Extrapolation and Expansion: Characteristics of Change Occurring in Mathematics Teaching Development Projects.

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Abstract

This paper commences by critically examining how mathematics teaching development projects based on the creation of communities of inquiry (CoI) are theorised from communities of practice theory (CPT) and cultural historical activity theory (CHAT). Two types of change, which can be developed from these sociocultural theories are articulated. Change as 'extrapolation' derived from CPT, and change as 'expansion' developed within CHAT. The differences between these types of change and their underlying principles are examined by contrasting conceptualisations of CPT and CHAT; attention focuses on mediation, goals, and agency. It is argued that the introduction of inquiry to CPT transforms the practice and entails a paradigm shift, with CoI as category within the critical paradigm. Expansion and extrapolation are illustrated using narrative evidence from a longitudinal case study of one school team that worked within a series of mathematics teaching developmental research projects over a period of six years. The paper emerges from the analysis and synthesis of a large volume of qualitative data accrued in teaching development projects.

Extrapolation and Expansion: Characteristics of Change Occurring in Mathematics Teaching Development Projects.

Extrapolation and *expansion* describe processes of change; the terms are rooted in communities of practice theory (CPT)[1] and cultural historical activity theory (CHAT) respectively. The purpose of this paper is to illuminate these terms and demonstrate how both are of value in the analysis and interpretation of events within mathematics teaching developmental research. Although both are sociocultural theories, CPT and CHAT have significant differences, especially regarding the notions of mediation, goal directedness and personal agency. In the first part of this paper these differences are used to argue that development as extrapolation (derived from CPT) is characterised by roots in prior experience, whereas expansion (coming from CHAT) is characterised by creative innovation that is essentially oriented towards future expectations. The second part of the paper uses data from mathematics teaching developmental research projects led by a team based in southern Norway to illustrate extrapolation and expansion. The projects are founded on principles of community and inquiry; that is CPT transformed to accommodate inquiry, in the learning and teaching of mathematics, together with inquiry as a tool in the development of teaching practice. It is asserted that this transformation and the development of a community of inquiry is not just a modification of CPT but constitutes a paradigm shift.

The illustrations in the second part of this paper arise from a longitudinal case study of one school teacher team that has worked within the projects over a period of six years. The longitudinal study is intended to explore and expose changes in teaching practice that occurred during the period of the projects' activity.

Inquiry

National tests and international studies (TIMSS and PISA) have exposed disappointing performance of Norwegian students in mathematics and stimulated a debate resulting in revised curricula for schools and teacher education, a national mathematics centre that focuses on teaching development, research, and researcher education. The projects from which the episodes used in the second part of this paper contribute to this effort through the development of teaching mathematics in school. In particular, through teaching development activity, we seek better learning opportunities and experiences in mathematics for students. Within the projects inquiry is promoted as an approach in mathematical activity, in learning and teaching mathematics, and in the development of teaching mathematics.

Inquiry approaches to teaching and learning mathematics are well established as, for example, Jaworski (2003) reveals by tracing the development of inquiry in education: generally from Dewey (1933), and in mathematics through the work on problem solving (Polya, 1945/1957; Schoenfeld, 1985; Mason, Burton & Stacey, 1982), and later through constructivist research and theorising (Davis, Maher & Noddings, 1990). Members of our research community in southern Norway have aligned with international developments in problem solving (e.g. Borgersen, Cestari & Bjuland, 2010) and inquiry (e.g. Fuglestad, 2010). The mathematics teaching developmental research reported in this paper is rooted in inquiry at three levels:

- Students' inquiry as an approach in learning *and understanding* mathematics.
- Teachers' inquiry into the processes of teaching and learning mathematics, *and understanding* and meeting their students' learning needs.
- Didacticians'[2] inquiry into the processes of developing mathematics teaching and learning in schools *and understanding* the needs, constraints and opportunities experienced by teachers as they implement inquiry approaches in their classrooms.

