CLOSED SURFACES AND CHARACTER VARIETIES

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Abstract. We give some algebraic characterizations for when the character variety techniques of Culler and Shalen can be used to construct a closed essential surface in a one-cusped hyperbolic 3-manifold.

1. Introduction

Suppose that N is a compact, irreducible 3-manifold with torus boundary and that X is an irreducible algebraic subset of the $\mathrm{SL}_2\mathbb{C}$-character variety for N. Given an element $\gamma \in \pi_1 N$, define $I_\gamma \in \mathbb{C}[\mathsf{X}]$ to be the regular function $\chi \mapsto \chi(\mu)$. The subring of the coordinate ring for X generated by these functions is called the trace ring for X, denoted $T(\mathsf{X})$. $T_{\mathbb{Q}}(\mathsf{X})$ is the smallest $\mathbb{Q}$-algebra in $\mathbb{C}[\mathsf{X}]$ which contains $T(\mathsf{X})$. Let $\mu \in \Gamma$ be a slope. Then $\mathbb{C}[\mathsf{X}]$, $T_{\mathbb{Q}}(\mathsf{X})$, and $T(\mathsf{X})$ have the structure of a $\mathbb{C}[I_\mu]$-module, a $\mathbb{Q}[I_\mu]$-module, and a $\mathbb{Z}[I_\mu]$-module, respectively. Let $\mathrm{Rk}^{\mathbb{C}}_{\mathsf{X}}(\mu)$, $\mathrm{Rk}^{\mathbb{Q}}_{\mathsf{X}}(\mu)$, and $\mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}}(\mu)$ denote the ranks of these modules. Let $\mathcal{S}$ be the set of slopes for N.

Theorem 1. *Let $\mathbb{X} \in \{\mathbb{C}, \mathbb{Q}, \mathbb{Z}\}$. Then*

(1) *The function $\mathrm{Rk}^{\mathbb{X}}_{\mathsf{X}} \colon \mathcal{S} \to \mathbb{Z}^+ \cup \{\infty\}$ is constant with value ∞ if and only if X detects a closed essential surface.*

(2) *If X does not detect a closed essential surface, then $\mathrm{Rk}^{\mathbb{X}}_{\mathsf{X}}(\mu) = \infty$ if and only if the slope μ is detected by X.*

In general, we have

$$\mathrm{Rk}^{\mathbb{C}}_{\mathsf{X}} \;\leq\; \mathrm{Rk}^{\mathbb{Q}}_{\mathsf{X}} \;\leq\; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}}. \tag{1}$$

It is easy to prove the following two propositions.

Proposition 1.1. *Suppose that N is a knot manifold and $H_1(N;\mathbb{Z}) \cong \mathbb{Z}$, generated by μ. Let $\mathsf{X}_A \subseteq \mathsf{X}(N)$ be the curve consisting of all abelian representations of $\pi_1 N$.*

$$\mathrm{Rk}^{\mathbb{C}}_{\mathsf{X}_A}(\mu) \;=\; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_A}(\mu) \;=\; 1.$$

Proposition 1.2. *If N is a knot manifold and X is a norm curve component of $\mathsf{X}(N)$ then $\mathrm{Rk}^{\mathbb{C}}_{\mathsf{X}}(\mu) \geq 2$ for every slope μ.*

As a corollary to Theorem 1 we have the following.

Corollary 1.3. *Suppose N is the exterior of a two-bridge knot and $\langle a, b \,|\, \omega b = a\omega\rangle$ is its standard presentation. If $\mathsf{X} \subset \mathsf{X}(N)$ is an irreducible algebraic component defined over $\mathbb{Q}$ then X is defined by an irreducible polynomial of the form*

$$I_{ab}^n - \sum_{j=0}^{n-1} p_j(I_a) I_{ab}^j.$$

The set $\{I_{ab}^j\}_0^{n-1}$ *is a free basis for* $\mathbb{C}[X]$ *as a* $\mathbb{C}[I_a]$*-module,* $T_{\mathbb{Q}}(\mathsf{X})$ *as a* $\mathbb{Q}[I_a]$*-module, and* $T(\mathsf{X})$ *as a* $\mathbb{Z}[I_a]$*-module.*

We have done computations for the following examples. Here X indicates that we are speaking about $\mathrm{SL}_2\mathbb{C}$-character varieties and Y indicates $\mathrm{PSL}_2\mathbb{C}$-character varieties. When $N \subset S^3$ is a knot exterior we reserve $\lambda, \mu \in \Gamma$ as a longitude, meridian pair. X_A (Y_A) indicates the algebraic set of abelian characters and if N is hyperbolic X_0 (Y_0) always denotes an algebraic component containing a discrete faithful character.

Example (5) is of particular interest. It shows that the inequalities (1) need not be equalities and, for a slope σ, $T(\mathsf{X})$ is torsion free as a $\mathbb{Z}[I_\sigma]$-module but is not necessarily free.

(1) **N is the exterior of a trefoil knot.** Then $\mathsf{X}(N) = \mathsf{X}_A \cup \mathsf{X}_0$ where X_0 is irreducible.
$$\mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_0}(\mu) \;=\; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{Y}_0}(\mu^2) \;=\; 1.$$
This shows that the converse to Proposition 1.1 does not hold.

(2) **N is the exterior of the figure-eight knot.**
$$\begin{aligned}\mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_0}(\mu) \;&=\; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{Y}_0}(\mu^2) \;=\; 2\\ \mathrm{Rk}^{\mathbb{C}}_{\mathsf{Y}_0}(\lambda) \;&=\; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{Y}_0}(\lambda) \;=\; 4\\ \mathrm{Rk}^{\mathbb{C}}_{\mathsf{Y}_0}(\mu^2\lambda) \;&=\; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{Y}_0}(\mu^2\lambda) \;=\; 5.\end{aligned}$$

(3) $\mathbf{N = M_{003}}$. Then $\Gamma = \langle a, b \,|\, aba^{-2}bab^3 \rangle$ and if $\mu = (b^2aba)^{-1}$ and $\lambda = (aba)^{-1}bab$ then μ and λ are primitive, peripheral, and generate the peripheral subgroup of N.
$$\mathrm{Rk}^{\mathbb{C}}_{\mathsf{X}_0}(\mu) \;=\; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_0}(\mu) \;=\; 4.$$
Also, $I_\mu \in T(\mathsf{Y}_0)$ and
$$\mathrm{Rk}^{\mathbb{C}}_{\mathsf{Y}_0}(\mu) \;=\; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{Y}_0}(\mu) \;=\; 2.$$
Note that, since Y_0 is a norm curve, these ranks achieve their minimum possible value. In contrast to examples (1) and (2), the $\mathrm{PSL}_2\mathbb{C}$-rank is strictly smaller than the $\mathrm{SL}_2\mathbb{C}$-rank (at μ). In this example, all $\mathbb{Z}[I_\mu]$-modules are free.

(4) **N is the exterior of the knot** $\mathbf{8_{20}}$. N is hyperbolic, so we consider X_0. The diagram in Figure 1 gives a presentation for Γ with generators $\gamma = tu$ and $\mu = b$.

The functions I_μ, I_γ, and $I_{\gamma\mu}$ give an embedding of X_0 into $\mathbb{C}^3$. I_γ and $I_{\gamma\mu}$ both satisfy irreducible integral dependencies in $\mathbb{Z}[I_\mu][x]$ of degree 5. It follows that $\mathcal{B} = \{I_\gamma^i I_{\gamma\mu}^j \,|\, 0 \le i, j \le 4\}$ is a generating set for $\mathbb{C}[\mathsf{X}_0]$ as a $\mathbb{C}[I_\mu]$-module, $T_{\mathbb{Q}}(\mathsf{X}_0)$ as a $\mathbb{Q}[I_\mu]$-module, and $T(\mathsf{X}_0)$ as a $\mathbb{Z}[I_\mu]$-module. Notice that $|\mathcal{B}| = 25$.

A Groebner basis argument shows that, in fact, relations amongst the elements of $\mathcal{B}$ are plentiful and $\{1, I_\gamma, I_\gamma^2, I_\gamma^3, I_\gamma^2 I_{\gamma\mu}, I_{\gamma\mu}\}$ is a free basis for each of the above modules. Hence,
$$\mathrm{Rk}^{\mathbb{C}}_{\mathsf{X}_0}(\mu) \;=\; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_0}(\mu) \;=\; 6$$
and $T(\mathsf{X}_0)$ is a free $\mathbb{Z}[I_\mu]$-module.

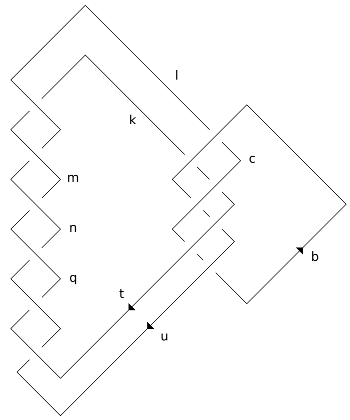

FIGURE 1. The knot 8_{20} labelled with Wirtinger generators

(5) **N is the once punctured hyperbolic torus bundle from Section 5 of [11].** Then $\Gamma = \langle \alpha, \beta, \tau \,|\, \tau\alpha\tau^{-1} = (\beta\alpha\beta)^{-1}, \tau\beta\tau^{-1} = \beta\alpha(\beta\alpha\beta)^{-3}\rangle$. The elements τ and $\lambda = [\alpha, \beta]$ form a basis for the peripheral subgroup of N. The functions

$$t = I_\tau \qquad u = I_{\alpha\tau} \qquad v = I_{\beta\tau} \qquad w = I_{\alpha\beta\tau}$$
$$x = I_\alpha \qquad y = I_\beta \qquad z = I_{\alpha\beta}$$

give an embedding of $\mathsf{X}(N)$ into $\mathbb{C}^7$. For $\epsilon \in \{0, 1\}$, we have an irreducible algebraic component $\mathsf{X}_\epsilon \subset \mathsf{X}(N)$ which contains a discrete faithful character. The projection

$$(t, y, z)\colon\ \mathsf{X}_0 \cup \mathsf{X}_1 \to \mathbb{C}^3$$

is an isomorphism onto its image, so we may identity X_ϵ with its image under this map.
(a) $\{1, z, z^2, z^3, y, zy, z^2y, z^3y\}$ is a free basis for $\mathbb{C}[\mathsf{X}_\epsilon]$ as a $\mathbb{C}[I_\lambda]$-module.
(b) $\{1, z, z^2, z^3, y, zy, z^2y, z^3y, t, zt, z^2t, z^3t, yt, zyt, z^2yt, z^3yt\}$ is a free basis for $T(\mathsf{X}_\epsilon)$ as a $\mathbb{Q}[I_\lambda]$-module.
(c) $\{1, y, y^2, y^3, t, u, v, w, x, z, yt, yu, yx, y^2x, xt, xu, vy\}$ is a basis for $T(\mathsf{X}_\epsilon)$ as a $\mathbb{Z}[I_\lambda]$-module. However, $T(\mathsf{X}_\epsilon)$ is torsion free but <u>not</u> free as a $\mathbb{Z}[I_\lambda]$-module.

The relation

$$\mathrm{Rk}^{\mathbb{Q}}_{\mathsf{X}_\epsilon}(\lambda) < \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_\epsilon}(\lambda)$$

reflects the fact that the inverse of the isomorphism (t, y, z) is not defined over $\mathbb{Z}$. The equality

$$\mathrm{Rk}^{\mathbb{Q}}_{\mathsf{X}_\epsilon}(\lambda) = 2 \cdot \mathrm{Rk}^{\mathbb{C}}_{\mathsf{X}_\epsilon}(\lambda)$$

reflects the fact that X_ϵ is not defined over $\mathbb{Q}$.

