domain for $\widehat{G}_n$. It follows that there are $2n$ distinct $\widehat{G}_n$-orbits of cuboctahedral tiles in $\mathcal{T}_n$. We also know that there are $2n$ distinct Γ_J-orbits of cuboctahedral tiles in $\mathcal{T}_n$. Since $\Gamma_J < \widehat{G}_n$, the orbits are the same. By similar reasoning, if $\mathcal{K} \subset \mathbb{H}^3$ is a hyperplane that separates two cuboctahedral tiles of $\mathcal{T}_n$ then $\widehat{G}_n \cdot \mathcal{K} = \Gamma_J \cdot \mathcal{K} = \Gamma_I \cdot \mathcal{K}$.

Now suppose that there is some $i \in \{1, \dots, n-1\}$ with $a_i \neq b_i$. Let i_0 be the minimal such i. It follows from the proof of Proposition 6.2 and the definition of Γ_T that there is a word w in c and m_1 so that if we replace $\mathsf{x}, \Gamma_I, \Gamma_J, \mathcal{T}_n$, and G_n by their w conjugates we have $\langle \overline{\Gamma}_{T_0}, \Gamma_{T_0}\rangle < \Gamma_I$ and the image of $\mathcal{H}/\Lambda$ is the i^{th} mutation sphere in $\mathbb{H}^3/\Gamma_I$. Since i_0 is the minimal index with $a_{i_0} = b_{i_0}$ we also know that $\langle \overline{\Gamma}_{T_0}, \Gamma_{T_0}^{\mathsf{m}_1}\rangle < \Gamma_J$ and $\mathcal{H}/\Lambda$ is the i^{th} mutation sphere in $\mathbb{H}^3/\Gamma_J$.

The cuboctahedron $\mathsf{xc}(\mathcal{P}_2)$ is a tile of $\mathcal{T}_n$ and $\mathsf{x} \in \widehat{G}_n$, so $\mathsf{xc}(\mathcal{P}_2)$ is in the Γ_J-orbit of $\mathsf{c}(\mathcal{P}_2)$. Hence, we may multiply x by some element of Γ_J to assume that $\mathsf{xc}(\mathcal{P}_2) = \mathsf{c}(\mathcal{P}_2)$. We also have $\mathsf{x}(\mathcal{H}) \in \Gamma_J \cdot \mathcal{H}$. The covering maps $\rho_I\colon \mathbb{H}^3 \to \mathbb{H}^3/\Gamma_I$

and $\rho_J\colon\, \mathbb{H}^3 \to \mathbb{H}^3/\Gamma_J$ restrict to give the commutative diagram,

$$\begin{array}{ccc}
\mathsf{c}(\mathcal{P}_2)\cup\mathcal{P}_2 & \xrightarrow{\quad\mathsf{x}\quad} & \mathsf{c}(\mathcal{P}_2)\cup\mathsf{x}(\mathcal{P}_2) \\
\Big\downarrow{\scriptstyle \rho_I} & & \Big\downarrow{\scriptstyle \rho_J} \\
C\big(\langle\overline{\Gamma}_{T_0},\Gamma_{T_0}\rangle\big) & \xrightarrow{\quad\phi_{\mathsf{x}}\quad} & \rho_J\big(\mathsf{c}(\mathcal{P}_2)\cup\mathsf{x}(\mathcal{P}_2)\big) \\
& \searrow \qquad \swarrow & \\
& \mathbb{H}^3/\widehat{G}_n. &
\end{array}$$

Observe that $\rho_I\big(\mathsf{c}(\mathcal{P}_2)\big)$ and $\rho_J\big(\mathsf{c}(\mathcal{P}_2)\big)$ are the images of $C(\overline{\Gamma}_{T_0})$ under the natural embeddings. Also, $\rho_I(\mathcal{P}_2)$ is the image of $C(\Gamma_{T_0})$ under the natural embedding. Using that $\rho_J\mathsf{x} = \phi_{\mathsf{x}}\rho_I$, that x preserves the set of planes $\Gamma_I\cdot\mathcal{H}$, and the fact that $a_{i_0}\neq b_{i_0}$ we may conclude that $\rho_J\big(\mathsf{x}(\mathcal{P})\big)$ is the image of $C\big(\Gamma_{T_0}^{\mathsf{m}_1}\big)$ under the natural embedding and that $\rho_J\big(\mathsf{c}(\mathcal{P}_2)\cup\mathsf{x}(\mathcal{P}_2)\big) = C\big(\langle\overline{\Gamma}_{T_0},\Gamma_{T_0}^{\mathsf{y}\mathsf{m}_1}\rangle\big)$ for some $\mathsf{y}\in\mathrm{Stab}_{\widehat{G}_n}(\mathsf{c}\mathcal{P}_2)$. In fact, since $\overline{\Gamma}_{T_0}$ and $\Gamma_{T_0}^{\mathsf{m}_1}$ meet cute along a hyperplane $\mathcal{K}$ if and only if $\mathcal{K}=\mathcal{H}$, we may take $\mathsf{y}=\mathsf{x}$. We now have

$$\begin{array}{ccccc}
\mathsf{c}(\mathcal{P}_2)\cup\mathcal{P}_2 & \xrightarrow{\ \mathsf{x}\ } & \mathsf{c}(\mathcal{P}_2)\cup\mathsf{x}(\mathcal{P}_2) & \xrightarrow{\ \mathsf{x}^{-1}\ } & \mathsf{c}(\mathcal{P}_2)\cup\mathcal{P}_2 \\
\Big\downarrow{\scriptstyle \rho_I} & & \Big\downarrow{\scriptstyle \rho_J} & & \Big\downarrow{\scriptstyle \rho_I} \\
C\big(\langle\overline{\Gamma}_{T_0},\Gamma_{T_0}\rangle\big) & \xrightarrow{\ \phi_{\mathsf{x}}\ } & C\big(\langle\overline{\Gamma}_{T_0},\Gamma_{T_0}^{\mathsf{x}\mathsf{m}_1}\rangle\big) & \xrightarrow{\ \phi_{\mathsf{x}}^{-1}\ } & C\big(\langle\overline{\Gamma}_{T_0},\Gamma_{T_0}^{\mathsf{m}_1}\rangle\big)
\end{array}$$

But this is impossible since, as we have observed (see, for example, the proof of Lemma 5.3), the equivalence classes of the ideal vertices of $\mathsf{c}(\mathcal{P}_2)\cup\mathcal{P}_2$ under the quotients by the groups $\langle\overline{\Gamma}_{T_0},\Gamma_{T_0}\rangle$ and $\langle\overline{\Gamma}_{T_0},\Gamma_{T_0}^{\mathsf{m}_1}\rangle$ are distinct. □

7. Incommensurable mutants

The description of Lemma 5.3 might lead one to suspect that the mutations (13)(24) and (12)(34) of $F^{(0)}$ act very differently at the level of Kleinian groups. Indeed, it follows from Proposition 7.1 below, together with Proposition 4.2, that $S^3 - L_n$ is incommensurable with the complement of any link obtained from it by the mutation (12)(34) along a subcollection of the $S^{(i)}$. In fact, we consider it likely that no two such mutants are commensurable unless they are isometric.