In the above 'understanding' is emphasised because the projects' goals are aligned with the substantial body of opinion, notably expressed within the principles of the NCTM standards (NCTM, 2000), that teaching and learning mathematics is based on understanding. Inquiry, and in particular, collaborative inquiry is an approach to the development of understanding in mathematics, “through inquiry, learners go beyond the use and application of algorithms and rules to develop understandings of general relationships in mathematics …” (Jaworski, 2003, p. 2). Inquiry, we assert, empowers students to function productively in ‘adidactical’ situations (Brousseau, 1997). We interpret ‘adidactical’ situations rather loosely as when students are engaged in tasks that can be solved using only the mathematical features of the task and without didactical markers or clues provided by the teacher or context in which the task is set. Such situations are essential in learning mathematics, which is special because mathematical objects (concepts) can be experienced only through some form of representation and not by direct experience of the objects (Duval, 2006). However, we also agree with Jaworski and emphasise that we do not see ‘inquiry’ as an easy solution for all the problems of teaching and learning mathematics, inquiry is presented as an approach to meet didactical challenges rather than a means of avoiding the challenges.

Community

Collaboration within communities is also a central principle in the developmental work we pursue in the projects. We draw on Wenger’s (1998) analysis of communities of practice (CoP) and assert ‘mutual engagement’ and ‘joint enterprise’ as necessary dimensions of participation in supporting teachers’ risk taking entailed in developing their practice, and sustaining developmental activity beyond the life of the projects. Wenger’s third dimension of CoP, ‘shared repertoire’, points to an important goal in the projects as we seek a professional discourse that focuses on, for example, inquiry, understanding, motivation, action, challenge, enjoyment, and success. Wenger’s development of communities of practice theory (CPT) thus provides a natural starting point for a theoretical framework for the projects’ implementation. Further, CPT offers a framework for analysis and evaluating the impact of the project, first in terms of community building using the above mentioned dimensions of CoP, and second teaching development by exploring issues of identity, which Wenger asserts "are an integral aspect of a social theory of learning and thus inseparable from issues of practice, community, and meaning" (1998, p. 145). Thus, the projects' success may be evidenced by the extent to which participants identify themselves with the project, and this can be illuminated by exploring their 'modes of belonging': 'engagement', 'imagination' and 'alignment' (Wenger, 1998).

Wenger's CPT provides a useful framework for our mathematics teaching developmental research, but as argued below, it does not meet all our needs. Teaching developmental research projects create relatively short lived CoPs, and a goal is to create the projects as communities within which teachers can participate and identify themselves. Identity formation perceived as a learning trajectory of participation within the project community can provide a means of analysing development. However, the projects exist alongside and intersect with the teachers' more stable and enduring professional CoP; that is, teaching mathematics in the community of their school, colleagues and classes. The projects seek better informed and equipped teachers within their school practice, but also seek to *transform the practice of mathematics teaching* within schools. The projects thus have a goal to influence practice in an intersecting community, and as such we have found it useful to look elsewhere in sociocultural theory for an additional framework for the operationalization and analysis of goal directed development.

Community of Inquiry

'Community of inquiry' (CoI) as a frame for the teaching and learning mathematics, and for the development of teaching mathematics, is well established, theoretically and empirically (e.g. Berg, 2009; Cochrane-Smith & Lytle, 1999; Elbers, 2003; Elbers & Streefland, 2000; Goos, 2004; Graven, 2004; Jaworski, 2006a, 2006b; Schoenfeld, 1996; Wells, 1999). For reasons briefly outlined in the preceding section, it is not surprising that many mathematics teaching developmental research projects that set out to create a CoI use CPT, as a framework to conceptualise community, practice, participation and other ideas central to CoI. However, some of those who have used CPT in this manner have found it necessary to 'extend' the theory. Graven (2004), for example, adds a dimension of 'confidence' in the panoply of teachers' professional knowledge, experience and practice. Also, indeed crucial to the argument in this paper, Jaworski (2006a) extends CPT by explaining that the introduction of inquiry transforms 'alignment', one of Wenger's (1998) 'modes of belonging', into *critical alignment*. Jaworski asserts that inquiry is 'a tool for developing practice' (Jaworski, 2005, p. 103), and that a CoI is not merely a 'type' of CoP; it is not meant that 'inquiry' is the practice. As will be argued below CoI theory is different; CPT is not only 'extended' but radically transformed by the introduction of inquiry. Moreover, it is possible to produce a mapping from Jaworski's (2006a) explanation of critical alignment to Friere's (1972) account of conscientization and thus argue that this transformation constitutes a paradigm shift (Goodchild, 2008, 2011).