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2. Algebraic Geometry

Throughout this section we take k to be an algebraically closed field. All varieties are defined over k. When X is an irreducible variety we write $k[\mathsf{X}]$ and $k(\mathsf{X})$ to denote the ring of regular functions on X and the function field for X respectively. If $p \in \mathsf{X}$, define $\mathcal{O}_{\mathsf{X},p}$ to be the ring of germs of functions which are regular on neighborhoods of p. We have a surjective homomorphism $k[\mathsf{X}] \to k$ given by $f \mapsto f(p)$. Let $\mathfrak{m}_p$ be the maximal ideal which is the kernel of this map. By Theorem 3.2 part (c) of [13], $\mathcal{O}_{\mathsf{X},p}$ is isomorphic to the localization $k[\mathsf{X}]_{\mathfrak{m}_p}$. We use this isomorphism to identify these two rings and think of $\mathcal{O}_{\mathsf{X},p}$ as a subring of $k(\mathsf{X})$.

Definitions 2.1. Suppose $A \subseteq B$ are commutative rings and $1_B \in A$.

(1) An element $b \in B$ is **integral over A** if there is a number $n \in \mathbb{Z}^+$ and $\{\alpha_i\}_0^{n-1} \subset A$ such that

$$b^n + \alpha_{n-1} b^{n-1} + \cdots + \alpha_0 \;=\; 0.$$

This equation is called an **integral dependence relation of b over A**.

(2) **B is integral over A** if every element of B is integral over A.

(3) The **integral closure $\overline{\mathbf{A}}$ of A in B** is the set of all elements of B which are integral over A.

(4) If the integral closure of A is A then A is **integrally closed**.

By Corollary 5.3 of [1], the integral closure of A in B is a ring. The following characterization of integral closures is useful.

Theorem 2 (Corollary 5.22 of [1])**.** *Suppose A is a subring of a field K. The integral closure of A is the intersection of all valuation rings of K that contain A.*

Definitions 2.2. Suppose that X is an affine variety.

(1) X is a **normal variety** if $k[\mathsf{X}]$ is integrally closed in $k(\mathsf{X})$.

(2) A **normalization of** X is an irreducible normal variety X^ξ and a regular birational map $\xi\colon\ \mathsf{X}^\xi \to \mathsf{X}$ where $k[\mathsf{X}^\xi]$ is integral over $\xi^*\left(k[\mathsf{X}]\right)$.

If X is an affine variety and $p \in \mathsf{X}$ then the quotient $\mathfrak{m}_p/\mathfrak{m}_p^2$ has the structure of a k-vector space.

Definitions 2.3. Suppose that X is an affine variety.

(1) A point $p \in \mathsf{X}$ is **non-singular** if $\dim\left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)$ equals the dimension of X.

(2) X is **non-singular** if every point in X is non-singular.

Theorem 1 in Chapter II, Section 5.1 of [19] states that non-singular affine varieties are normal.

It is clear that if $\mathfrak{m}_p$ is principal then $\dim\left(\mathfrak{m}_p/\mathfrak{m}_p^2\right) = 1$. Theorem 2 in Chapter II, Section 5.1 of [19] implies that if X is a normal affine algebraic curve and $p \in \mathsf{X}$ then $\mathfrak{m}_p$ is principal. Thus, we have the following well-known theorem.

Theorem 3. *Suppose X is an irreducible affine algebraic curve. X is non-singular if and only if X is normal.*

The following theorem is a direct consequence of Theorems 4 and 5 in Chapter II, Section 5.2 in [19] (and the first part of the proof of Theorem 4).

Theorem 4. *If* X *is an irreducible affine algebraic curve then* X *has a normalization* $\xi\colon \mathsf{X}^\xi \to \mathsf{X}$. *Moreover, the normalization is unique, affine, and its coordinate ring is the integral closure of* $\xi^*(k[\mathsf{X}])$.

The projective coordinates on projective space $\mathbb{P}^n$ give distinguished open affine subsets $\{U_i\}_0^n$ which cover $\mathbb{P}^n$. So if $\mathsf{X} \subset \mathbb{P}^n$ is an irreducible projective variety then we have the distinguished affine open subsets $U_i \cap \mathsf{X}$ which cover X. We say that X is non-singular if $U_i \cap \mathsf{X}$ is non-singular for every $i = 0, \ldots, n$.

Definition 2.4. Suppose that X is an irreducible affine curve and let $\xi\colon \mathsf{X}^\xi \to \mathsf{X}$ the normalization of X. As in [13][Chapter 6], there is a smooth projective curve $\widetilde{\mathsf{X}}$ (unique up to isomorphism) so that X^ξ is isomorphic to an open set in $\widetilde{\mathsf{X}}$. The projective variety $\widetilde{\mathsf{X}}$ is called the **smooth projective model for** X.

Henceforth, we identify X^ξ with its image in $\widetilde{\mathsf{X}}$ and we let $\iota\colon \widetilde{\mathsf{X}} \dashrightarrow \mathsf{X}^\xi$ be the rational map which is defined as the identity on X^ξ.

Definition 2.5. The **ideal points** of X are the points in the set $\mathcal{I}(\mathsf{X}) = \widetilde{\mathsf{X}} - \mathsf{X}^\xi$. (It follows from the uniqueness of normalizations and smooth projective models that $\mathcal{I}(\mathsf{X})$ is well-defined.)

We have dominant birational maps

$$\widetilde{\mathsf{X}} \overset{\iota}{\dashrightarrow} \mathsf{X}^\xi \xrightarrow{\ \xi\ } \mathsf{X}$$

and the induced maps

$$k\big(\widetilde{\mathsf{X}}\big) \xleftarrow{\ \iota^*\ } k\left(\mathsf{X}^\xi\right) \xleftarrow{\ \xi^*\ } k(\mathsf{X})$$

are isomorphims. Moreover, $\xi^*\big(k[\mathsf{X}]\big) \subseteq k[\mathsf{X}^\xi]$.

Now suppose that Y is an affine variey and $\varphi\colon \mathsf{X} \to Y$ is a regular map with $\overline{\varphi(\mathsf{X})} = Y$. Then $\varphi^*\colon k(Y) \to k(\mathsf{X})$ is a field monomorphism and $\varphi^*(k[Y]) \subseteq k[\mathsf{X}]$.

Definition 2.6. A **hole** in $\varphi\colon \mathsf{X} \to Y$ is a point $\hat{p} \in \mathcal{I}(\mathsf{X})$ such that $(\varphi\xi\iota)^*(k[Y]) \subseteq \mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$.

Remark 1. If $\hat{p}$ is a hole in $\varphi\colon \mathsf{X} \to Y$ then, since $\hat{p} \notin \mathsf{X}^\xi$, there is an element $f \in k[\mathsf{X}]$ with $(\xi\iota)^*(f) \notin \mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$. Also, if φ is a birational map then no hole is equivalent to surjective.

Lemma 2.7. *If* $\varphi\colon \mathsf{X} \to Y$ *has a hole then* $k[\mathsf{X}]$ *is not integral over* $\varphi^*(k[Y])$.

Proof. Take a distinguished open affine set $U_i \subset \mathbb{P}^n$ with $\hat{p} \in U_i$. Set $\mathsf{X}_{\hat{p}} = U_i \cap \widetilde{\mathsf{X}}$. By Theorem 3, $\mathsf{X}_{\hat{p}}$ is a non-singular affine curve so $k[\mathsf{X}_{\hat{p}}]$ is integrally closed. Recall that $\mathcal{O}_{\mathsf{X}_{\hat{p}},\hat{p}}$ is isomorphic to the localization $k[\mathsf{X}_{\hat{p}}]_{\mathfrak{m}_{\hat{p}}}$. Proposition 5.13 of [1] gives that this localization is integrally closed. The inclusion induced isomorphism $k(\widetilde{\mathsf{X}}) \to k(\mathsf{X}_{\hat{p}})$ restricts to an isomorphism $\mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}} \to \mathcal{O}_{\mathsf{X}_{\hat{p}},\hat{p}}$. Hence $\mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$ is integrally closed.

The point $\hat{p}$ is a hole in $\varphi\colon \mathsf{X} \to Y$ so $(\varphi\xi\iota)^*(k[Y]) \subseteq \mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$. Since $\mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$ is integrally closed, every element of $k(\widetilde{\mathsf{X}})$ which is integral over $(\varphi\xi\iota)^*(k[Y])$ is an element of $\mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$. We have $f \in k[\mathsf{X}]$ with $(\xi\iota)^*(f) \notin \mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$, hence $(\xi\iota)^*(k[\mathsf{X}])$ is not integral over $(\varphi\xi\iota)^*(k[Y])$ and so $k[\mathsf{X}]$ is not integral over $\varphi^*(k[Y])$. □

Theorem 5. *Suppose* X *is an irreducible affine algebraic curve and* $\varphi\colon \mathsf{X} \to Y$ *is a regular map with* $\overline{\varphi(\mathsf{X})} = Y$. *Then* φ *has no hole if and only if* $k[\mathsf{X}]$ *is integral over* $\varphi^*(k[Y])$.

Proof. By Lemma 2.7, we need only show that if φ has no hole then $k[\mathsf{X}]$ is integral over $\varphi^*(k[Y])$.

Assume, to the contrary, that $k[\mathsf{X}]$ is not integral over $\varphi^*(k[Y])$. Since ξ^* is injective, this implies that $k[\mathsf{X}^\xi]$ is not integral over $(\varphi\xi)^*(k[Y])$. Taking integral closures in $k(\mathsf{X}^\xi)$ we have

$$k[\mathsf{X}^\xi] \;=\; \overline{k[\mathsf{X}^\xi]} \;\supsetneq\; \overline{(\varphi\xi)^*(k[Y])}.$$

Take $f \in k[\mathsf{X}^\xi] - \overline{(\varphi\xi)^*(k[Y])}$. By Theorem 2, there is a valuation ring $R \subset k(\mathsf{X}^\xi)$ with $f \notin R$ and $k \subset \overline{(\varphi\xi)^*(k[Y])} \subseteq R$.

Claim: $\iota^*(R) = \mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$ for some $\hat{p} \in \widetilde{\mathsf{X}}$.

Using the claim, $\iota^*(f) \notin \mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$ so $\hat{p} \in \mathcal{I}(\mathsf{X})$. Moreover, $(\varphi\xi)^*k[Y] \subseteq R$ so $(\varphi\xi\iota)^*(k[Y]) \subseteq \iota^*(R) = \mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$. That is, $\hat{p}$ is a hole in φ, a contradiction.

The claim follows from Corollary 6.6 of [13] since it gives an open set X_R in $\widetilde{\mathsf{X}}$ and a point $\hat{p} \in \mathsf{X}_R$ such that $\iota^*(R) = \mathcal{O}_{\mathsf{X}_R,\hat{p}} = \mathcal{O}_{\widetilde{\mathsf{X}},\hat{p}}$. □

It is well-known that, in this setting, $k[\mathsf{X}]$ is integral over $f^*(k[Y])$ if and only if $k[\mathsf{X}]$ is a finitely generated $f^*(k[Y])$-module, see for example Proposition 5.1 and Corollary 5.2 of [1]. Hence, we have the following immediate corollary.

Corollary 2.8. *Suppose* X *is an irreducible affine algebraic curve and* $\varphi\colon \mathsf{X} \to Y$ *is a regular map with* $\overline{\varphi(\mathsf{X})} = Y$. *Then* φ *has no hole if and only if* $k[\mathsf{X}]$ *is a finitely generated* $\varphi^*(k[Y])$*-module.*

Remarks 1. Suppose that $k[\mathsf{X}]$ is a finitely generated $\varphi^*(k[Y])$-module.