We lack the tools to prove the assertion above — mutants are notoriously difficult to distinguish because they share so many invariants. Examples are the invariant trace field (which in this case is the field $\mathbb{Q}(i,\sqrt{2})$) and (probably) the Bloch invariant (we will prove this particular case in Proposition 7.2 below), as remarked in 1 below the statement of Theorem 3. However, in this section we will describe families of mutants whose members have different cusp parameters and are therefore mutually incommensurable.

Proposition 7.1. *For fixed n and any $I=(a_0,\dots,a_n)\in\{0,1,2\}^{n+1}$ such that $a_i=2$ for some i, Γ_I has a nonintegral trace.*

Proof. Let $I=(a_0,\dots,a_n)\in\{0,1,2\}^{n+1}$ satisfy the hypothesis, and let i_0 be such that $a_{i_0}=2$. If $i_0=1$ or n, Proposition 5.8 implies that Γ_I contains an element with the same trace as the matrix below, or its complex conjugate.

$$\mathsf{m}_2\mathsf{t}\mathsf{m}_2^{-1}\mathsf{g} \;=\; \begin{pmatrix} -1+\sqrt{2}+11i+i\sqrt{2} & 1-7\sqrt{2}-16i-2i\sqrt{2} \\ \frac{1}{5}(2-\sqrt{2}-21i-2i\sqrt{2}) & \frac{1}{5}(-2+12\sqrt{2}+31i+4i\sqrt{2}) \end{pmatrix}$$

The trace of $\mathsf{m}_2\mathsf{t}\mathsf{m}_2^{-1}\mathsf{g}$ is not an algebraic integer, since the ring of integers of $\mathbb{Q}(i,\sqrt{2})$ is $\mathbb{Z}[i,\sqrt{2}]$. In the case $i_0 = 1$, the conjugate of this element by m_2, $\mathsf{t}\mathsf{m}_2^{(0)}\mathsf{g}(\mathsf{m}_2^{(0)})^{-1}$, is contained in Γ_I, since $\mathsf{m}_2^{(0)} = \mathsf{m}_2$ has order 2. In the case $i_0 = n$, the conjugate of $\overline{\mathsf{m}_2\mathsf{t}\mathsf{m}_2^{-1}\mathsf{g}} = \mathsf{m}_2\bar{\mathsf{t}}\mathsf{m}_2^{-1}\bar{\mathsf{g}}$ by c^{-2n} is contained in Γ_I.

In all other cases, Proposition 5.8 implies that Γ_I contains an element with the same trace as the matrix below.

$$\begin{aligned} \bar{\mathsf{h}}(\mathsf{m}_2\mathsf{h}\mathsf{m}_2^{-1}) &= \begin{pmatrix} -2i\sqrt{2} & -3+i\sqrt{2} \\ -3-i\sqrt{2} & 3i\sqrt{2} \end{pmatrix} \begin{pmatrix} -3i\sqrt{2} & 15-5i\sqrt{2} \\ \frac{3+i\sqrt{2}}{5} & 2i\sqrt{2} \end{pmatrix}, \\ &= \begin{pmatrix} -71/5 & -20-30i\sqrt{2} \\ \frac{18}{5}(-2+3i\sqrt{2}) & 55 \end{pmatrix} \end{aligned}$$

The trace of this matrix is evidently not an algebraic integer. □

For fixed n and any $I \in \{0,1,2\}^{n+1}$, since M_n and M_I decompose along totally geodesic surfaces into isometric pieces, they have the same volume. (In fact, [30, Theorem 1.3] asserts that hyperbolic volume is always invariant under mutation.) It would follow from the classical "Dehn invariant sufficiency" conjecture that any two hyperbolic manifolds with the same volume are scissors congruent (again see [25], for instance). In our situation we will verify this explicitly.

Proposition 7.2. *For fixed n and any $I \in \{0,1,2\}^{n+1}$, M_n and M_I have the same Bloch invariant.*

Proof. Recall from Lemma 2.4 that $F^{(0)}$ inherits a triangulation Δ_F from the fundamental domain $\mathcal{F}$ for the action of Λ on $\mathbf{H}$ pictured in Figure 5. From the figure, one finds that Δ_F has six edges, each a geodesic arc joining cusps of F. For example, the geodesic joining 0 and ∞ projects to an edge which joins cusp 1 to cusp 2. Of the other five edges, one joints 3 to 4, two join 2 to 4, and for each of 2 and 4 there is an edge joining it to itself.

Since $\mathsf{m}_1 \in \mathrm{PSL}_2(\mathbb{Z})$ it preserves the Farey tesselation of $\mathcal{H}$, which restricts on $\mathcal{F}$ to the triangulation pictured in Figure 5. Therefore ϕ_{m_1} preserves Δ_F. On the other hand, since ϕ_{m_2} exchanges 1 with 2 and 3 with 4 it does not preserve Δ_F. For instance, if e is the edge joining 2 to itself then $\phi_{\mathsf{m}_2}(e)$ joins 1 to itself.

It follows from the paragraph above that for $I = (a_0,\ldots,a_n) \in \{0,1,2\}^{n+1}$ such that some $a_i = 2$, the decomposition of M_I into octahedra and cuboctahedra is not an "ideal polyhedral decomposition". This is because for such i, the copies of $C(\Gamma_S)$ or $C(\Gamma_T)$ on either side of $F^{(i)}$ are glued by $\phi_{\mathsf{m}_2^{(i)}}$, which is conjugate to ϕ_{m_2} by ϕ_{i+1} and hence does not preserve Δ_F. Since Δ_F is the triangulation inherited from the external faces of the octahedron and cuboctahedron (by Lemma 2.4 again), it follows that external faces of the octahedra or cuboctahedra on either side of $F^{(i)}$ do not match up with each other.