Critical inspection of CPT has drawn attention to several 'gaps' when the theory is applied to mathematics teaching and development. However, before going further it must be noted that we do not suggest these are weaknesses of CPT per se, they concern what adaptations of CPT are necessary before it can be used it as an adequate theoretical perspective for teaching developmental research. Three key issues have attracted attention, mediation, goals and agency, which will be outlined below. First, we note that Kanes and Lerman (2008) draw attention to characteristic differences in the way fundamental concepts are explained within different articulations of CPT, in particular Wenger's (1998) 'Communities of Practice', and Lave and Wenger's (1991) 'Situated Learning'. Consequently, a discussion that is based on the notion of a single, unified CPT could be fundamentally flawed – especially if it also includes other original contributions such as Lave

(1988, 1996), and Scribner and Cole (1981). The assumption here is that there is sufficient common ground between these articulations of CPT to establish a basis for the present discussion.

Mediation.

CPT lacks a theory of mediation and teaching (Graven, 2004; Jaworski, 2007; Kanes & Lerman, 2008). Graven claims that Wenger (1998) ‘undermines the role of teaching’ (Graven, 2004, p. 185), and Jaworski (2007), uses Lave’s (1996) argument of ‘teaching as learning in practice’ to assert that CPT ‘is unhelpful in characterizing or analyzing mathematics teaching’ (Jaworski, 2007, p. 1691). Nevertheless, Jaworski acknowledges that CPT is useful as a framework for ‘characterizing and analyzing learning: for example teachers’ learning of mathematics teaching’ (2007, p. 1691). More fundamentally, Kanes and Lerman (2008) assert that CPT lacks a theory of mediation and a clear theorisation of tools and artefacts, as a result, they argue, the theory is difficult to apply to developmental and change processes. It is possible that the foregoing assertions are somewhat ‘fundamentalist’ in their interpretation of CPT and a more liberal stance can be taken, such as that adopted by Berg (2009), who associates Wenger’s ‘negotiation of meaning’ with the notion of ‘mediation of meaning’ (Kozulin, 2003).

Goals.

The theorisation of goals in CPT is inimical to problematising purposeful development. Scribner and Cole (1981) offer a definition of practice as “a recurrent, goal-directed sequence of activities” (p. 236), that is, it is the practice which is goal directed rather than the participants within the practice. Lave (1988) asserts that a study of goals is not appropriate because personal goals can be changed in the course of action and only known in retrospect. This can be understood in the context of teaching mathematics, a teacher might share with students the goal of the lesson as that the students will develop an understanding of *derivative*, say, but for the students who do not yet possess any knowledge of the derivative concept, the learning goal cannot make sense. It is only after the event and due reflection that students can be aware of the goal of their actions. However, it may be reasonable for students to have as a learning goal the resolution of a mathematical problem embedded in an adidactical situation created by the teacher, with the expectation that the milieu, also part of the teacher's creation, will provide sufficient feedback to confirm that the problem has indeed been solved (Brousseau, 1997). That is, the goal is the resolution of tension between knowing and not-knowing represented by the problem and new insight that enables the resolution, rather than an image of what the resolution might be. Nardi (1996) also draws attention to the treatment of goals in CPT as one of the defining characteristics that separates the theory from other sociocultural theories. From the perspective of teaching development, the possibility of goal directed action is crucial; developmental activity is not a process of thrashing around in the dark in the hope of hitting on something worthwhile, it is rather the purposeful integration of theory, thought experiment and practical engagement with the intention of achieving a desired outcome.

Agency.