1. Since X is irreducible, $k[\mathsf{X}]$ has no zero divisors. Hence $k[\mathsf{X}]$ is a torsion free $\varphi^*(k[Y])$-module.
2. We can be more concrete about a basis for $k[\mathsf{X}]$ as a $\varphi^*(k[Y])$-module. Let $\{x_i\}_1^m$ be coordinate functions for X. By Theorem 5, each x_i is integral over $\varphi^*(k[Y])$. Let n_i be the degree of an integral dependence for x_i and define $\mathcal{S} = \{x_1^{\alpha_1} \cdots x_m^{\alpha_m} \mid 0 \leq \alpha_i < n_i\}$. Every element of $k[\mathsf{X}]$ may be expressed as a $\varphi^*(k[\mathsf{X}])$-linear combination of the elements from the finite set $\mathcal{S}$. (See Proposition 2.16 of [1].)

3. Character varieties and closed essential surfaces

Definition 3.1. A **knot manifold** is a connected, compact, irreducible, orientable 3-manifold whose boundary is an incompressible torus.

Let N be a knot manifold and $\Gamma = \pi_1(N)$. Denote the set of $\mathrm{SL}_2\mathbb{C}$-representations of Γ as $\mathsf{R}(N)$ and the set of characters of representations in $\mathsf{R}(N)$ as $\mathsf{X}(N)$. Let $t\colon \mathsf{R}(N) \to \mathsf{X}(N)$ be the map which takes representations to their characters. It is shown in [10] that $\mathsf{R}(N)$ and $\mathsf{X}(N)$ are affine algebraic sets defined over $\mathbb{C}$ and the map t is regular. Henceforth, we will refer to $\mathsf{R}(N)$ and $\mathsf{X}(N)$ as the representation variety and character variety for N.

Culler and Shalen have revealed deep connections between essential surfaces in N and the character variety $\mathsf{X}(N)$. For more background see [10], the survey article [20], or Chapter 1 of [9].

Definitions 3.2. Suppose that X is a non-empty algebraic subset of $\mathsf{X}(N)$.

(1) Given $\gamma \in \Gamma$, the **trace function** for γ on X is the regular function $I_\gamma \in \mathbb{C}[\mathsf{X}]$ defined by $I_\gamma(\chi) = \chi(\gamma)$.
(2) Let $T(\mathsf{X})$ be the subring (with 1) in $\mathbb{C}[\mathsf{X}]$ generated by $\{I_\gamma | \gamma \in \Gamma\}$. $T(\mathsf{X})$ is called the **trace ring** for X.

The following proposition gives that, as a ring, $T(\mathsf{X})$ is finitely generated.

Proposition 3.3 (Proposition 4.4.2 of [20]). *Let $\{\gamma_i\}_1^n$ be a generating set for Γ. Then $T(\mathsf{X})$ is generated, as a ring, by the constant function 1 along with the functions in the set*

$$\left\{ I_{\gamma_{j_1}\cdots\gamma_{j_k}} \;\middle|\; 1 \leq k \leq n \text{ and } 1 \leq j_1 < \cdots < j_k \leq n \right\}.$$

The trace ring is a subring of the $\mathbb{C}$-algebra $\mathbb{C}[\mathsf{X}]$. In fact, $T(\mathsf{X})$ generates the $\mathbb{C}$-algebra $\mathbb{C}[\mathsf{X}]$. Thus, a generating set $\{\gamma_i\}_1^n$ for Γ gives an embedding of X into $\mathbb{C}^{2^n-1}$ by taking the functions I_V to be coordinate functions. It is straightforward to see that under this embedding, $\mathsf{X}(N)$ is cut out by polynomials with coefficients in $\mathbb{Z}$.

We have a regular map $\partial\colon \mathsf{X}(N) \to \mathsf{X}(\partial N)$ from $\mathsf{X}(N)$ to the character variety for the peripheral subgroup (well-defined up to conjugation) of Γ given by restricting characters. For a non-empty algebraic subset X of $\mathsf{X}(N)$, let $\partial\mathsf{X}$ denote the Zariski closure of $\partial(\mathsf{X})$.

Assume now that $\mathsf{X} \subseteq \mathsf{X}(N)$ is an irreducible affine curve and let

$$\widetilde{\mathsf{X}} \overset{\iota}{\dashrightarrow} \mathsf{X}^\xi \xrightarrow{\;\xi\;} \mathsf{X}$$

be the corresponding maps and varieties as defined in Section 2.

Definition 3.4. The unoriented isotopy class of an essential simple closed curve in ∂N is called a **slope**.

We will usually identify a slope with a primitive peripheral element $\mu \in \Gamma$ which is represented by a loop which is freely homotopic to a curve representing the slope. Although, such a μ is only well-defined up to conjugation and inversion, the value of the corresponding trace function is independent of these choices.

The following theorem is well known and fundamental, see Theorem 2.2.1 and Proposition 2.3.1 of [10].

Theorem 6. *For every ideal point $\hat{x}$ of X we get an associated non-empty collection $\{\Sigma_i\}$ of essential surfaces in N.*

(1) *This collection contains a closed surface if and only if*
$$(\partial\xi\iota)^* \left(\mathbb{C}[\partial\mathsf{X}]\right) \;\subseteq\; \mathcal{O}_{\widetilde{\mathsf{X}},\hat{x}}.$$
(2) *Otherwise there exists a unique slope $\mu \in \Gamma$ such that $I_\mu \in \mathcal{O}_{\widetilde{\mathsf{X}},\hat{x}}$. In this case, every component of every $\partial\Sigma_i$ represents the slope μ.*

Definitions 3.5.

(1) Suppose that $\mathsf{X} \subseteq \mathsf{X}(N)$ is an irreducible affine curve, $\hat{x}$ is an ideal point of X, and $\{\Sigma_i\}$ is the collection of essential surfaces in N given by $\hat{x}$ and Theorem 6. A surface Σ is **associated to $\hat{\mathbf{x}}$** if $\Sigma \in \{\Sigma_i\}$.

(2) Suppose that X is an algebraic subset of $\mathsf{X}(N)$. A surface Σ is **detected by** X if there is an ideal point $\hat{x}$ of an irreducible affine curve in X so that Σ is associated to $\hat{x}$.
(3) If an essential surface $\Sigma \subset N$ is detected by X (associated to $\hat{x}$) and $\partial\Sigma \neq \emptyset$ the slope represented by a component of $\partial\Sigma$ is **detected by** X (associated to $\hat{x}$). The slope is **strongly** or **weakly** detected (associated) depending on whether Σ arrises in case (1) or case (2) of Theorem 6, respectively.

Remark 2. It is natural to ask if, whenever N contains a closed essential surface, there is a closed essential surface in N detected by $\mathsf{X}(N)$. The author is not certain if the answer is known to be no, however there are compelling reasons to believe that the answer is no, see for example [5] or Remark 5.1 in [11].

We conclude this section with some applications of the work in Section 2.

Theorem 7. *Suppose N is a knot manifold and X is an irreducible algebraic subset of $\mathsf{X}(N)$. The following are equivalent.*

(1) *X does not detect a closed essential surface.*
(2) *$\dim(\mathsf{X}) = \dim(\partial\mathsf{X}) = 1$ and $\partial\colon \mathsf{X} \to \partial\mathsf{X}$ does not have a hole.*
(3) *$\mathbb{C}[\mathsf{X}]$ is integral over $\partial^*(\mathbb{C}[\partial\mathsf{X}])$.*
(4) *$\mathbb{C}[\mathsf{X}]$ is a finitely generated $\partial^*(\mathbb{C}[\partial\mathsf{X}])$-module.*

Proof. As mentioned earlier, conditions (3) and (4) are equivalent by Proposition 5.1 and Corollary 5.2 of [1].

If $\dim(\mathsf{X}) > \dim(\partial\mathsf{X})$ then $\mathbb{C}[\mathsf{X}]$ is transcendental over $\partial^*(\mathbb{C}[\partial\mathsf{X}])$ and so condition (3) cannot hold. The inequality $\dim(\mathsf{X}) > \dim(\partial\mathsf{X})$ also implies that X contains a curve C to which Theorem 6 applies yielding a closed essential surface detected by X. Hence, conditions (2) and (3) both imply that $\dim(\mathsf{X}) = \dim(\partial\mathsf{X})$.

We have established that each of the four conditions implies that $\dim(\mathsf{X}) = \dim(\partial\mathsf{X})$. By Proposition 2.4 of [6], $\dim(\mathsf{X}) \geq 1$. Also $\dim(\partial\mathsf{X}) \leq 1$, otherwise, for any slope α on ∂N, there is a curve in X to which Theorem 6 can be applied to give an essential surface in N with boundary slope α. This would contradict Hatcher's theorem [15] that N has only finitely many boundary slopes. So if any condition (1) through (4) holds, then $\dim(\mathsf{X}) = \dim(\partial\mathsf{X}) = 1$.

The theorem now follows from Theorems 5, 6, and Corollary 2.8. □

For $\gamma \in \Gamma$, let $\mathbb{C}[I_\gamma]$ denote the $\mathbb{C}$-subalgebra of $\mathbb{C}[\mathsf{X}]$ generated by I_γ. Consider the regular map $I_\gamma\colon \mathsf{X} \to \mathbb{C}$ and the induced map $I_\gamma^*\colon \mathbb{C}[x] \to \mathbb{C}[\mathsf{X}]$. If I_γ is non-constant on X then I_γ^* is injective and so $\mathbb{C}[I_\gamma]$ is naturally isomorphic to $\mathbb{C}[x]$. Moreover, $\mathbb{Z}[x]$ is naturally isomorphic to the $\mathbb{Z}$-submodule $\mathbb{Z}[I_\gamma] \subseteq T(\mathsf{X})$ generated by all finite powers of I_γ. For instance, if μ is a slope which is not detected by X, since $\dim(\mathsf{X}) \geq 1$, Theorem 6 implies that I_μ is non-constant on X.

Theorem 8. *Suppose N is a knot manifold, X is an irreducible algebraic subset of $\mathsf{X}(N)$, and $\mu \in \Gamma$ is a slope. The following are equivalent.*

(1) *X does not detect a closed essential surface and μ is not detected by X.*
(2) *$\mathbb{C}[\mathsf{X}]$ is integral over $\mathbb{C}[I_\mu]$.*
(3) *As a $\mathbb{C}[I_\mu]$-module, $\mathbb{C}[\mathsf{X}]$ is a finitely generated and free.*

Proof. As with conditions (3) and (4) of the previous theorem, we know that (3) implies (2) and that (2) implies that $\mathbb{C}[\mathsf{X}]$ is a finitely generated $\mathbb{C}[I_\mu]$-module. $\mathbb{C}[\mathsf{X}]$

is an integral domain so, as a $\mathbb{C}[I_\mu]$-module, $\mathbb{C}[\mathsf{X}]$ is torsion free. $\mathbb{C}[I_\mu]$ is a PID so $\mathbb{C}[X]$ is a free $\mathbb{C}[I_\mu]$-module.

It remains to show that (1) and (2) are equivalent.

Assume first that (2) holds. Theorem 7 shows that X does not detect a closed essential surface and $\dim(\mathsf{X}) = 1$. Hence I_μ is not constant. We have the regular map $I_\mu\colon \mathsf{X} \to \mathbb{C}$ and $\overline{I_\mu(\mathsf{X})} = \mathbb{C}$. Since $\mathbb{C}[\mathsf{X}]$ is integral over $\mathbb{C}[I_\mu]$, Theorem 5 shows that $I_\mu\colon \mathsf{X} \to \mathbb{C}$ does not have a hole. By Theorem 6, μ is not strongly detected by X. Since X does not detect a closed essential surface, μ cannot be weakly detected by X.