It is possible to rectify this by gluing "flat" tetrahedra between copies of $C(\Gamma_S)$ and/or $C(\Gamma_T)$ joined by the mutation ϕ_{m_2}. If $\mathcal{T}$ is a flat tetrahedron glued to, say, $C(\Gamma_S)$ along two adjacent triangles in $\partial C(\Gamma_S)$, then $C(\Gamma_S) \cup \mathcal{T}$ is homeomorphic to $C(\Gamma_S)$, but in the induced triangulation of the boundary, the edge separating the triangles along which $\mathcal{T}$ is glued has been replaced by an edge joining their two opposite vertices. For a more thorough exposition, see [26, §4].

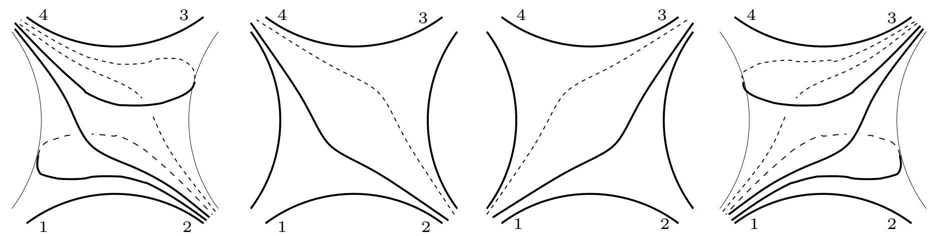

FIGURE 11. Interpolating between Δ_T, on the left, and $\phi_{\mathsf{m}_2}(\Delta_T)$.

Figure 11 illustrates a process by which Δ_F may be changed to its image under ϕ_{m_2} by a sequence of moves on edges. The edges of Δ_F are pictured on the left in bold. Moving left-to-right, at each stage two edges are replaced by edges transverse to them and disjoint from the remaining edges. After three such moves, the original triangulation has been changed to its image under ϕ_{m_2}.

Now suppose $I = (a_1, \dots, a_n) \in \{0,1,2\}^{n+1}$. For each $i < n$ such that $a_i = 2$, replace $C(\Gamma_T^{(i+1)})$ by its union with 6 flat tetrahedra, glued successively along $\partial_- C(\Gamma_T^{(i+1)})$ to realize the change of triangulations illustrated in Figure 11. The result is homeomorphic to $C(\Gamma_T^{(i+1)})$, since adding a flat tetrahedron does not change the homeomorphism type, but the gluing induced by $\phi_{\mathsf{m}_2^{i+1}}$ now preserves the triangulation.

It follows from the above that the Bloch invariant $\beta(M_I)$ may be calculated using the resulting polyhedral decomposition. This differs from the original by the addition of the cross ratio parameters of the flat tetrahedra. Each of these is equal to 2, since the triangulation of F is a projection of the Farey tessellation of $\mathbf{H}$. But in the Bloch group, $2\cdot[2] = 0$ is a consequence of the relation $[z] = \left[\frac{z}{z-1}\right]$. Since the number of flat tetrahedra is a multiple of 6, the sum of their cross ratio parameters contributes nothing to the Bloch invariant. □

The proposition below tracks the change of cusp parameters under mutation. To simplify our task, we restrict our attention to complements of links obtained by mutating only with (12)(34) along a subcollection of the $S^{(i)}$ and note in passing that since those obtained by mutating only with (13)(24) are commensurable with M_n, their cusp parameters are $\mathrm{PGL}_2(\mathbb{Q})$-equivalent to those of M_n.

Proposition 7.3. *For $I = (t_0, t_1, \dots, t_n) \in \{0,2\}^{n+1}$ and $j \in \{0,1,\dots,n\}$, define*

$$c_j = \sum_{k=0}^{j} \frac{t_k}{2} \pmod 2.$$

Let T_1 be a cross section of the cusp of M_I such that $T_1 \cap C(\Gamma_S) = p_S(A_1)$ (as defined in Lemma 4.10), and let T_2 be a cross section of the cusp of M_I with $T_2 \cap C(\Gamma_S) = p_S(A_2)$. Up to the action of $\mathrm{PGL}_2(\mathbb{Q})$, their complex moduli are as *follows.*

$$m(T_1) = i\left[1 + 2\sum_{j=1}^{n} \frac{2\sqrt{2}}{5^{c_{j-1}}} + \frac{1}{5^{c_n}}\right]$$
$$m(T_2) = i\left[\frac{1}{5} + 2\sum_{j=1}^{n} \frac{2\sqrt{2}}{5^{(1-c_{j-1})}} + \frac{1}{5^{(1-c_n)}}\right]$$

Proof. To simplify notation, we will identify A_k and $p_S(A_k)$ and view $A_k \subset C(\Gamma_S)$ for $k = 1, 2$. Recall the decomposition of M_I, along the surfaces $F^{(j)}$, into a union of isometric copies of $C(\Gamma_S)$ and $C(\Gamma_T)$ as described in Proposition 5.5:

$$C(\Gamma_S) \cup C(\Gamma_T^{(1)}) \cup \ldots \cup C(\Gamma_T^{(n)}) \cup C(\overline{\Gamma}_S) \to M_I$$

We will denote by l_j the gluing map supplied by Proposition 5.5, taking $\partial_+ C(\Gamma_T^{(j)})$ to $\partial_- C(\Gamma_T^{(j+1)})$ when $1 \leq j < n$. The map l_0 takes $\partial C(\Gamma_S)$ to $\partial_- C(\Gamma_T^{(1)})$, and $l_n \colon \partial_+ C(\Gamma_T^{(n)}) \to \partial C(\overline{\Gamma}_S)$.

Recall that for each j between 1 and n and k between 1 and 4, the annular cusp cross section $DB_k^{(j)}$ of $C(\Gamma_T^{(j)})$ is defined by $\phi_j \circ p_T(DB_k)$. Thus each of T_1 and T_2 meets each of the $C(\Gamma_T^{(j)})$ in a collection of annular cusp cross sections parallel to a subcollection of the $DB_k^{(j)}$, $k \in \{1, 2, 3, 4\}$. Similarly, each of $T_1 \cap C(\overline{\Gamma}_S)$ and $T_2 \cap C(\overline{\Gamma}_S)$ is parallel to one of the annular cross sections $\overline{A}_1$ or $\overline{A}_2$.