Closely connected to the place of goals in CPT is that of personal agency. Wenger (1998) explains that CPT, in his development, aims at a central position between the extremes of social theory that focus on social structures and individual agency, so perhaps it is not surprising that he does not linger on this theme. However, inquiry and critical reflection are tools intended to enhance and direct an individual's capacity to act *on* their situation. Moreover, Greene (1988), taking Dewey's writings as a starting point, argues that inquiry is a means of the individual breaking free from the constraints of socially reproductive practices. Agency is not only about the choices that are entailed in aligning (or not) to practice, from a critical perspective it is about exposing and challenging those aspects of practice that interfere with development.

Development as 'Extrapolation' in a Community of Inquiry.

Notwithstanding the foregoing, CPT is useful in explaining a form of development in participation in practice as will be illustrated using examples the analysis of the longitudinal case study in the second part of this paper. It is possible to infer a theory of development from Wenger's (1998) discussion of learning, identity and modes of belonging. Wenger lists twelve principles of learning within a social perspective (1998, pp. 226-228), here the focus is directed on the principle of learning as a transformation of identity and 'ability to participate in the world by changing all at once who we are, our practices, and our communities' (1998, p. 227). Identity is about the relationship between the individual and the practice and focuses attention on the individual (Wenger, 1998). Identity formation and learning, Wenger asserts, are a matter of three modes of belonging: *engagement*, *imagination* and *alignment*. Engagement relates to participation in a practice as a member of the CoP and alignment is about 'the coordination of … energies, actions and practices. … we do what it takes to play our part' (Wenger, 1998, p. 179). Thus engagement and alignment are about an individual adapting her/himself to the practice. Imagination, however, is a mode of belonging that enables an individual to perceive her/himself in an extended (beyond actual experience) historical, spatial and social context of practice, engagement and alignment. Wenger explains, 'My use of the concept of imagination refers to a process of expanding our self by transcending our time and space and creating new images of the world and ourselves' (Wenger, 1998, p. 176). The word 'expansion' will be used in a different context below and it will be necessary to mark a distinction between the way the word is used in each context, hence attention is drawn to the object of expansion, 'self', and the outcome, 'new images of the world and ourselves'. Wenger uses the word 'extrapolating' in the context of imagining beyond one's present position, "creating images of the world and seeing connections through time and space by extrapolating from our own experience" (1998, p. 173). Imagination, as a mode of belonging, is about the creation of new images based on what is known and experienced, either directly or vicariously through other participants. We want to demonstrate through examples from data arising within the projects that the notion of 'extrapolation' as a mode of development within CPT, and CoI is very useful in analysing outcomes from projects' activity.

We offer our own definition of a category of development in mathematics teaching practice that we refer to as 'extrapolation' based on Wenger's articulation of CPT. Although we are led to the term through Wenger's work, and use categories of CPT developed by Wenger, the use of the term in the context of mathematics teaching development is our own.

The professional practice to which teachers align themselves is essentially conservative and reactionary; as Fullan explains, teachers belong to:

> An educational system which is fundamentally conservative. The way that teachers are trained, the way that schools are organised, the way that the educational hierarchy operates, and the way that education is treated by political decision-makers results in a system that is more likely to retain the status quo than to change (Fullan, 1993, p. 3).

Teachers seek 'normal desirable states of student activity' (Brown & McIntyre, 1993), and students have their expectations and exert strong pressures on teachers to conform. Under these conditions it is unsurprising that teaching development is generally slow, or as Jaworski describes in the context of teachers' engaging in research into their own practice, 'evolutionary' (Jaworski, 1998). In such circumstances when teaching is aligned to practice the notion of development as extrapolation appears appropriate.