Now suppose that (1) holds. As in the proof of Theorem 7, we have that $\dim(\mathsf{X}) = 1$. Consider the regular map $I_\mu\colon \mathsf{X} \to \mathbb{C}$. Either $\overline{I_\mu(\mathsf{X})} = \mathbb{C}$ or I_μ is constant on X. But since $\dim(\mathsf{X}) = 1$, Theorem 6 implies that if I_μ is constant on X then μ is strongly detected by X or X detects a closed essential surface. So we must have $\overline{I_\mu(\mathsf{X})} = \mathbb{C}$.

By Theorem 5, it suffices to show that the regular map I_μ does not have a hole. Again, we appeal to Theorem 6 and notice that if I_μ has a hole then either μ is strongly detected or X detects a closed essential surface. □

Definition 3.6. For an irreducible algebraic subset $\mathsf{X} \subseteq \mathsf{X}(N)$ and a slope μ, define the $\mathbb{C}$**-rank of** X **at** μ to be the rank of $\mathbb{C}[\mathsf{X}]$ as a $\mathbb{C}[I_\mu]$-module. We will denote this rank as $\mathrm{Tr}^{\mathbb{C}}_{\mathsf{X}}(\mu)$.

Remarks 2.

1. (1) $\mathrm{Tr}^{\mathbb{C}}_{\mathsf{X}}$ is a function from the set of slopes of N to $\mathbb{Z}^+ \cup \{\infty\}$. By Theorem 8, X detects a closed essential surface if and only if $\mathrm{Tr}^{\mathbb{C}}_{\mathsf{X}}(\mu) = \infty$ for every slope μ. If X does not detect a closed essential surface, $\mathrm{Tr}^{\mathbb{C}}_{\mathsf{X}}(\mu) = \infty$ if and only if μ is detected by X.
2. (2) By Lemma 1.4.4 of [9], there are only finitely many boundary slopes strongly detected by X, so we can always choose μ so that it is not such a slope to decide whether or not X detects a closed essential surface.

For X an irreducible algebraic component of $\mathsf{X}(N)$ we define X° to be the union of the irreducible components X' of $\mathsf{X}(N)$ such that

1. (1) X' detects a closed essential surface if and only if X does and
2. (2) for every slope μ, X' detects μ if and only X detects μ.

Proposition 3.7. *If* X *is an irreducible algebraic component of* $\mathsf{X}(N)$ *then* X° *is defined over* $\mathbb{Q}$.

Proof. By a theorem of Weil [16, Ch. III, Thm. 7], it suffices to prove that X° is invariant under the action of $\mathrm{Aut}(\mathbb{C})$ on $\mathsf{X}(N)$ (see [6, Prop. 2.3] and [3, Section 5]).

First, we assume that X does not detect a closed essential surface. Theorem 8 implies that X° is the union of the irreducible algebraic components X' of $\mathsf{X}(N)$ such that, for every slope μ, $\mathbb{C}[\mathsf{X}']$ is integral over $\mathbb{C}[I_\mu]$ if and only if $\mathbb{C}[\mathsf{X}]$ is integral over $\mathbb{C}[I_\mu]$.

Let $\phi \in \mathrm{Aut}(\mathbb{C})$, X' an irreducible component of X°, and μ a slope. Define $\mathsf{X}'' = \phi(\mathsf{X}')$. The set $\phi(\mathsf{X}')$ is an irreducible algebraic component of $\mathsf{X}(N)$. To prove the proposition we need only show that $\mathbb{C}[\mathsf{X}'']$ is integral over $\mathbb{C}[I_\mu]$ if and only if $\mathbb{C}[\mathsf{X}']$ is integral over $\mathbb{C}[I_\mu]$.

The automorphism ϕ determines an automorphism on any complex polynomial ring by acting on the coefficients. This automorphism descends to an isomorphism $\phi\colon \mathbb{C}[\mathsf{X}'] \to \mathbb{C}[\mathsf{X}'']$. Since $\phi\colon \mathbb{C} \to \mathbb{C}$ restricts to the identity on $\mathbb{Q}$ and I_μ is represented by a polynomial with integer coefficients we have $\phi(I_\mu) = I_\mu$. Therefore if $p \in \mathbb{C}[I_\mu]$ then $\phi(p) \in \mathbb{C}[I_\mu]$. So if $f \in \mathbb{C}[\mathsf{X}']$ is integral over $\mathbb{C}[I_\mu]$, we can apply ϕ to an integral dependence relation for f over $\mathbb{C}[I_\mu]$ to obtain one for $\phi(f)$ over $\mathbb{C}[I_\mu]$. To prove the converse, we apply ϕ^{-1} to any integral dependence relation for $\mathbb{C}[\mathsf{X}'']$ over $\mathbb{C}[I_\mu]$.

The case when X does detect a closed essential surface follows from the above argument along with Lemma 5.3 of [3]. □

For an algebraic set $\mathsf{X} \subseteq \mathsf{X}(N)$, let $T_{\mathbb{Q}}(\mathsf{X})$ denote the smallest $\mathbb{Q}$-algebra that contains $T(\mathsf{X})$.

Theorem 9. *Suppose N is a knot manifold, X is an irreducible algebraic subset of $\mathsf{X}(N)$, and $\mu \in \Gamma$ is a slope. The following are equivalent.*

1. *X does not detect a closed essential surface and μ is not detected by X.*
2. *$T_{\mathbb{Q}}(\mathsf{X})$ is integral over $\mathbb{Q}[I_\mu]$.*
3. *As a $\mathbb{Q}[I_\mu]$-module, $T_{\mathbb{Q}}(\mathsf{X})$ is a finitely generated and free.*

Proof. The first paragraph of the proof of Theorem 8 shows that (2) and (3) are equivalent. Theorem 8 shows that (2) implies (1).

It remains to show that (1) implies (2). Suppose then that X does not detect a closed essential surface and μ is not detected by X. $T_{\mathbb{Q}}(\mathsf{X})$ is generated by $T(\mathsf{X})$, so it suffices to show that elements of $T(\mathsf{X})$ are integral over $\mathbb{Q}[I_\mu]$.

Let $\{\mathsf{X}_j\}_1^n$ be the set of irreducible algebraic components of X° and take $f \in T(\mathsf{X})$. The map $\mathbb{C}[\mathsf{X}^\circ] \to \mathbb{C}[\mathsf{X}]$ induced by inclusion is given by restriction and is surjective. Take $F \in \mathbb{C}[\mathsf{X}^\circ]$ such that $F|_{\mathsf{X}} = f$. Let $\pi\colon \mathsf{X}^\circ \to \mathbb{C}^2$ be the map (F, I_μ). Let (x, y) be the coordinates on $\mathrm{Im}(\pi)$ determined by $x\pi = F$ and $y\pi = I_\mu$.

Fix $j \in \{1, \ldots, n\}$. By Theorem 7, $I_\mu\colon \mathsf{X}_j \to \mathbb{C}$ is non-constant and $\overline{\pi(\mathsf{X}_j)}$ is an irreducible plane curve. Let $P_j \in \mathbb{C}[x, y]$, be an irreducible polynomial which defines $\overline{\pi(\mathsf{X}_j)}$. X° is defined over $\mathbb{Q}$ (Proposition 3.7) and π is given by polynomials with $\mathbb{Z}$-coefficients, so we may assume that

$$P \;=\; \prod_{j=1}^{n} P_j \;\in \mathbb{Z}[x, y].$$

For each j, Theorem 8 gives that $F|_{\mathsf{X}_j}$ is integral over $\mathbb{C}[I_\mu]$. This means that there is a polynomial in $\mathbb{C}[x, y]$ which is zero on $\overline{\pi(\mathsf{X}_j)}$ and monic as a polynomial in $(\mathbb{C}[x])[y]$. Select one such polynomial p_j with minimal degree. Then p_j is irreducible and must differ from P_j only by multiplication by some $\alpha_j \in \mathbb{C}$. We have

$$P \;=\; \prod P_j \;=\; \left(\prod \alpha_j\right) \cdot \left(\prod p_j\right).$$

Hence $\prod \alpha_j \in \mathbb{Z}$ and $(\prod \alpha_j^{-1}) \cdot P$ gives an integral dependence relation for f over $\mathbb{Q}[I_\mu]$.

□

Definition 3.8. For an irreducible algebraic subset $\mathsf{X} \subseteq \mathsf{X}(N)$ and a slope μ, define the **$\mathbb{Q}$-rank of X at** μ to be the rank of $T_{\mathbb{Q}}(\mathsf{X})$ as a $\mathbb{Q}[I_\mu]$-module. We will denote this rank as $\mathrm{Tr}_{\mathsf{X}}^{\mathbb{Q}}(\mu)$.

It will require more work to show that there are related theorems which work over $\mathbb{Z}$. We begin by establishing some lemmas concerning valuations.

4. Valuations

Lemma 4.1. *Suppose $\nu\colon F^\times \to \Lambda$ is a valuation on the field F. Assume that*

$$cx^k \;=\; \sum_{i=0}^{k-1} a_i x^i$$

for some $c, x, a_i \in F$ and $k \in \mathbb{Z}^+$ such that $c \neq 0$, $\nu(c) = 0$, and $\nu(a_i) \geq 0$ for every i. Then $\nu(x) \geq 0$.

Proof. If every a_i is zero then $x = 0$ and so $\nu(x) = \infty$, so we may assume that at least one of the a_i's is non-zero. Let j be such that $\nu(a_j) + j\nu(x)$ is minimal in the set $\{\nu(a_i) + i\nu(x) \,|\, 0 \leq i < k\}$. We have

$$k\nu(x) \;=\; \nu(c) + k\nu(x) \;\geq\; \nu(a_j) + j\nu(x)$$

so

$$(k-j)\nu(x) \;\geq\; \nu(a_j) \;\geq\; 0.$$

Hence $\nu(x) \geq 0$. □

Lemma 4.2. *If ν is a non-trivial valuation on $\mathbb{Q}$ then ν is a p-adic valuation.*

Proof. Let R be the valuation ring corresponding to ν and let $\mathfrak{m}$ be the maximal ideal in R. We know that $\mathbb{Z} \subset R$. Note that $\mathfrak{m} \cap \mathbb{Z}$ is prime (maximal) in $\mathbb{Z}$ so ν is p-adic. □

The following lemma is essentially Lemma 2.2 of [7].

Lemma 4.3. *Suppose $p \in \mathbb{Z}$ is a prime and $h \in \mathbb{Z}[t]$ is irreducible over $\mathbb{Z}$ and of the form*

$$h(t) \;=\; c_0 + c_1 t + \cdots + c_{k-1} t^{k-1} + c_k p^r t^k$$

where $(c_k, p) = 1$ and $c_k \neq \pm 1$. Let $b \in \mathbb{C}$ be any root of h. Then there is a valuation ν on $\mathbb{Q}(b)$ with $\nu(p) = 0$ and $\nu(b) < 0$.

Proof. Since $(c_k, p) = 1$ and $h \in \mathbb{Z}[t]$ we know that r is an integer and at least zero. Now, for $i \in \{0, \ldots, k-1\}$, define $a_i = c_i p^{k-1-i}$. We have $p^{(k-1)r} c_i = a_i (p^r)^i$, so

$$p^{(k-1)r} h(t) \;=\; a_0 + a_1(p^r t) + \cdots + a_{k-1}(p^r t)^{k-1} + c_k (p^r t)^k$$

Set $d = p^r b$ and

$$h^*(t) \;=\; a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + c_k t^k.$$

Then $h^* \in \mathbb{Z}[t]$ and $h^*(d) = 0$. We have

$$c_k d^k \;=\; \sum_{i=0}^{k-1} a_i d^i.$$

Let ν_p be an extension of the p-adic valuation to $\mathbb{Q}(b)$. Since $(c_k, p) = 1$, $\nu_p(c_k) = 0$. Also, for every i, $a_i \in \mathbb{Z}$ so $\nu(a_i) \geq 0$. Hence, Lemma 4.1 implies that $\nu_p(d) \geq 0$.