In the proof of Proposition 4.15 we see that if $1 \leq j < n$, $1 \leq k \leq 4$, and $t_j = 0$, then $l_j = \iota_+^{(j)} (\iota_-^{(j)})^{-1}$ takes $\partial_+ DB_k^{(j)}$ to $\partial_- DB_k^{(j+1)}$. Thus by the description above, if $t_j = 2$ then l_j acts on the indices k by the permutation $(12)(34)$. The proof of Proposition 4.15 likewise implies that if $t_0 = 0$, then $l_0(\partial A_k) = \partial_- DB_k^{(1)} \sqcup \partial_- DB_{k+2}^{(1)}$ for $k = 1, 2$; hence if $t_0 = 2$, then $l_0(\partial A_k) = \partial_- DB_{3-k}^{(1)} \sqcup \partial_- DB_{5-k}^{(1)}$. A similar dichotomy holds for l_n.

Remark. The definitions of the annular cusp cross-sections in Lemmas 4.10 and 4.11 depended on a particular collection of horospheres centered at the ideal vertices of $\mathcal{P}_1$ and $\mathcal{P}_2$. These give rise to a particular collection of horospherical cross-sections of the cusps of $F^{(0)}$, which is not preserved by ϕ_{m_2}.

It would thus be more accurate to say, for example, that when $t_j = 2$ and $1 \leq j < n$, $l_j(\partial_+ DB_1^{(j)})$ is a cusp cross-section of $\partial_- C(\Gamma_T^{(j+1)})$ *parallel* to $\partial_- DB_2^{(j+1)}$. Since we are interested in computing moduli, and these are unaffected by similarities, we have ignored this distinction above and will continue to do so below.

Claim. For each $j \in \{1, \ldots, n\}$,

$$T_1 \cap C(\Gamma_T^{(j)}) = \begin{cases} DB_1^{(j)} \cup DB_3^{(j)} & \text{if } c_{j-1} = 0 \\ DB_2^{(j)} \cup DB_4^{(j)} & \text{if } c_{j-1} = 1. \end{cases}$$

Furthermore, $T_1 \cap C(\overline{\Gamma}_S) = \overline{A}_1$ if $c_n = 0$ and $\overline{A}_2$ if $c_n = 1$.

Proof of claim. This is proved by induction on j. In the base case $j = 1$, since $c_0 = t_0/2$ and $T_1 \cap C(\Gamma_S) = A_1$, the conclusion in this case follows directly from the dichotomy in the behavior of l_0 recorded above the claim.

Suppose now that the claim holds for some $j < n$, and note that therefore $T_1 \cap M_T^{(j)}$ has components $DB_k^{(j)}$ and $DB_{k'}^{(j)}$, where $k, k' \in \{1, 2, 3, 4\}$ have the same parity, which is opposite that of c_{j-1}. By definition, c_j has the opposite parity from c_{j-1} if and only if $t_j = 2$. By the above, l_j acts on the indices of the $DB_k^{(j-1)}$ changing parity if and only if $t_j = 2$ as well. Hence the claim holds for $j+1$.

So, by induction, the claim holds for each $j \leq n$. The final statement in the claim follows by an argument that mimics the argument used in the inductive step. $\square$

The moduli of A_1, A_2, $\overline{A}_1$, and $\overline{A}_2$ are described in Lemma 4.10, and those of the $DB_j^{(i)}$ are described in Lemma 4.12. Using these descriptions and Lemma 4.9, the claim above shows that the imaginary part of $m(T_1)$ is as described in the statement of the proposition. The description of the imaginary part of $m(T_2)$ follows similarly.

Now recall the definitions of the arcs a_1 and $db_k^{(j)}$ for $1 \leq j \leq n$ and $k = 1, 3$, and the collections of arcs $\mathcal{A}_2$ and $D\mathcal{B}_k^{(j)}$ for $1 \leq j \leq n$ and $k = 2, 4$, from the proof of Proposition 4.15. For our purposes here, we additionally define $\mathcal{A}_1$ to be a collection of five arcs evenly spaced around A_1, each perpendicular to ∂A_1 at each of its endpoints, such that $a_1 \in \mathcal{A}_1$. We analogously define $D\mathcal{B}_k^{(j)}$ as a collection of evenly spaced arcs in $DB_k^{(j)}$ containing $db_k^{(j)}$ for $1 \leq j \leq n$ and $k = 1, 3$.

Claim. If $t_0 = 0$ then $l_0(\partial \mathcal{A}_k) = \partial_- D\mathcal{B}_k^{(1)} \cup \partial_- D\mathcal{B}_{k+2}^{(1)}$ for $k = 1, 2$, and if $t_0 = 2$ then $l_0(\partial \mathcal{A}_k) = \partial_- D\mathcal{B}_{3-k}^{(1)} \cup \partial_- D\mathcal{B}_{5-k}^{(1)}$. Similarly, for $1 \leq j \leq n-1$,

$$l_j(\partial_+ D\mathcal{B}_k^{(j)}) = \partial_- D\mathcal{B}_k^{(j+1)} \text{ for } k = 1, 2, 3, 4, \text{ if } t_j = 0,$$

$$l_j(\partial_+ D\mathcal{B}_k^{(j)}) = \begin{cases} \partial_- D\mathcal{B}_{3-k}^{(j+1)} & \text{for } k = 1, 2 \\ \partial_- D\mathcal{B}_{7-k}^{(j+1)} & \text{for } k = 3, 4 \end{cases}, \text{if } t_j = 2.$$

Also, if $t_n = 0$ then $l_n^{-1}(\partial \overline{\mathcal{A}}_k) = \partial_+ D\mathcal{B}_k^{(n)} \cup \partial_+ D\mathcal{B}_{k+2}^{(n)}$ for $k = 1, 2$, and if $t_n = 2$ then $l_n^{-1}(\partial \overline{\mathcal{A}}_k) = \partial_+ D\mathcal{B}_{3-k}^{(n)} \cup \partial_- D\mathcal{B}_{5-k}^{(n)}$.

In the discussion above the first claim, we recorded the analogous dichotomy to that of the claim above for the action of the gluing maps l_j on boundaries of annular cusp cross-sections. The substance of this claim is thus that the gluing maps preserve arc endpoints.