We suggest the following characteristics of development as extrapolation: Extrapolation is *contingent*, because it is dependent on the situation, circumstances or milieu of the enterprise that motivates or supports change. It is *reactive* to immediate experienced situations and events. Extrapolation is *progressive* in the sense that it is dependent on what is known and contiguous with current practice; and *incremental* because it does not progress far from the known. In the context of social practice, engagement and alignment lead to the assumption of rules which extrapolation is likely to *confirm* because the extension of self, practice or community inevitably 'fits' with existing conceptions. Development that is an extrapolation of practice tends to *conform* to the norms of practice, because participants are aligned to the practice. It is essentially *reproductive* and *safe* because it does not entail significant movement from engagement in the routine and common expectations of practice and does not challenge the underlying assumptions of participation. New ideas, arising from participants' imagination are *adopted* into practice, modified and *assimilated* in order to avoid the risks inherent in allowing imagination to lead one into a dysfunctional practice. In CPT, Wenger explains that imagination can be a driving force for innovation and change: "it can recast the present and show it as holding unsuspected possibilities" (1998, p. 178). However, as Wenger observes imagination "runs the risk of losing touch with the sense of social efficacy by which our experience of the world can be interpreted as competence" (1998, p.178). Given the nature of the education system outlined above and classroom practice we believe that teachers are averse to risk taking and having their competence questioned. The foregoing characteristics are based upon our personal experience of teaching development activity, reports from others such as those cited above and other attempts at teaching development such as efforts in French didactical engineering to move didactical designs from the experimental setting into general school practice (Artigue, 2009). We do not wish to assert that all the suggested characteristics need to be evident, nor that the existence of any one characteristic should be sufficient to assert that observed development is an extrapolation; we present them as tell-tale signs that need to be interpreted within a holistic analysis of didactical practice.

The argument presented here is that CPT offers a theoretical model of practice, and development within practice that explains some types of change in teaching. CPT is useful where teaching assimilates new ideas, materials or tools into existing practice rather than being transformed as the practice accommodates and adapts to new possibilities; examples can be found in (Cohen, 1990; Cuban, 2001; Hennessy, Ruthven, & Brindley, 2005). However, mathematics teaching developmental research seeks fundamental transformation of practice and thus needs a critical dimension that challenges the assumed rules. This critical dimension is provided through the addition of *inquiry*, which transforms ‘alignment’ into *critical alignment* (Jaworski, 2006a). A theory that can be applied to such transformation is required; Cultural historical activity theory (CHAT) fulfils this purpose.

Cultural Historical Activity Theory

CHAT brings to the foreground mediation, goal directed action and agency (Roth & Lee, 2007); thus it should not be surprising that it is used as a theoretical framework for developmental activity in teaching (e.g. Engeström, 1994; Gordon & Fittler, 2004; Roth & Tobin, 2005). Very briefly, CHAT theorises goal directed actions (of individuals or groups) as the substance of historically enduring object oriented human activity. Actions are mediated, that is actions are both enabled and gain meaning through, cultural tools and artefacts – the most basic of these being language. Other sources of mediation are the community, rules, and division of labour, which are the social and cultural context of the activity. Engeström (1987) presents an extended version of Vygotsky’s (1978) notion of mediated action with the person or group in goal directed action on some object in dialectical relationship with mediating artefacts, rules, community and division of labour, thus representing an ‘extended activity system'. Elsewhere, we have developed an account of mediated action in the developmental research projects (Berg, Fuglestad, Goodchild & Sriraman, submitted), and thus do not extend the discussion of this point here. The immediate purpose here is to outline a model of learning and development as 'expansion' based on Engeström's writing (1987, 1999, 2001), rather than provide a thorough introduction to CHAT, which can be found elsewhere, for example Roth and Lee (2007).

Development as 'Expansion' in a Community of Inquiry.

Expansion refers to the transformation of the entire activity system (Engeström, 2001), of one or more dimensions within the system and/or the dialectical relations between the dimensions. Leont'ev asserts "activity is the nonadditive molar unit of life … activity is not a reaction or aggregate of reactions, but a system with its own structure, its own internal transformations, and its own development" (1979, p. 46). Expansion is about transformation rather than reproduction, as Engeström asserts "Expansive learning activity produces culturally new patterns of activity. Expansive learning at work produces new forms of work activity" (2001, p. 139). Expansion is explained by Engeström as taking place in cycles of activity which are successively dominated by internalisation and externalisation. Consider the trajectory of a teacher's career. Initially the novice mathematics teacher is in a phase of internalization as s/he learns the basic craft of the practice, how to interpret the curriculum, how to manage classes, how to respond to students' questions, how to organise learning experiences, how to prepare students for examinations and so on (this might be compared with Wenger's notion of an inward trajectory of participation). In the first instance these may be learned in the form of responsive behaviours, but alongside, the teacher also begins to internalize the underlying principles and structures of teaching mathematics in school that give meaning to the regular actions of the craft (possibly, alignment), and incidentally provide a foundation for personal development in the form of extrapolation. However, the internalisation of the basic principles and structures expose long standing contradictions and tensions, or double binds within the activity (Engeström, 2001). Following the example of mathematics teaching, a double bind might be experienced from the apparent contradiction of a demanding curriculum, high stakes examinations, classes including the full attainment range, pressure from teachers' peers to conform to school norms, public and political pressure expressed through the media to improve student performance, and advice from professional groups to teach for understanding. Extrapolation of participation within the practice will not resolve this double bind; the resolution requires culturally new patterns of work activity, that is, expansion (CPT does not help us to address creative responses to the contradictions within practice adequately).