Since h is irreducible over $\mathbb{Z}$ and $\mathbb{Q}(b) = \mathbb{Q}(d)$, it follows that h^* is irreducible over $\mathbb{Z}$. Hence, neither b nor d is integral over $\mathbb{Z}$. By Corollary 5.2 of [1], we have a valuation ν on $\mathbb{Q}(b)$ with $\nu(d) < 0$. The restriction of ν to $\mathbb{Q}$ cannot be the p-adic valuation and (again using Lemma 4.1) cannot be trivial. By Lemma 4.2, $\nu|_{\mathbb{Q}} = \nu_q$ for some prime $q \in \mathbb{Z}$ which is different from p. Therefore, $\nu(p) = 0$. Also

$$\nu(b) \;=\; \nu(dp^{-r}) \;=\; \nu(d) - r\nu(p) \;=\; \nu(d) < 0.$$

□

5. $T(\mathsf{X})$ as a $\mathbb{Z}[I_\mu]$-module

Throughout this section, N is a knot manifold and Γ is its fundamental group. We use some terminology inspired by [18].

Definition 5.1. A representation $\rho\colon\, \Gamma \to \mathrm{SL}_2\mathbb{C}$ is **algebraic non-integral** if the image of ρ is in $\mathrm{SL}_2 F$, where F is a number field and $\chi_\rho(\gamma)$ is not an algebraic integer for some $\gamma \in \Gamma$. If ρ is an algebraic non-integral representation we abbreviate this by saying that ρ is an **ANI-representation**.

Similar to Theorem 6, the following result again follows from the work of Culler and Shalen. See Lemma 2 of [18].

Theorem 10. *For every ANI-representation ρ of Γ there is an associated non-empty collection $\{\Sigma_i\}$ of essential surfaces in N. Suppose that $\mu \in \Gamma$ is a slope which is not a boundary slope. The set $\{\Sigma_i\}$ contains a closed surface if and only if $\chi_\rho(\mu)$ is an algebraic integer.*

Definitions 5.2.

(1) Suppose that ρ is an ANI-representation of Γ and $\{\Sigma_i\}$ is the collection of essential surfaces in N given by ρ and Theorem 10. A surface Σ is **associated to** ρ if $\Sigma \in \{\Sigma_i\}$.

(2) Suppose that X is an algebraic subset of $\mathsf{X}(N)$. A surface Σ is **ANI-detected by** X if there is an ANI-representation ρ of Γ whose character lies on X and Σ is associated to ρ.

Proposition 5.3. *If X is an irreducible algebraic subset of $\mathsf{X}(N)$ and N contains a closed essential surface which is ANI-detected by X, then N contains a closed essential surface detected by X.*

Proof. Assume that X does not detect a closed essential surface and suppose for a contradiction that there is a closed essential surface in N associated to an ANI-representation $\rho\colon\, \Gamma \to \mathrm{SL}_2 F$ with $\chi_\rho \in \mathsf{X}$. Take $\gamma \in \Gamma$ with $\chi_\rho(\gamma)$ non-integral and choose $\mu \in \Gamma$ peripheral, primitive, and not a boundary slope.

By Theorems 2 and 10, there is a valuation ν on F with $\nu(I_\gamma(\chi_\rho)) < 0$ and $\nu(I_\mu(\chi_\rho)) \geq 0$. By Theorem 9, there is a polynomial

$$f(x,y) \;=\; \alpha x^r + \sum_{j=0}^{r-1} q_j(y) x^j$$

with $\alpha \in \mathbb{Z} - \{0\}$, $q_j \in \mathbb{Z}[y]$, and $f(I_\gamma, I_\mu) = 0$ on X. Since $\nu(I_\mu(\chi_\rho)) \geq 0$, we know that $\nu(q_j(I_\mu(\chi_\rho))) \geq 0$. But then, Lemma 4.1 implies that $\nu(I_\gamma(\chi_\rho)) \geq 0$, a contradiction. □

Example 1. *In [8] Cooper and Long ask whether there is a one-cusped manifold whose character variety detects a closed essential surface. This was answered by Tillmann in [21] where he made a calculation which shows that the* *character variety of the Kinoshita-Terasaka knot (11n42) detects a closed essential surface. Together with data collected by Goodman, Heard, and Hodgson* [12], *http://www.ms.unimelb.edu.au/~snap/, Proposition 5.3 gives many more examples.*

Suppose that N is a one-cusped hyperbolic 3-manifold and let X_0 be an algebraic component of $\mathsf{X}(N)$ which contains the character of a discrete faithful representation ρ_0. By Theorem 10 and Proposition 5.3, if ρ_0 is an ANI-representation then N contains a closed essential surface detected by $\mathsf{X}(N)$.

For all manifolds in the Callahan-Hildebrand-Weeks census of cusped hyperbolic manifolds with up to 7 *tetrahedra* [4]*, and for all complements of hyperbolic knots and links up to 12 crossings, Goodman, Heard, and Hodgson use the computer program* `Snap` *to attempt to (among other things) decide whether or not the holonomy representation is an ANI-representation. They find 252 such manifolds, 21 of which are complements of knots in S^3 (this is almost certainly not a complete list). The knots are 9a30, 9a31, 10a89, 10a96, 10a103, 11n97, 12n156, 12n245, 12n246, 12n260, 12n494, 12n508, 12n518, 12n600, 12n602, 12n604, 12n605, 12n694, 12n888, 12a1205, and 12a1288.*

Roughly, the following theorem is proven by applying Theorem 9 and arguing as in [7].

Theorem 11. *Suppose N is a knot manifold, X is an irreducible algebraic subset of $\mathsf{X}(N)$, and $\mu \in \Gamma$ is a slope. The following are equivalent.*

1. *X does not detect a closed essential surface and μ is not detected by X.*
2. *$T(\mathsf{X})$ is integral over $\mathbb{Z}[I_\mu]$.*
3. *$T(\mathsf{X})$ is a finitely generated $\mathbb{Z}[I_\mu]$-module.*

Proof. As with Theorem 9, we need only argue that (1) implies (2). Suppose that (1) is true and take $\gamma \in \Gamma$. Our goal is to show that I_γ is integral over $\mathbb{Z}[I_\mu]$, so we may assume that I_γ is not everywhere zero on X. By Theorem 7, $\dim(\mathsf{X}) = 1$ and by Theorem 8, I_μ must be non-constant on X.

By Theorem 9, there is a polynomial $f(y, z) \in \mathbb{Z}[y, z]$ of the form

$$f(y,z) \;=\; \alpha y^r + \sum_{j=0}^{r-1} q_j(z) y^j$$

with $q_j \in \mathbb{Z}[z]$ such that $\alpha \neq 0$, $f(I_\gamma, I_\mu)$ is the zero function on X, and the greatest common divisor of the coefficients of f is one. Choose f among all such polynomials to have smallest possible total degree. Then, since I_μ and I_γ are not everywhere zero on X, neither y nor z divides f.

Take $k \in \mathbb{N}$ large enough such that $f(y, xy^{-k})$ has no two terms with the same degree in y. Take $l \in \mathbb{Z}$ as small as possible such that

$$g(x,y) \;=\; y^l f(y, xy^{-k})$$

is a polynomial in $\mathbb{Z}[x, y]$. Then $g(x, y)$ is of the form

$$g(x,y) \;=\; \alpha y^s + \sum_{j=0}^{s-1} a_j x^{t_j} y^j$$

where the t_j's are distinct integers, the greatest common divisor of the a_j's is one, $g(I_\mu I_\gamma^k, I_\gamma) = 0$ on X, and neither x nor y divides g.

We know that I_μ is non-constant on X so there must be a point $\hat{x} \in \widetilde{\mathsf{X}}$ with $I_\mu(\hat{x}) = 0$. If $I_\mu I_\gamma^k$ is constant on X then $I_\mu = K I_\gamma^{-k}$ for some $K \in \mathbb{C}$. Hence, $\hat{x}$ is an ideal point of X. By Theorem 2.2.1 and Proposition 2.3.1 of [10], either X detects a closed essential surface or an essential surface with boundary slope μ. Both are contradictions so $I_\mu I_\gamma^k$ must be non-constant on X. Therefore, we may choose a prime p such that $\gcd(\alpha, p) = 1$ and for every root b of $g(p, y)$ there is a character $\chi_\rho \in \mathsf{X}$ with $\chi_\rho(\mu) \cdot \chi_\rho(\gamma)^k = p$ and $\chi_\rho(\gamma) = b$.

Take
$$g(p, y) \;=\; \eta \cdot \prod_{j=0}^{u} h_j(y)$$
a complete factorization of $g(p, y)$ over $\mathbb{Z}$. So $\eta \in \mathbb{Z}$ and each h_j is irreducible over $\mathbb{Z}$. We claim that α divides η. The leading coefficient of $g(p, y)$ is α. If we denote the the leading coefficient of h_j as c_j, we have $\alpha = \eta c_0 \cdots c_u$. If α doesn't divide η then there is some c_i with $\gcd(\alpha, c_i) > 1$. Write $c_i = p^t d$ where $\gcd(p, d) = 1$. Since $\gcd(\alpha, p) = 1$ we must have $\gcd(\alpha, d) > 1$, in particular $d \neq \pm 1$. Let b be a root of h_i. By our choice of p, there is $\chi_\rho \in \mathsf{X}$ which corresponds to the solution $g(p, b)$. Using Hilbert's Nullstellensatz, we may assume that the entries of the matrices in $\rho(\pi_1 N)$ lie in a number field F. By Lemma 4.3, we have a valuation ν on $\mathbb{Q}(b)$ with $\nu(p) = 0$ and $\nu(b) < 0$. Take an extension of ν to F. We have $\nu(I_\gamma(\chi_\rho)) < 0$ and
$$0 \;=\; \nu\left(I_\mu(\chi_\rho) I_\gamma(\chi_\rho)^k\right) \;=\; \nu\left(I_\mu(\chi_\rho)\right) + k \cdot \nu\left(I_\gamma(\chi_\rho)\right).$$
Since $k \geq 0$ we must have $\nu(I_\mu(\chi_\rho)) \geq 0$. Hence, N contains a closed essential surface which is ANI-detected by X. Proposition 5.3 shows that this is a contradiction, so we must have that α divides η.

The constant η divides every coefficient of $g(p, y)$ and α divides η, so α divides every coefficient of $g(p, y)$. But the coefficient for z^j is $a_j p^{t_j}$ and $\gcd(\alpha, p) = 1$. Hence α divides each a_j. The only integers that divide every a_j are ± 1. Therefore $f(I_\gamma, I_\mu)$ is an integral dependence relation for I_γ over $\mathbb{Z}[I_\mu]$. □

Definition 5.4. For an irreducible algebraic subset $\mathsf{X} \subseteq \mathsf{X}(N)$ and a slope μ, define the **$\mathbb{Z}$-rank of X at** μ to be the rank of $T(\mathsf{X})$ as a $\mathbb{Z}[I_\mu]$-module. We will denote this rank as $\mathrm{Tr}_{\mathsf{X}}^{\mathbb{Z}}(\mu)$.