Proof of claim. Suppose first that $t_j = 0$, so by its definition $l_j = \iota_+^{(j)}(\iota_-^{(j)})^{-1}$. The proof of Proposition 4.15 directly addresses the cases of $\mathcal{A}_2$, $\overline{\mathcal{A}}_2$, and $D\mathcal{B}_k^{(j)}$, where $k = 2$ or 4. In the remaining case of $\mathcal{A}_1$, the definition implies that $\partial \mathcal{A}_1$ consists of ten points, five evenly spaced around each component of ∂A_1, with each such collection containing a point of ∂a_1. Also by definition, $\partial_- D\mathcal{B}_k^{(1)}$ is a collection of five points spaced evenly around $\partial_- DB_k^{(1)}$, one of which is $\partial_- db_k^{(1)}$ for $k = 1, 3$. By the proof of Proposition 4.15, $\iota_+^{(0)}(\iota_-^{(0)})^{-1}$ takes ∂a_1 to $\partial_- db_1^{(1)} \cup \partial_- db_3^{(1)}$; hence the entire collection $\partial \mathcal{A}_1$ is taken to $\partial_- D\mathcal{B}_1^{(1)} \cup \partial_- D\mathcal{B}_3^{(1)}$ since $\iota_+^{(0)}(\iota_-^{(0)})^{-1}$ is an isometry. The remaining cases when $t_j = 0$, $j \geq 1$, follow similarly.

To illustrate the case $t_j = 2$ we focus on the subcase $1 \leq j < n$. When $t_0 = 2$, l_j takes $\partial_+ DB_1^{(j)}$ to $\partial_- DB_2^{(j+1)}$, for example. The crucial observation here is that $l_0(\partial_+ db_1^{(j)})$ is in $\partial_- D\mathcal{B}_2^{(j+1)}$. This holds because by definition, $\partial_+ db_1^{(j)}$ is a point in the edge of the triangulation Δ_T which exits the ideal vertex 1. (This is the top edge in Figure 11.) Although ϕ_{m_2} does not preserve Δ_T, it preserves this edge, exchanging its endpoints at 1 and 2. Since $\partial_- D\mathcal{B}_2^{(j+1)}$ has a point in each edge which exits 2, it contains $\phi_{\mathsf{m}_2}(\partial_+ db_1^{(j)})$. Since the points of $\partial_+ D\mathcal{B}_1^{(j)}$ are evenly spaced around $\partial_+ DB_1^{(j)}$ and the same is true for $\partial_- D\mathcal{B}_2^{(j+1)}$, it follows that $l_0(\partial_+ D\mathcal{B}_1^{(j)}) = \partial_- D\mathcal{B}_2^{(j+1)}$.

Since ϕ_{m_2} takes the edge of Δ_T to itself and exchanges its endpoints, $l_0(\partial_+ db_3^{(j)}) \in \partial_- D\mathcal{B}_4^{(j+1)}$ in this case. Then it follows from "even-spacedness" that $l_0(\partial_+ D\mathcal{B}_3^{(j)}) = \partial_- D\mathcal{B}_4^{(j+1)}$. The same argument implies that $\partial_- db_1^{(j+1)} \in l_0(\partial_+ D\mathcal{B}_2^{(j+1)})$ and therefore that $l_0(\partial_+ D\mathcal{B}_2^{(j)}) = \partial_- D\mathcal{B}_1^{(j+1)}$, and similarly that $l_0(\partial_+ D\mathcal{B}_4^{(j)}) = \partial_- D\mathcal{B}_3^{(j+1)}$. The same sequence of observations, applied to $\partial\mathcal{A}_k$ and $\partial\overline{\mathcal{A}}_k$, $k = 1, 2$, completes the claim. □

The second claim implies that the set

$$\mathcal{A}_1 \ \cup \ \mathcal{A}_2 \ \cup \bigcup_{j,k} D\mathcal{B}_k^{(j)} \ \cup \ \overline{\mathcal{A}}_1 \ \cup \ \overline{\mathcal{A}}_2$$

consists of a disjoint union of closed geodesics, some in T_1 and some in T_2, each meeting any of the geodesics $F^{(j)} \cap T_1$ or $F^{(j)} \cap T_2$ perpendicularly in up to five points. That $m(T_1)$ and $m(T_2)$ have imaginary part equal to 0 now follows as in the proof of Proposition 4.15. □

Proposition 7.3 allows us to describe arbitrarily large subfamilies of the M_I which have different cusp parameters.

Corollary 7.4. *For $0 \le k \le n$, let $I_k = (t_0, t_1, \ldots, t_n)$ be defined by $t_i = 0$ for $i \ne k$, and $t_k = 2$. The cusp parameters of M_{I_k} are not $\mathrm{PGL}_2(\mathbb{Q})$–equivalent to those of $M_{I_{k'}}$ for $k \ne k'$, when both are less than $(n+1)/2$.*

Proof. By Proposition 7.3, the cusps of M_{I_k} have moduli described below.

$$m(T_1) = i\left[\frac{6}{5} + \frac{4}{5}(n+4k)\sqrt{2}\right] \qquad m(T_2) = i\left[\frac{6}{5} + \frac{4}{5}(5n-4k)\sqrt{2}\right]$$

Since $m(T_1)$ and $m(T_2)$ are both of the form described in Lemma 4.16 for any k, if the cusp parameters of M_{I_k} are equivalent to those of $M_{I_{k'}}$, then one of the two cases below holds.

$$n + 4k = n + 4k' \qquad \text{and} \qquad 5n - 4k = 5n - 4k'$$
$$n + 4k = 5n - 4k' \qquad \text{and} \qquad 5n - 4k = n + 4k'$$

In the first case, $k = k'$, and in the second, $k' = n - k$. Thus as long as k and $k' < (n+1)/2$ are unequal, their cusp parameters are as well. □

There are also arbitrarily large subfamilies which share cusp parameters, even among complements of links obtained by mutating only with (12)(34). We do not know if these are commensurable.

Corollary 7.5. *For $0 \le k < n$, let $I_k = (t_0, \ldots, t_n)$ be defined by $t_i = 0$ for $i \ne k, k+1$, and $t_k = t_{k+1} = 2$. For each k, the cusp parameters of M_{I_k} are*

$$m(T_1) = i\left[2 + 4\left(n - \frac{4}{5}\right)\sqrt{2}\right] \qquad m(T_2) = i\left[\frac{2}{5} + \frac{4}{5}(n+4)\sqrt{2}\right],$$

up to the action of $\mathrm{PGL}_2(\mathbb{Q})$.

Appendix A. Proof of Lemma 2.6

Following Morgan [22], we define a *pared manifold* to be a pair (M, P), where M is a compact, orientable, irreducible 3-manifold with nonempty boundary which is not a 3-ball, and $P \subseteq \partial M$ is the union of a collection of disjoint incompressible annuli and tori satisfying the following properties:

- Every noncyclic abelian subgroup of $\pi_1 M$ is conjugate into the fundamental group of a component of P.
- Every map $\phi\colon (S^1 \times I, S^1 \times \partial I) \to (M, P)$ which induces an injection on fundamental groups is homotopic as a map of pairs to a map ψ such that $\psi(S^1 \times I) \subset P$.