Externalisation begins in the form of discrete, creative individual innovations, one might conceive of these as experimental actions, external to the activity system; Cole and Engeström (1993) refer to these as 'violations'. As experimental activity produces desirable outcomes so the innovations become internalised in the activity and a new phase of internalisation begins. Whereas internalisation might be accompanied or followed by extrapolation as a rather natural feature of participation, expansion, as Engeström presents it, requires mediation. In a CoI the mediation is provided by inquiry. Cole and Engeström assert:

> The new activity structure does not emerge out of the blue. It requires reflective analysis of the existing activity structure – participants must learn to know and understand what they want to transcend. And the creation of a new activity system requires the reflective appropriation of advanced models and tools that offer ways out of the internal contradictions. (1993, p. 40).

Expansion as a model of development has been worked out through CHAT, which is based upon a notion of goal directed action. However, whereas individual innovations may be goal directed towards resolving the internal contradictions of the activity, the eventual outcome of an expansive cycle cannot be predicted in advance (Cole & Engeström, 1993). Ironically, given the denial of personal goals in CPT, the converse may be true of extrapolation, because of its tendency to confirm and reproduce assumptions of structures and practices.

We attempt to outline characteristics of development as expansion to contrast with the characteristics of extrapolation outlined above. Critical alignment and inquiry are asserted to raise awareness of and *critically challenge* the contradictions of practice and provide a means for participants to effect transformation. Expansion is thus the outcome of purposeful, *goal-directed*, *experimental* and *innovative* action. Practitioners are encouraged, and empowered through inquiry led action to be *creative*, and *transformative.* However, such action carries *risks* because creativity, experimentation and innovation require practitioners to move beyond the familiar, safe ground of practice. Planning lessons entails a thought experiment, if based on prior experience and practice, and using approaches in which the teacher has developed some expertise, and furthermore the lesson is consistent with students' expectations, then the implementation and outcome of the planned lesson may be reasonably predicted. It is the unpredictability of innovation, and the unavoidable tensions that arise when implementation is not informed by prior experience and students' expectations are not met, that can undermine theoretically sound action. In the context of mathematics teaching developmental activity where one group (didacticians) are encouraging another group (teachers) to experiment with approaches not fully understood by the teachers the risk of failure is acute, as exposed in the account of one teaching experiment reported by Jaworski, Goodchild, Eriksen and Daland (2011). In the case of expansion, practice *adapts* to *accommodate* innovation rather than rigidly constraining new ideas to conform. Again we observe that one characteristic observed may not be sufficient to claim development to be expansive, nor will necessarily all be evident. The characteristics of extrapolation and expansion as forms of development are summarised and contrasted in table 1.

Extrapolation	**Expansion**
Contingent	Goal directed
Reactive	Experimental
Progressive	Innovative
Incremental	Creative
Conforming to and confirming rules and structures of practice	Critically-challenging rules and structures of practice
Reproductive	Transformative
Safe	Risky
Characterised by assimilation and adoption	Characterised by accommodation and adaptation

Table 1. Differences between extrapolation and expansion as forms of development

To demonstrate how extrapolation and expansion are both of value in explaining outcomes of mathematics teaching development activity we use two instantiations of development of a school mathematics teacher team that participated in a series of projects. We first set out the underlying methodologies, and then outline the data upon which the instantiations of development are constructed.