Corollary 5.5. *Suppose N is a knot manifold, X is an irreducible algebraic subset of $\mathsf{X}(N)$ which is defined over $\mathbb{Q}$. The following are equivalent.*

1. *X does not detect a closed essential surface and μ is not detected by X.*
2. *Every $f \in T(\mathsf{X})$ satisfies a monic irreducible polynomial with coefficients in $\mathbb{Z}[I_\mu]$.*

Proof. By Theorem 8, need only argue that (1) implies (2). Assume that X does not detect a closed essential surface and let f be a non-zero element of $T(\mathsf{X})$. By Theorem 7, X is a curve. Since X does not detect a closed essential surface, I_μ is non-constant on X. Let V be the Zariski closure of the image of X under the map (f, I_μ). V must be an irreducible affine curve defined over $\mathbb{Q}$. Let $p \in \mathbb{Q}[x, y]$ be a defining equation for V. By multiplying by an integer we may assume $p \in \mathbb{Z}[x, y]$. The proof of Theorem 11, shows that p is an integral dependence relation of f over $\mathbb{Z}[I_\mu]$. Since V is irreducible, p is irreducible. □

Definition 5.6. A 3-manifold is called **small** if it does not contain a closed essential surface.

Corollary 5.7. *Let X be an irreducible algebraic subset of $\mathsf{X}(N)$, where N is a small knot manifold. A slope μ is detected by X if and only if $T(\mathsf{X})$ is a finitely generated $\mathbb{Z}[I_\mu]$-module.*

6. $\mathrm{PSL}_2\mathbb{C}$-character varieties

The theorems from Sections 3 and 5, also apply in the $\mathrm{PSL}_2\mathbb{C}$ setting after making appropriate definitions and using virtually identical arguments. See [2] for details in what follows.

As usual, let N be a compact irreducible orientable 3-manifold with torus boundary and let $\Gamma = \pi_1 N$. Denote the set of $\mathrm{PSL}_2\mathbb{C}$-representations of Γ as $\overline{\mathsf{R}}(N)$. $\mathrm{PSL}_2\mathbb{C}$ acts on $\overline{\mathsf{R}}(N)$ by conjugation. Let $\bar{t}\colon\ \overline{\mathsf{R}}(N) \to \mathsf{Y}(N)$ denote the corresponding algebro-geometric quotient. $\overline{\mathsf{R}}(N)$ and $\mathsf{Y}(N)$ are affine algebraic sets and the map $\bar{t}$ is a surjective regular map. $\mathsf{Y}(N)$ is called the $\mathrm{PSL}_2\mathbb{C}$-character variety for N and a point in $\mathsf{Y}(N)$ is called a $\mathrm{PSL}_2\mathbb{C}$-character. As with the $\mathrm{SL}_2\mathbb{C}$-character variety, every irreducible algebraic component of $\mathsf{Y}(N)$ has dimension at least one [6].

We would like to relate $\mathsf{Y}(N)$ to $\mathrm{SL}_2\mathbb{C}$-character varieties and define $T(\mathsf{Y})$ where Y is an irreducible algebraic subset of $\mathsf{Y}(N)$. First we claim that there is a well-defined equivalence class of $\mathbb{Z}_2$-central extensions

$$1 \to \mathbb{Z}_2 \to \hat{\Gamma} \to \Gamma \to 1 \tag{2}$$

and a finite regular map from the $\mathrm{SL}_2\mathbb{C}$-character variety $\mathsf{X}(\hat{\Gamma})$ of $\hat{\Gamma}$ to $\mathsf{Y}(N)$ which contains Y in its image.

First note that if $\bar{\rho}_1, \bar{\rho}_2 \in \overline{\mathsf{R}}(N)$ have the same image under $\bar{t}$ then the representations are either both reducible or are both irreducible. Hence we may refer to the points of Y as being either reducible or irreducible.

Suppose that every element of Y is reducible and choose $\bar{\chi} \in \mathsf{Y}$ smooth in $\mathsf{Y}(N)$. There is a diagonal representation $\bar{\rho} \in \bar{t}^{-1}(\bar{\chi})$. This representation determines an extension (2) and a lift $\hat{\rho}\colon\ \hat{\Gamma} \to \mathrm{SL}_2\mathbb{C}$ of $\bar{\rho}$. Moreover, the isomorphism class of $\hat{\Gamma}$ is independent of our choices of $\bar{\chi}$ and $\bar{\rho}$. The natural epimorphisms $\mathrm{SL}_2\mathbb{C} \to \mathrm{PSL}_2\mathbb{C}$ and $\hat{\Gamma} \to \Gamma$ together induce a regular map $\phi\colon\ \mathsf{X}(\hat{\Gamma}) \to \mathsf{Y}(N)$. In fact, the image of ϕ is the quotient of $\mathsf{X}(\hat{\Gamma})$ under the natural action of $H^1(\hat{\Gamma};\mathbb{Z}_2)$. Moreover, $\mathsf{Y} \subseteq \mathrm{Im}(\phi)$.

Otherwise Y contains an irreducible character which is a smooth point of $\mathsf{Y}(N)$. Let $\bar{\chi}$ be such a point and $\bar{\rho} \in \bar{t}^{-1}$. Exactly as before, we obtain an extension (2) and a lift $\hat{\rho}$ of $\bar{\rho}$ to $\hat{\Gamma}$. Again, the isomorphism class of $\hat{\Gamma}$ is independent of our choices, we obtain a finite regular map $\phi\colon\ \mathsf{X}(\hat{\Gamma}) \to \mathsf{Y}(N)$, the image of ϕ is the quotient of $\mathsf{X}(\hat{\Gamma})$ by $H^1(\hat{\Gamma};\mathbb{Z}_2)$, and $\mathsf{Y} \subseteq \mathrm{Im}(\phi)$.

In either case, the induced map $\phi^*\colon\ \mathbb{C}[\mathrm{Im}(\phi)] \to \mathbb{C}[\mathsf{X}(\hat{\Gamma})]$ is injective. We define the trace ring $T(\mathrm{Im}(\phi))$ to be

$$T(\mathrm{Im}(\phi)) \;=\; (\phi^*)^{-1}\left(\mathrm{Im}(\phi^*) \cap T\big(\mathsf{X}(\hat{\Gamma})\big)\right).$$

The inclusion $\mathsf{Y} \to \mathrm{Im}(\phi)$ induces an epimorphism $\mathbb{C}[\mathrm{Im}(\phi)] \to \mathbb{C}[\mathsf{Y}]$. Define $T(\mathsf{Y})$ to be the image of $T(\mathrm{Im}(\phi))$ under this epimorphism.

Note that if $\gamma \in \Gamma$ and $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$ are the preimages of γ then $I^2_{\hat{\gamma}_1} = I^2_{\hat{\gamma}_2}$. It follows that there is a well-defined squared trace function $I^2_\gamma \in T(\mathsf{Y})$.

Now, in Sections 3 and 5, we can use the $\mathrm{PSL}_2\mathbb{C}$-definitions rather than $\mathrm{SL}_2\mathbb{C}$-definitions and use the squared trace function I_μ^2 whenever I_μ is not in the $\mathrm{PSL}_2\mathbb{C}$-trace ring, to obtain corresponding facts in the $\mathrm{PSL}_2\mathbb{C}$-setting.

Remark 3. The equivalence classes of extensions (2) are in bijective correspondence with the elements of $H^2(\Gamma; \mathbb{Z}_2)$.

7. Ranks and examples

Throughout this section we assume that N is a knot manifold and $\Gamma = \pi_1 N$. This section begins an investigation into the meaning of the ranks defined in the previous section. For an irreducible algebraic set $\mathsf{X} \subseteq \mathsf{X}(N)$, observe that

$$\mathrm{Rk}_{\mathsf{X}}^{\mathbb{C}} \;\leq\; \mathrm{Rk}_{\mathsf{X}}^{\mathbb{Q}} \;\leq\; \mathrm{Rk}_{\mathsf{X}}^{\mathbb{Z}}.$$

Among other things, we will see examples which show that these inequalities need not be equalities.

If R is a ring, M is an R-module, and $S \subset M$ then we write $\langle S \rangle_R$ to denote the R-module generated by S.

7.1. Abelian curves.

Suppose that $H_1(N;\mathbb{Z}) \cong \mathbb{Z}$. There is a curve $\mathsf{X}_A \subseteq \mathsf{X}(N)$ consisting of all characters of representations which factor through $H_1(N;\mathbb{Z})$. Let $\mu \in \Gamma$ be a peripheral element whose image in $H_1(N;\mathbb{Z})$ generates $H_1(N;\mathbb{Z})$.

Every character in X_A is the character of a diagonal representation and the image subgroup in $\mathrm{SL}_2\mathbb{C}$ is generated by the image of μ. It follows that every trace function $I_\mu \in T(\mathsf{X}_A)$ is represented by a polynomial in $\mathbb{Z}[I_\mu]$. Using the single coordinate I_μ on X_A, we have $\mathsf{X}_A = \mathbb{C}$. In particular, we have the following well-known proposition.

Proposition 7.1. *Suppose that N is a knot manifold and $H_1(N;\mathbb{Z}) \cong \mathbb{Z}$. Let $\mathsf{X}_A \subseteq \mathsf{X}(N)$ be the curve consisting of all abelian representations of $\pi_1 N$. Then*

1. *X_A has exactly one ideal point. This ideal point detects the unique boundary slope which is trivial in $H_1(N;\mathbb{Z})$.*
2. *$T(\mathsf{X}_A) = \mathbb{Z}[I_\mu]$.*

Hence, $T(\mathsf{X}_A)$ is a free $\mathbb{Z}[I_\mu]$-module with rank one. Together with Theorem 8, this shows that X_A does not detect a closed essential surface.

7.2. Norm curves.

Recall the following definition from [3].

Definition 7.2. An algebraic curve component X of $\mathsf{X}(N)$ is called a **norm curve component** if $I_\gamma \colon \mathsf{X} \to \mathbb{C}$ is non-constant for every non-trivial peripheral element of Γ.

For example, if N is hyperbolic then there is a component X_0 of $\mathsf{X}(N)$ which contains the character of a discrete faithful representation. By Proposition 3.1.1 of [10], X_0 is a norm curve component.

Proposition 7.3. *Suppose that N is a knot manifold and X is an irreducible algebraic subset of $\mathsf{X}(N)$. Let μ be a slope. If X is a norm curve component then $\mathrm{Rk}_{\mathsf{X}}^{\mathbb{C}}(\mu) \geq 2$.*

Proof. Suppose that X is a norm component and that $\mathrm{Rk}_X^{\mathbb{C}}(\mu) < 2$. By Theorem 7, $\dim(\mathsf{X}) = 1$. Hence, $\mathrm{Rk}_{\mathsf{X}}^{\mathbb{C}}(\mu) = 1$ and we have $f \in \mathbb{C}[X]$ which generates $\mathbb{C}[X]$ as a $\mathbb{C}[I_\mu]$-module. We may assume f is the constant function 1 or of the form I_γ for some $\gamma \in \Gamma$.

We claim that we may assume that $f = 1$. Otherwise, we have a polynomial $p \in \mathbb{C}[I_\mu]$ such that $I_\gamma = 1/p$. If p is constant then we can take $f = 1$. If p is not constant then take $\hat{x} \in \widetilde{\mathsf{X}}$ such that $p(\hat{x}) = 0$. Since $I_\gamma^{-1}(\hat{x}) = 0$, $I_\gamma \notin \mathcal{O}_{\widetilde{\mathsf{X}},\hat{x}}$ and $\hat{x}$ is an ideal point which either detects a closed essential surface or the slope μ (Theorem 6). This is a contradiction.

We have established that $\mathbb{C}[\mathsf{X}] = \mathbb{C}[I_\mu]$. Hence X has exactly one ideal point. On the other hand, [3][Proposition 4.5] and Theorem 6 show that X strongly detects at least two boundary slopes which is impossible. □

7.3. **Two-bridge knots.** Suppose that K is a two bridge knot. Define $N = S^3 - K$ and $\Gamma = \pi_1 N$. Let $\langle \mu, \beta \,|\, \omega\beta = \mu\omega \rangle$ be the standard presentation for Γ as given in Section 4.5 of [17]. The element $\mu \in \Gamma$ is a meridian for the knot.