This definition is intended to capture the topology of the compact manifold obtained by truncating the cusps of the convex core of a geometrically finite hyperbolic 3-manifold by open horoball neighborhoods. Indeed, Corollary 6.10 of [22] asserts that if (M, P) is obtained in this way, where P consists of the collection of boundaries of the truncating horoball neighborhoods, then (M, P) is a pared manifold.

Lemma 2.6 from the body of this paper asserts that if (M, P) has the pared homotopy type of a geometrically finite hyperbolic manifold $\mathbb{H}^3/\Gamma$ where Γ is not Fuchsian and $\partial C(\Gamma)$ is totally geodesic, then $M - P$ is homeomorphic to $C(\Gamma)$. The key point of the proof is that the geometric conditions on Γ ensure that (M, P) is an acylindrical pared manifold. Then Johannson's Theorem [13], that pared homotopy equivalences between acylindrical pared manifolds are homotopic to pared homeomorphisms, applies. We expand on this below.

It is worth noting that Lemma 2.6 fails in more general circumstances. Canary-McCullough give examples of this phenomenon in [7], where for instance they describe homotopy equivalent non-Fuchsian geometrically finite manifolds with incompressible convex core boundary which are not homeomorphic (Example 1.4.5). Their memoir [7] is devoted to understanding the ways in which homotopy equivalences of hyperbolic 3-manifolds can fail to be homeomorphic to homeomorphisms, and Lemma 2.6 follows quickly from results therein.

The treatment of Canary-McCullough itself uses the theory of *characteristic submanifolds* of manifolds with *boundary pattern* developed in [13]. The characteristic submanifold of a manifold with boundary pattern is a maximal collection of disjoint codimension–zero submanifolds, each of which is an interval bundle or Seifert–fibered space embedded reasonably with respect to the boundary pattern. Rather than attempting to establish all of the notation necessary to define this formally, we refer the interested reader to [13] and [7]. Here we simply transcribe the relevant theorem of [7], which places strong restrictions on the topology of the characteristic submanifold of a pared manifold whose boundary pattern is determined by the pared locus.

For the purposes of Lemma 2.6 we exclude from consideration certain pared manifolds which never arise from convex cores of geometrically finite hyperbolic 3-manifolds. We say (M, P) is *elementary* if it is homeomorphic to one of $(T^2 \times I, T^2 \times \{0\})$, $(A^2 \times I, A^2 \times \{0\})$, or $(A^2 \times I, \emptyset)$, where T^2 and A^2 denote the torus and annulus, respectively; otherwise (M, P) is nonelementary. Define $\partial_0 M := \overline{M - P}$. We say an annulus properly embedded in $M - P$ is *essential* in (M, P) if it is incompressible and boundary–incompressible in $M - P$. For a codimension–0 submanifold V embedded in M, we denote by $\mathrm{Fr}(V)$ the *frontier* of V (that is, its topological boundary in M), and note that $\mathrm{Fr}(V) = \overline{\partial V - (V \cap \partial M)}$. With notation thus established, the following theorem combines the definition of the characteristic submanifold with Theorem 5.3.4 of [7].

Theorem. *Let (M, P) be a nonelementary pared manifold with $\partial_0 M$ incompressible. Select the fibering of the characteristic submanifold so that no component is an I–bundle over an annulus or Möbius band.*

(1) *Suppose V is a component of the characteristic submanifold which is an I–bundle over a surface B. Then each component of the associated ∂I–bundle is contained in $\partial_0 M$, each component of the associated I–bundle over ∂B is either a component of P or a properly embedded essential annulus, and B has negative Euler characteristic.*
(2) *Suppose V is a Seifert fibered component of the characteristic submanifold. Then V is homeomorphic either to $T^2 \times I$ or to a solid torus. If V is homeomorphic to $T^2 \times I$, then one of its boundary components lies in P, the other components of $V \cap \partial M$ are annuli in $\partial_0 M$, and all components of $\mathrm{Fr}(V)$ are properly embedded essential annuli. If V is a solid torus, then $V \cap \partial M$ has at least one component, each an annulus either containing a component of P or contained in $\partial_0 M$. The components of $\mathrm{Fr}(V)$ are properly embedded essential annuli.*

The characteristic submanifold contains regular neighborhoods of all components of P.

The key claim in the proof of Lemma 2.6 is a further restriction on the characteristic submanifold of (M, P), in the case that M is obtained from the convex core of a non-Fuchsian geometrically finite manifold with totally geodesic convex core boundary by removing horoball neighborhoods of the cusps. P is the union of the boundaries of these neighborhoods.

Claim. (M, P) as above is nonelementary, and $\partial_0 M$ is incompressible. The characteristic submanifold of (M, P) consists only of (Seifert fibered) regular neighborhoods of the components of P, each of whose boundary has a unique component of intersection with ∂M.

We prove the claim below, but assuming it for now, the proof of Lemma 3 proceeds as follows. A representation as given in the statement of the lemma induces a pared homotopy equivalence between (M, P) and the pared manifold (N, Q) obtained by truncating $C(\Gamma)$ with open horoball neighborhoods. Since $C(\Gamma)$ has totally geodesic convex core boundary, (N, Q) is as described by the claim; hence (M, P) is as well (see Theorem 2.11.1 of [7], for example). Johansson's Classification Theorem (cf. [7], Theorem 2.9.10) implies that the original pared homotopy equivalence is homotopic to one which maps the complement of the characteristic submanifold of (M, P) homeomorphically to the complement of the characteristic submanifold of (N, Q). It follows from the claim that these are homeomorphic to $M - P$ and $N - Q$, respectively, and the lemma follows.

Proof of claim. As was mentioned above, the elementary pared manifolds do not arise from geometrically finite hyperbolic manifolds. Since (M, P) is obtained from the convex core of a geometrically finite manifold with totally geodesic convex core boundary, the following are known not to occur:

(1) A compressing disk for $\partial_0 M$.

(By definition $\partial_0 M$ lifts to a geodesic hyperplane in $\mathbb{H}^3$, hence the induced map $\pi_1 \partial M_0 \to \pi_1 M$ is injective.)

(2) An *accidental parabolic*; that is, an incompressible annulus properly embedded in M with one boundary component in P and one in $\partial_0 M$, which is not parallel to P.