Methodology

We report episodes from a school team that participated in three mathematics teaching developmental research projects led by a team of didacticians at the University of Agder over the period 2004-2010. Not all the teachers took part in all three projects; however the continuous involvement of the school opens the possibility of a longitudinal case study that can be explored for evidence of development. Two projects ran in parallel over the period

2004-2007: Learning Communities in Mathematics (LCM) and ICT and Mathematics Learning (ICTML). Following these a 'binary' project ran for the period 2007-2010, Teaching Better Mathematics and Learning Better Mathematics (TBM/LBM). TBM focused principally on researching teaching development and is mainly the concern of didacticians. LBM focused on the development of teaching and learning mathematics in schools and kindergartens and is led collaboratively by representatives of school authorities, teachers from participating schools and didacticians.[3] The common theoretical principles and developmental goals of the projects are outlined below and afterwards, the essential differences between the projects will be explained.

The projects have shared three basic associated developmental goals, rooted in inquiry as outlined above. These were: the improvement of students' learning opportunities and performance in mathematics; the development of mathematics teaching; and the development of a didactical environment in which teachers and didacticians collaborate in the development of teaching and learning mathematics in school (Jaworski, 2006b). Each of these developmental goals has an associated research question that relates to how the desired developmental goal may be achieved. The examples used in this paper emerged from a longitudinal study of the data that sought to address the question:

> *What changes in mathematics teaching in a school team that has participated in the projects over a period of six years?*

We want to make it clear that we are using the examples here to illustrate the two categories of development – extrapolation and expansion, not to provide the empirical evidence that would constitute an answer to this research question.

Developmental research entails interconnected cycles of research and development (Goodchild, 2008, Gravemeijer, 1994). Research seeks knowledge and principles of general applicability, whereas the developmental activity is localised in particular settings. Research activity is concerned with taking global theories of learning and development and interpreting these at a local level to inform developmental activity. The developmental activity is concerned with the interpretation of local theory in terms of thought experiment (i.e. planning) and implementation. Systematic observation and analysis of data arising from developmental activity is then used to examine and inform theory; it is within this phase that this paper is located. The theoretical foundations of the projects have been outlined above, that is a transformed CPT and CoI. Operationalisation of the projects has included, principally:

- Teams of at least three mathematics teachers in each school and the active support of the school principal.
- Developmental workshops for which didacticians take responsibility for detailed planning on the basis of teachers' reflections and advice about needs and focus.
- School meetings, including both teachers and didacticians.
- Observation in classrooms by teachers and didacticians, followed by joint reflection on classroom episodes.

The workshops comprised teachers reporting activities from their own classrooms, didacticians making theoretically informed presentations about, for example, inquiry approaches to mathematics teaching and issues regarding learning different topics within mathematics (such as mental calculation, algebra, proof, and so on). In addition there have been opportunities for teachers to meet in small groups, across schools and grade levels to work on mathematical problems, discuss pedagogical issues and begin planning for their own classes. The projects have also included events aimed at disseminating experiences, including three national conferences – two arranged by the project and one arranged by the Research Council of Norway to 'showcase' the projects. Other local events have included workshops with an 'open invitation' for all teachers in the participating authorities' schools. The team of didacticians has remained fairly stable over the six years, varying in number between 10 and 13.

An important consideration in the projects has been the relationship between teachers and didacticians and the roles they take (Cestari, Daland, Eriksen, & Jaworski, 2005). This has taken account of responsibilities, as well as practical, developmental and ethical principles. Teachers are responsible for their classes, they have experience of teaching and learning that must be acknowledged and respected. Didacticians have to take responsibility for activities that teachers cannot undertake due to their regular workload, such as the detailed planning of workshops. Any development of teaching and learning will occur because teachers make it happen. The projects are concerned with the development of teachers' knowledge of teaching and learning mathematics, students' knowledge of mathematics, and didacticians knowledge of the developmental process. The research is about exploring the developmental processes rather than finding out what teachers or learners of mathematics 'do'. Consequently, all participants are inquirers into their own practice, all are thus 'researchers' and the projects are based upon a principle of co-learning agreement (Wagner, 1997).