Suppose that X is an algebraic component of $\mathsf{X}(N)$ which is defined over $\mathbb{Q}$. Let $x = I_\mu$ and $y = I_{\mu\beta}$ in $T(\mathsf{X})$. It is shown in [14] that N is small and a is not a boundary slope. By Corollary 5.5, we have a set of polynomials $\{p_j\}_{j=0}^n$ in $\mathbb{Z}[x]$ where X is given as the zero set in $\mathbb{C}^2$ of

$$P(x, y) \;=\; y^n - \sum_{j=0}^{n} p_j(x) y^j$$

and P is irreducible in $\mathbb{C}[x, y]$. Let $\mathcal{B} = \{y^j\}_{j=0}^{n-1} \subset T(\mathsf{X})$. Since P is zero in $T(\mathsf{X})$, $\mathcal{B}$ generates $T(\mathsf{X})$ as a $\mathbb{Z}[x]$-module, $T_{\mathbb{Q}}(\mathsf{X})$ as a $\mathbb{Q}[x]$-module, and $\mathbb{C}[\mathsf{X}]$ as a $\mathbb{C}[x]$-module. In particular,

$$\mathrm{Rk}_{\mathsf{X}}^{\mathbb{C}}(\mu) \;\le\; \mathrm{Rk}_{\mathsf{X}}^{\mathbb{Q}}(\mu) \;\le \mathrm{Rk}_{\mathsf{X}}^{\mathbb{Z}}(\mu) \;\le\; n. \tag{3}$$

Proposition 7.4. *$\mathcal{B}$ is a free basis for $\mathbb{C}[\mathsf{X}]$ as a $\mathbb{C}[x]$-module, $T_{\mathbb{Q}}(\mathsf{X})$ as a $\mathbb{Q}[x]$-module, and $T(\mathsf{X})$ as a $\mathbb{Z}[x]$-module.*

Proof. First, we induct on k to show that $M_k = \langle 1, y, \ldots, y^k \rangle_{\mathbb{C}[x]}$ is free of rank $k+1$ whenever $k \le n-1$. The base case is immediate since $M_0 = \mathbb{C}[x]$.

Since $M_k = M_{k-1} + \langle y^k \rangle$, to see that $M_k = M_{k-1} \oplus \langle y^k \rangle$, it is enough to show that $M_{k-1} \cap \langle y^k \rangle = \{0\}$. Let f be an element of $M_{k-1} \cap \langle y^k \rangle$. Then we have $\{p_j\}_{j=0}^k \subset \mathbb{C}[x]$ such that

$$f \;=\; \sum_{j=0}^{k-1} p_j(x) y^j \;=\; p_k(x) y^k$$

Hence,

$$-p_k(x) y^k + \sum_{j=0}^{k-1} p_j(x) y^j \;\in\; (P) \subset \mathbb{Q}[x, y].$$

But $k < n$ and P is irreducible, so each p_j must be zero.

Therefore, $\mathcal{B}$ is a free basis for $\mathbb{C}[\mathsf{X}]$ as a $\mathbb{C}[x]$-module and using the inequality (3) we have

$$\mathrm{Rk}_{\mathsf{X}}^{\mathbb{C}}(\mu) \;=\; \mathrm{Rk}_{\mathsf{X}}^{\mathbb{Q}}(\mu) \;= \mathrm{Rk}_{\mathsf{X}}^{\mathbb{Z}}(\mu) \;=\; n.$$

$\mathbb{Q}[x]$ is a PID, so $\mathcal{B}$ is also a free basis for $T_{\mathbb{Q}}(\mathsf{X})$ as a $\mathbb{Q}[x]$-module.

To see that $\mathcal{B}$ is a free basis for $T(\mathsf{X})$ as a $\mathbb{Z}[x]$-module, it remains only to show that this module is free. To do this, we show that $\langle 1, y, \ldots, y^{k-1}\rangle_{\mathbb{Z}[x]} \cap \langle y^k\rangle_{\mathbb{Z}[x]} = \{0\}$, whenever $k \leq n-1$. If f is in this intersection, then it is in $M_{k-1} \cap \langle y^k\rangle_{\mathbb{C}[x]}$ which we have already seen is trivial. □

Example 2. *Let K be a trefoil knot. A calculation shows that $\mathsf{X}(N) = \mathsf{X}_A \cup \mathsf{X}_0$, where X_0 is an irreducible curve. The polynomial P which defines X_0 is $y-1$. Therefore*

$$\mathrm{Rk}^{\mathbb{C}}_{\mathsf{X}_0}(\mu) \;=\; \mathrm{Rk}^{\mathbb{Q}}_{\mathsf{X}_0}(\mu) \;=\; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_0}(\mu) \;=\; 1.$$

Proposition 7.3 reaffirms the well-known fact that X_0 is not a norm curve.

Example 3. *Let K be the figure-eight knot. A calculation shows that $\mathsf{X}(N) = \mathsf{X}_A \cup \mathsf{X}_0$, where X_0 is an irreducible curve. The polynomial P which defines X_0 is $y^2 + (-1-x^2)y + (-1+2x^2)$. Therefore*

$$\mathrm{Rk}^{\mathbb{C}}_{\mathsf{X}_0}(\mu) \;=\; \mathrm{Rk}^{\mathbb{Q}}_{\mathsf{X}_0}(\mu) \;=\; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_0}(\mu) \;=\; 2.$$

$H^1(N;\mathbb{Z}_2) \cong \mathbb{Z}_2$ with generator σ. $H^1(N;\mathbb{Z}_2)$ acts on X_0 by $\sigma(x,y) = (-x,y)$. It follows that $T(\mathsf{Y}_0)$ can be identified with the subring (with 1) of $T(\mathsf{X}_0)$ generated by x^2 and y. Moreover, Y_0 can be identified with the zeros of $y^2+(-1-\xi)y+(-1+2\xi)$ where $\xi = x^2$. We have

$$\mathrm{Rk}^{\mathbb{C}}_{\mathsf{Y}_0}(\mu) \;=\; \mathrm{Rk}^{\mathbb{Q}}_{\mathsf{Y}_0}(\mu) \;=\; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{Y}_0}(\mu) \;=\; 2.$$

Let $\lambda = [\beta,\mu^{-1}][\mu,\beta^{-1}]$. Then λ commutes with μ and represents the boundary slope determined by a Seifert surface. Since $\sigma(\lambda)$ is the identity, $l = I_\lambda \in T(\mathsf{Y}_0)$. If we take (l,ξ,y) as coordinates on Y_0 then the ideal determined by Y_0 is generated by the following polynomials in $\mathbb{C}[l,\xi,y]$.

$$\begin{aligned}
p_1 &= (-1-4l) + (-11+4l)y + (16-l)y^2 - 7y^3 + y^4\\
p_2 &= 2 - l - 5\xi + \xi^2\\
p_3 &= 1 - 2\xi + y + \xi y - y^2\\
p_4 &= (2l) + \xi + (5-l)y + 5y^2 + y^3.
\end{aligned}$$

Let $M = \langle 1, y, y^2, y^3\rangle_{\mathbb{C}[l]}$. The first polynomial describes the image of Y_0 under the map $\chi \mapsto (L(\chi), y(\chi))$. Hence, p_1 is irreducible and M is free of rank 4. The last polynomial shows that $\xi^i y^j \in M$ for every $i,j \in \mathbb{Z}^+$. It follows that

$$\mathrm{Rk}^{\mathbb{C}}_{\mathsf{Y}_0}(\lambda) \;=\; \mathrm{Rk}^{\mathbb{Q}}_{\mathsf{Y}_0}(\lambda) \;=\; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{Y}_0}(\lambda) \;=\; 4.$$

Let $s = I_{\mu^2\lambda}$. Then $s \in T(\mathsf{Y}_0)$. Similar calculations show that $\{1, y, y^2, y^3, \xi\}$ generates $\mathbb{C}[\mathsf{Y}_0]$ as a $\mathbb{C}[s]$-module and that

$$\mathrm{Rk}^{\mathbb{C}}_{\mathsf{Y}_0}(\mu^2\lambda) \;=\; \mathrm{Rk}^{\mathbb{Q}}_{\mathsf{Y}_0}(\mu^2\lambda) \;=\; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{Y}_0}(\mu^2\lambda) \;=\; 5.$$

7.4. **Other examples.**

Example 4. *Let N be the* `Snappea` *census manifold M_{003} from [4]. N is a finite volume hyperbolic 3-manifold with a single cusp. $\langle a, b \,|\, aba^{-2}bab^3\rangle$ is a presentation for $\Gamma = \pi_1 N$ and the elements $\mu = (b^2aba)^{-1}$ and $\lambda = (aba)^{-1}bab$ together generate a peripheral subgroup. Define*

$$m = I_\mu \qquad x = I_a \qquad y = I_b \qquad z = I_{ab}$$

Let X_0 *be an irreducible algebraic component of* $\mathsf{X}(N)$ *which contains the character of a discrete faithful representation and let* Y_0 *be the image of* X_0 *in the* $PSL_2\mathbb{C}$*-character variety. We can take* (m, x, y, z) *as coordinates on* X_0*. The ideal in* $\mathbb{C}[m, x, y, z]$ *determined by* X_0 *is generated by the following three polynomials*

$$\begin{aligned} p_1 &= z^4 - mz^2 - z^2 + 1 \\ p_2 &= -z^2 + m + y \\ p_3 &= -z^3 + mz + z + x. \end{aligned}$$

The first polynomial is irreducible and gives that $\langle 1, z, z^2, z^3\rangle_{\mathbb{C}[m]}$ *is free of rank 4. The last two polynomials show that* $x, y \in \langle 1, z, z^2, z^3\rangle_{\mathbb{C}[m]}$*. It follows that* $\mathbb{C}[\mathsf{X}_0] = \langle 1, z, z^2, z^3\rangle_{\mathbb{C}[m]}$ *and*

$$\mathrm{Rk}^{\mathbb{C}}_{\mathsf{X}_0}(\mu) \ = \ \mathrm{Rk}^{\mathbb{Q}}_{\mathsf{X}_0}(\mu) \ = \ \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_0}(\mu) \ = \ 4.$$

Let $\zeta = z^2$*. It is straightforward to check that* $m \in T(\mathsf{Y}_0)$ *and that* (m, ζ) *can be taken as coordinates on* Y_0*. As such,* Y_0 *is the zero locus of the polynomial* $\zeta^2 + (-m-1)\zeta + 1$*. It follows that*

$$\mathrm{Rk}^{\mathbb{C}}_{\mathsf{Y}_0}(\mu) \ = \ \mathrm{Rk}^{\mathbb{Q}}_{\mathsf{Y}_0}(\mu) \ = \ \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{Y}_0}(\mu) \ = \ 2.$$

We see that these ranks achieve the smallest possible value for the rank of a norm curve. In contrast to Example 3, $\mathrm{Rk}^{\mathbb{C}}_{\mathsf{Y}_0}(\mu) < \mathrm{Rk}^{\mathbb{C}}_{\mathsf{X}_0}(\mu)$*.*

Example 5. *Let* N *be the exterior of the knot* 8_{20}*. The knot* 8_{20} *is hyperbolic, so we let* X_0 *be an irreducible algebraic component of* $\mathsf{X}(N)$ *which contains a discrete faithful character. As outlined in the introduction to this paper,* Γ *is generated by a pair of elements* μ *and* γ *where* μ *is a meridian. Let* $x = I_\mu$*,* $y = I_\gamma$*, and* $z = I_{\gamma\mu}$*. The functions* x, y, z *give an embedding of* X_0 *into* $\mathbb{C}^3$*.*