(Every essential curve on $\partial_0 M$ that is not boundary-parallel is homotopic to a geodesic, but an element of $\pi_1(M)$ corresponding to an accidental parabolic has translation length 0.)

(3) A *cylinder*; that is, a properly embedded essential annulus in $M - P$.

(The double DM of M across $\partial_0 M$ is a hyperbolic manifold and the double of a cylinder in M is an essential torus in DM.)

We show that if the characteristic manifold has any components other than those listed in the claim then at least one of the above facts cannot hold.

For a component V of the characteristic submanifold which is an I–bundle over a surface B, at least one component of the associated I–bundle over ∂B must be properly embedded, since otherwise we would have $M = V$ and it is well known that an I–bundle over a surface does not admit a hyperbolic structure with totally geodesic convex core boundary unless the convex core is a Fuchsian surface. But this annulus violates fact 2 or 3. Thus there are no I–bundle components of the characteristic submanifold.

If V is a Seifert fibered component of the characteristic submanifold homeomorphic to $T^2 \times I$, then one component of ∂V is a torus $P_1 \subset P$, and all other components of $\partial V \cap \partial M$ are annulli in $\partial_0 M$. If this second class is nonempty, then each component of $\mathrm{Fr}(V)$ is an essential annulus properly embedded in $M - P$. This is not possible by fact 3, so $\partial V \cap \partial M$ consists only of P_1 and V is a regular neighborhood of P_1.

If V is a solid torus and $V \cap \partial M$ contains a component of P, then a similar argument shows that this is the unique component of $\partial V \cap \partial M$, so in this case V is a regular neighborhood of an annular component of P. If on the other hand $V \cap \partial M$ does *not* contain any components of P, then it has at least two components, for otherwise a meridional disk of V determines a boundary compression of the annulus $\mathrm{Fr}(V)$ in $M - P$. But then any component of $\mathrm{Fr}(V)$ violates fact 3. □

Appendix B. Mutations

For basic facts about mapping class groups used in this appendix, see [10].

Let S be a 2-sphere and $\{p_i\}_1^\infty$ a sequence of distinct points in S. Let $\mathrm{Mod}(S, \{p_i\}_1^n)$ be the mapping class group of $(S, \{p_i\}_1^n)$, that is, the set of mapping classes which preserve the set $\{p_i\}_1^n$. Let $\mathrm{PMod}(S, \{p_i\}_1^n)$ be the subgroup of $\mathrm{Mod}(S, \{p_i\}_1^n)$ which acts trivially on the set of marked points.

Let S_n be the permutation group for the set $\{p_i\}_1^n$ and let $V < S_4$ be the subgroup generated by the even elements of order 2 in S_4. Then $V \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

We get a representation $\theta\colon \mathrm{Mod}(S, \{p_i\}_1^4) \to S_4$ by considering the action on $\{p_i\}_1^4$. This map is part of the exact sequence

$$1 \longrightarrow \mathrm{PMod}\left(S, \{p_i\}_1^4\right) \longrightarrow \mathrm{Mod}\left(S, \{p_i\}_1^4\right) \longrightarrow S_4 \longrightarrow 1.$$

Definition B.1. A *mutation* of $(S, \{p_i\}_1^4)$ is an order 2 element of $\theta^{-1}(V)$.

Let σ be the regular tetrahedron with edges of unit length and fix a bijection $\{p_i\}_1^4 \to \sigma^{(0)}$. This bijection gives an isomorphism $\hat{}\colon\ S_4 \to \mathrm{Isom}(\sigma)$. Let $\phi\colon\ S \to \sigma$ be a homeomorphism which extends the above bijection. This gives a monomorphism $\psi\colon\ S_4 \to \mathrm{Mod}\big(S, \{p_i\}_1^4\big)$ by $g \mapsto [\phi^{-1}\hat{g}\phi]$ and $\phi(g)$ acts on the marked points of S as the permutation g. Hence ψ splits the above sequence.

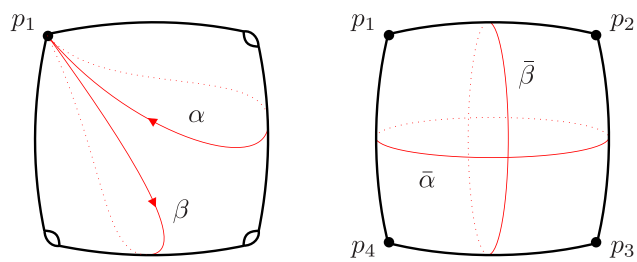

FIGURE 12. The Dehn twists on the curves $\bar{\alpha}$ and $\bar{\beta}$ represent the mapping classes of the images of $[\alpha], [\beta] \in \pi_1\big(S - \{p_i\}_2^4\big)$ under the push map.

Proposition B.2. *The mutations of $\big(S, \{p_i\}_1^4\big)$ are the three nontrivial elements of $\psi(V)$. In particular, every mutation is determined uniquely by its image under θ.*

Proof. Let α and β be the oriented curves in $S - \{p_2, p_3, p_4\}$ based at p_1 and shown in Figure 12. Let $\bar{\alpha}$ and $\bar{\beta}$ be the curves in S shown on the right in Figure 12. If γ is a simple closed curve in a surface let T_γ denote the Dehn twist along γ and $[T_\gamma]$ the element of the mapping class group represented by this Dehn twist.

We first claim that $\mathrm{PMod}\big(S, \{p_i\}_1^4\big)$ is free of rank two generated by $[T_{\hat{\alpha}}]$ and $[T_{\hat{\beta}}]$. Consider the Birman Exact sequence

$$1 \longrightarrow \pi_1\big(S - \{p_i\}_2^4\big) \xrightarrow{push} \mathrm{PMod}\big(S, \{p_i\}_1^4\big) \xrightarrow{\quad forget \quad} \mathrm{PMod}\big(S, \{p_i\}_2^4\big) \longrightarrow 1.$$

The push map is an isomorphism since $\mathrm{PMod}\big(S, \{p_i\}_2^4\big)$ is trivial. The images of $[\alpha]$ and $[\beta]$ are $[T_{\bar{\alpha}}]$ and $[T_{\hat{\beta}}]$ respectively.

Suppose that $h \in \mathrm{PMod}\big(S, \{p_i\}_1^4\big)$ and $g \in V$. We claim that $\psi(g)h = h\psi(g)$. This is because h is represented by a product of Dehn twists in $\bar{\alpha}$ and $\bar{\beta}$ while $\psi(g)$ preserves the isotopy classes of both curves.