Let $\mathcal{G} = \{g_j\}_1^5 \subset \mathbb{Z}[x, y, z]$ *where*

$$\begin{aligned} g_1 &= \ 1 + 5y + 7y^2 + 2y^3 - 2y^4 - y^5 - 2x^2 - 6yx^2 - 3y^2x^2 + y^4x^2 + x^4 + yx^4 \\ g_2 &= \ -x - 3yx - y^2x - y^3x + x^3 + yx^3 + y^2z + y^3z \\ g_3 &= \ -1 - 4y - 3y^2 + y^3 + y^4 + x^2 + 2yx^2 - y^3x^2 + y^2xz \\ g_4 &= \ x + 3yx + 2y^2x - x^3 - yx^3 - z - 2yz - y^2z + x^2z \\ g_5 &= \ 2 + 6y - 2y^3 - 3x^2 - yx^2 + 2y^2x^2 + xz - 3yxz + z^2. \end{aligned}$$

$\mathcal{G}$ *is a Groebner basis for the ideal* $(\mathcal{G})$ *with respect to the pure lexicographic order on monomials in* $\mathbb{C}[x, y, z]$ *determined by the relationship* $y < x < z$*. Furthermore,* $(\mathcal{G})$ *is the ideal for* $\mathsf{X}_0 \subset \mathbb{C}^3$*.*

We use the symbol $\equiv$ *to indicate congruence modulo* $(\mathcal{G})$ *in* $\mathbb{C}[x, y, z]$ *and we write* $LM(f)$ *for the leading monomial of* $f \in \mathbb{C}[x, y, z]$ *with respect to our chosen monomial order. For* $1 \le j \le 5$*, let* $m_j = LM(g_j)$*. Then* $(m_j)_1^5 = (yx^4, y^3z, y^2xz, x^2z, z^2)$ *and* $m_j < m_{j+1}$*.*

The polynomial g_1 *shows that* y *is integral over* $\mathbb{Z}[x]$ *and a quick calculation shows that*

$$z^5 - 2xz^4 + (-2+3x^2)z^3 + (12x - 9x^3 + x^5)z^2 + (-18x^2 + 10x^4 - x^6)z + (6x^3 - 2x^5) \ \in \ (\mathcal{G}).$$

Hence, z *is also integral over* $\mathbb{Z}[x]$*. This shows that the set* $\mathcal{B} \ = \ \left\{ y^i z^j \mid 0 \le i, j \le 4 \right\}$ *is a generating set for* $\mathbb{C}[\mathsf{X}_0]$ *as a* $\mathbb{C}[x]$*-module,* $T_{\mathbb{Q}}(\mathsf{X}_0)$ *as a* $\mathbb{Q}[x]$*-module, and* $T(\mathsf{X}_0)$

as a $\mathbb{Z}[x]$-module. However, $\mathcal{B}$ is to large to be a basis for any of these modules. Let $\mathcal{B}' = \{1, y, y^2, y^3, z, y^2z\}$.

Theorem 12. *$\mathcal{B}'$ is a free basis for $\mathbb{C}[\mathsf{X}_0]$ as a $\mathbb{C}[x]$-module, $T_{\mathbb{Q}}(\mathsf{X}_0)$ as a $\mathbb{Q}[x]$-module, and $T(\mathsf{X}_0)$ as a $\mathbb{Z}[x]$-module. In particular,*

$$\mathrm{Rk}^{\mathbb{C}}_{\mathsf{X}_0}(\mu) \;=\; \mathrm{Rk}^{\mathbb{Q}}_{\mathsf{X}_0}(\mu) \;=\; \mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_0}(\mu) \;=\; 6.$$

Corollary 7.5. *X_0 does not detect a closed essential surface nor does it detect the slope μ.*

The proof of Theorem 12 follows from the following lemmas.

Lemma 7.6. *If $0 \le k \le 4$ then the rank of $\langle 1, y, \ldots, y^k\rangle_{\mathbb{C}[x]}$ is $k+1$.*

Proof. The proof uses the same argument as Proposition 7.4. □

Lemma 7.7. *If $k \ge 1$ then there is a polynomial $P \in \mathbb{C}[x, y]$ such that*

$$x^{2k}z \;\equiv\; P + z(1 + 2y + y^2)^k.$$

Proof. Induct on k and use the polynomial g_4. □

Lemma 7.8. *If $k \ge 3$ then there is a polynomial $P \in \mathbb{C}[x, y]$ such that*

$$y^k z \;\equiv\; P \pm y^2 z.$$

Proof. Induct on k and use the polynomial g_2. □

Lemma 7.9. *If $k \ge 1$ then there is a polynomial $P \in \mathbb{C}[x, y]$ and $d_0, \ldots, d_4 \in \mathbb{C}$ such that*

$$x^k z \;\equiv\; P + d_0 z + d_1 yz + d_2 y^2 z + d_3 xz + d_4 xyz.$$

Proof. The proof follows from Lemmas 7.7 and 7.8 along with the polynomial g_3. □

Lemma 7.10. *Suppose $p \in \mathbb{C}[x, y]$, $g \in \mathbb{C}[x]$, and $d_0, \ldots, d_4 \in \mathbb{C}$. Define*

$$f \;=\; p + z\left(g + d_0 + d_1 y + d_2 y^2 + d_3 x + d_4 xy\right).$$

If $f \in (\mathcal{G})$ then $f = p$.

Proof. Induct on k using Lemma 7.9 and that $\mathcal{G}$ is a Groebner basis. □

Lemma 7.11. *For every $k \in \mathbb{Z}^+$,*

$$\langle 1, y, \ldots y^k, z\rangle_{\mathbb{C}[x]} \;=\; \langle 1, \ldots, y^k\rangle_{\mathbb{C}[x]} \oplus \langle z\rangle_{\mathbb{C}[x]}.$$

Proof. Use Lemma 7.10. □

Lemma 7.12. *$\langle \mathcal{B}'\rangle_{\mathbb{C}[x]}$ is a free $\mathbb{C}[x]$-module with rank 6.*

Proof. Use g_3 along with Lemmas 7.6 and 7.11. □

Now to prove Theorem 12, we need only show that $\mathcal{B}'$ spans the modules in question. This is straightforward using the polynomials $\mathcal{G}$.

Example 6. *N is the once punctured hyperbolic torus bundle from Section 5 of [11]. For more details in the calculations that follow, see [11].*

$\pi_1 N = \Gamma = \langle \alpha, \beta, \tau \,|\, \tau\alpha\tau^{-1} = (\beta\alpha\beta)^{-1}, \tau\beta\tau^{-1} = \beta\alpha(\beta\alpha\beta)^{-3}\rangle$. The elements τ and $\lambda = [\alpha, \beta]$ form a basis for the peripheral subgroup of N. The functions

$$t = I_\tau \qquad u = I_{\alpha\tau} \qquad v = I_{\beta\tau} \qquad w = I_{\alpha\beta\tau}$$
$$x = I_\alpha \qquad y = I_\beta \qquad z = I_{\alpha\beta} \qquad l = I_\lambda$$

give an embedding of $\mathsf{X}(N)$ into $\mathbb{C}^8$. For $\epsilon \in \{0, 1\}$, we have an irreducible algebraic component $\mathsf{X}_\epsilon \subset \mathsf{X}(N)$ which contains a discrete faithful character.

Theorem 13.

(1) $\mathrm{Rk}^{\mathbb{C}}_{\mathsf{X}_\epsilon}(\lambda) = 8$.
(2) $\mathrm{Rk}^{\mathbb{Q}}_{\mathsf{X}_\epsilon}(\lambda) = 16$.
(3) $\mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_\epsilon}(\lambda) = 17$.

As a $\mathbb{Z}[l]$-module, $T(\mathsf{X}_\epsilon)$ is torsion free but not free.

We prove Theorem 13 in steps. Define

$$\mathcal{B}_{\mathbb{C}} = \{1, z, z^2, z^3, y, zy, z^2y, z^3y\}$$
$$\mathcal{B}_{\mathbb{Q}} = \{1, z, z^2, z^3, y, zy, z^2y, z^3y, t, zt, z^2t, z^3t, yt, zyt, z^2yt, z^3yt\}$$
$$\mathcal{B}_{\mathbb{Z}} = \{1, y, y^2, y^3, t, u, v, w, x, z, yt, yu, yx, y^2x, xt, xu, vy\}.$$

We will show that these sets are bases for the modules in question.

Let $\mathcal{I}_\epsilon$ be the ideal for $\mathsf{X}_\epsilon \subset \mathbb{C}^8$. The following polynomials are in $\mathcal{I}_\epsilon$.

$$8u + tyz - ty^3z \qquad\qquad 4v - 6ty + ty^3$$
$$4w + 2tz - ty^2z \qquad\qquad 2x - yz$$

This shows that the projection π, given by $(l, t, u, v, w, x, y, z) \mapsto (l, t, y, z)$, restricts to an isomorphism from X_ϵ onto its image. We have

$$\pi(\mathsf{X}_\epsilon) \;=\; V(y^2z^2 - 2z^2 - 8,\; 2t + (-1)^\epsilon iz^2,\; 8 + 2l - 2y^2 - z^2).$$

Hence, $\mathbb{C}[\mathsf{X}_\epsilon]$ is generated, as a $\mathbb{C}$-algebra, by y and z. Also, $T_{\mathbb{Q}}(\mathsf{X}_\epsilon)$ is generated, as $\mathbb{Q}$-algebra by t, y, and z.

Proposition 7.13. *$\mathcal{B}_{\mathbb{C}}$ is a basis for $\mathbb{C}[\mathsf{X}_\epsilon]$ as a $\mathbb{C}[l]$-module and $\mathcal{B}_{\mathbb{Q}}$ is a basis for $T_{\mathbb{Q}}(\mathsf{X}_\epsilon)$ as a $\mathbb{Q}[l]$-module.*

Proof. (Outline.) A straightforward, yet tedious, argument shows that $\mathcal{B}_{\mathbb{C}}$ and $\mathcal{B}_{\mathbb{Q}}$ generate the modules in question. We use Groebner bases to show that the modules spanned by $\mathcal{B}_{\mathbb{C}}$ and $\mathcal{B}_{\mathbb{Q}}$ are free of rank 8 and 16 respectively. □

Proposition 7.14. *$\mathcal{B}_{\mathbb{Z}}$ is a basis for $T(\mathsf{X}_\epsilon)$ as a $\mathbb{Z}[l]$-module. Moreover, this module is torsion free but not free.*

Proof. (Outline.)

(1) Another straightforward, yet tedious, argument shows that $\mathcal{B}_{\mathbb{Z}}$ generates the $\mathbb{Z}[l]$-module $T(\mathsf{X}_\epsilon)$.
(2) Define

$$M_1 \;=\; \left\langle 1, y, y^2, y^3, u, v, w, x, z, yt, yu, yx, y^2x, xt \right\rangle_{\mathbb{Z}[l]}$$

and

$$M_2 \;=\; \langle t, xu, vy\rangle_{\mathbb{Z}[l]}\,.$$

(3) Use Proposition 7.13 to show that M_1 is free of rank 14.
(4) Argue that $T(\mathsf{X}_\epsilon) = M_1 \oplus M_2$.
(5) If $T(\mathsf{X}_\epsilon)$ is free then M_2 is projective. Show that M_2 is not projective and thus $T(\mathsf{X}_\epsilon)$ is not free.
(6) Use Proposition 7.13 to show that $M_1 + \langle t, xu\rangle_{\mathbb{Z}[l]}$ is free of rank 16.
(7) By (5), $\mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_\epsilon}(\lambda) > 16$. By (1), $\mathrm{Rk}^{\mathbb{Z}}_{\mathsf{X}_\epsilon}(\lambda) \leq 17$.

□

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