Now suppose that m is a mutation. Then there is some $h \in \mathrm{PMod}\big(S, \{p_i\}_1^4\big)$ and $g \in V$ so that $m = \psi(g)h$. The mutation has order two so we have

$$id \;=\; m^2 \;=\; \big(\psi(g)h\big)^2 \;=\; \psi(g^2)h^2 \;=\; h^2.$$

But $\mathrm{PMod}\big(S, \{p_i\}_1^4\big)$ is a free group, so $h = id$ and $m = \psi(g)$. ☐

REFERENCES

[1] C. Adams and E. Schoenfeld. Totally geodesic Seifert surfaces in hyperbolic knot and link complements. I. *Geom. Dedicata*, 116:237–247, 2005.

[2] Colin Adams, Hanna Bennett, Christopher Davis, Michael Jennings, Jennifer Kloke, Nicholas Perry, and Eric Schoenfeld. Totally geodesic Seifert surfaces in hyperbolic knot and link complements. II. *J. Differential Geom.*, 79(1):1–23, 2008.

[3] Colin C. Adams. Thrice-punctured spheres in hyperbolic 3-manifolds. *Trans. Amer. Math. Soc.*, 287(2):645–656, 1985.

[4] Iain R. Aitchison and J. Hyam Rubinstein. Geodesic surfaces in knot complements. *Experiment. Math.*, 6(2):137–150, 1997.

[5] H. Bass. Finitely generated subgroups of GL_2. In *The Smith conjecture (New York, 1979)*, volume 112 of *Pure Appl. Math.*, pages 127–136. Academic Press, Orlando, FL, 1984.

[6] Armand Borel. Cohomologie de SL_n et valeurs de fonctions zeta aux points entiers. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(4):613–636, 1977.

[7] R. D. Canary and D. McCullough. Homotopy equivalences of 3-manifolds and deformation theory of Kleinian groups. *Mem. Amer. Math. Soc.*, 172(812):xii+218, 2004.

[8] Johan L. Dupont. *Scissors congruences, group homology and characteristic classes*, volume 1 of *Nankai Tracts in Mathematics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2001.

[9] D. B. A. Epstein and R. C. Penner. Euclidean decompositions of noncompact hyperbolic manifolds. *J. Differential Geom.*, 27(1):67–80, 1988.

[10] B. Farb and Margalit D. A primer on mapping class groups. Online. http://www.math.utah.edu/ margalit/primer, 2009.

[11] O. Goodman, D. Heard, and C. Hodgson. Commensurators of cusped hyperbolic manifolds. Preprint. arXiv:0801.4815, January 2008. to appear, Exper. Math.

[12] Allen Hatcher. Hyperbolic structures of arithmetic type on some link complements. *J. London Math. Soc. (2)*, 27(2):345–355, 1983.

[13] K. Johannson. *Homotopy equivalences of* 3*-manifolds with boundaries*, volume 761 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[14] S. Kojima. Polyhedral decomposition of hyperbolic manifolds with boundary. In *Proc. Workshops Pure. Math.*, volume 10, pages 37–57. 1990.

[15] S. Kojima. Geometry of hyperbolic 3-manifolds with boundary. *Kodai Math. J.*, 17(3):530–537, 1994. Workshop on Geometry and Topology (Hanoi, 1993).

[16] C. J. Leininger. Small curvature surfaces in hyperbolic 3-manifolds. *J. Knot Theory Ramifications*, 15(3):379–411, 2006.

[17] M. Macasieb and T. Mattman. Commensurability classes of (-2,3,n) pretzel knot complements. Preprint. arXiv:0804.0112, April 2008.

[18] C. Maclachlan and A. W. Reid. Parametrizing Fuchsian subgroups of the Bianchi groups. *Canad. J. Math.*, 43(1):158–181, 1991.

[19] C. Maclachlan and A. W. Reid. *The arithmetic of hyperbolic 3-manifolds*, volume 219 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003.

[20] B. Maskit. On Klein's combination theorem. III. In *Advances in the Theory of Riemann Surfaces (Proc. Conf., Stony Brook, N.Y., 1969)*, Ann. of Math. Studies, No. 66, pages 297–316. Princeton Univ. Press, Princeton, N.J., 1971.

[21] W. Menasco and A. W. Reid. Totally geodesic surfaces in hyperbolic link complements. In *Topology '90 (Columbus, OH, 1990)*, volume 1 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 215–226. de Gruyter, Berlin, 1992.

[22] J. W. Morgan. On Thurston's uniformization theorem for three-dimensional manifolds. In *The Smith conjecture (New York, 1979)*, volume 112 of *Pure Appl. Math.*, pages 37–125. Academic Press, Orlando, FL, 1984.

[23] W. D. Neumann and A. W. Reid. Amalgamation and the invariant trace field of a Kleinian group. *Math. Proc. Cambridge Philos. Soc.*, 109(3):509–515, 1991.

[24] W. D. Neumann and A. W. Reid. Arithmetic of hyperbolic manifolds. In *Topology '90 (Columbus, OH, 1990)*, volume 1 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 273–310. de Gruyter, Berlin, 1992.

[25] Walter D. Neumann. Hilbert's 3rd problem and invariants of 3-manifolds. In *The Epstein birthday schrift*, volume 1 of *Geom. Topol. Monogr.*, pages 383–411 (electronic). Geom. Topol. Publ., Coventry, 1998.

[26] Walter D. Neumann and Jun Yang. Bloch invariants of hyperbolic 3-manifolds. *Duke Math. J.*, 96(1):29–59, 1999.

[27] L. Paoluzzi and B. Zimmermann. On a class of hyperbolic 3-manifolds and groups with one defining relation. *Geom. Dedicata*, 60(2):113–123, 1996.

[28] J. G. Ratcliffe. *Foundations of hyperbolic manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

[29] Alan W. Reid and Genevieve S. Walsh. Commensurability classes of 2-bridge knot complements. *Algebr. Geom. Topol.*, 8(2):1031–1057, 2008.

[30] D. Ruberman. Mutation and volumes of knots in S^3. *Invent. Math.*, 90(1):189–215, 1987.

[31] W. P. Thurston. The geometry and topology of 3-manifolds. mimeographed lecture notes, 1979.

Department of Mathematical Sciences, University of Montana
E-mail address: Eric.Chesebro@mso.umt.edu

Department of MSCS, University of Illinois at Chicago
E-mail address: jdeblois@math.uic.edu