LCM included teams of teachers from eight participating schools, two primary (grades 1-8), two lower secondary (grades 9-10), two combined primary/secondary (grades 1-10) and two upper secondary (grades 11-13). ICTML took a specific focus on the use of ICT as a tool for teaching and learning mathematics and comprised the two lower secondary and two combined schools participating in LCM. The binary project TBM/LBM included one each of the primary, lower secondary and combined schools that continued from LCM; and one new school from each of these classifications. Also the project included four kindergartens and two new upper secondary schools in addition to the continuation of the upper secondary teachers from the LCM project. Thus TBM/LBM comprised four kindergartens, two primary schools, two lower secondary schools, two combined schools and three upper secondary schools.[4] Apart from the inclusion of kindergartens and some changes in school participation there were other significant differences between the LCM, ICTML and TBM/LBM projects that reflected the knowledge gained from the accumulating experience. Whereas in the former projects, it was, initially, the didacticians who set the goals and steered the activity, in LBM leaders from the school authorities and school teams met regularly to discuss progress and suggestions for future activities. The involvement of the school authorities was welcomed as it introduced a level of recognition and encouragement of the teachers' activity within the projects. An unanticipated consequence was that the introduction of additional levels of management and steering combined with the contractual agreement being made with the school authorities rather than the individual schools appeared to weaken the relationship between didacticians and schools. The inclusion of kindergartens and additional schools also stretched didacticians' capacity to make the desired regular visits to observe in classrooms and contribute to school team meetings.

Unit of analysis.

The analysis must meet the needs of two contrasting theoretical perspectives, CPT and CHAT, which emphasise different categories of practice or activity, thus the definition of a satisfactory unit of analysis presents a challenge. Within CPT the unit of analysis may be 'participation' as the dialectical relation between participant and social practice, which enables the analyst to expose evidence of trajectories of participation, identity formation and the modes of belonging that form the basis of extrapolation. From the CHAT perspective the unit of analysis could be 'mediated action' (Wertsch, 1998, p. 24) that focuses on the dialectical and mediational links between actor(s) and object of activity, including cultural artefacts, rules, community and division of labour. Here the focus would be on structural entailments, mediation, goals and agency, and the contradictions in activity that provide the creative context for expansion. To expose evidence of both extrapolation and expansion, a unit of analysis that meets the demands of both CPT and CHAT is necessary. This is achieved by defining the unit of analysis from first principles, based on Vygotsky's assertions that a unit of analysis must "possess all the basic characteristics of the whole" (Vygotsky, 1987, p.45, in Wertsch, 1998, p. 26), and "contain the dynamics under examination" (Edwards, 2007, p.96).

In our work we focus on the development of mathematics teaching within institutional settings, and to understand this the unit of analysis is the complex of dialectical relationships between teachers, students, mathematics ,and the social, cultural, historical, and institutional entailments of teaching in school. To be clear, 'institutional' is used to summarise both internal categories, such as organisation, spacio-temporal and administrative structures, and external categories such as political and economic conditions experienced through, for example, the curriculum and funding policies. Mathematics teaching is, we believe, based on teachers' rationality and the constraints and affordances of their practice. To bring forth evidence of changes in teaching it is necessary to take the very broad field of view outlined above.

Data.

This wide ranging 'unit' of analysis raises questions about how any piece of data can possibly contain all the necessary dimensions of evidence, and the form of analytical process can be adopted. All of the data we use is qualitative; and most is in the form of recordings (video and audio) from naturally occurring situations, such as workshops and meetings that took place with a developmental intent rather than for the purpose of research. In other words, the research purposes were served in the act of recording the event, not in the occurrence of the event per se. Events take place in a social, historical and institutional context; utterances may also describe events that took place in another context. The analysis thus takes account of what is said and the wider context in which utterances are made.

Change is a slow process and evidence of change must arise from the examination of practices over long periods. The examples that are used to illustrate this paper come from over six years of project activity, and the data base that has been accumulated is sizeable including several hundred hours of video and audio recordings from workshops, meetings in school and university, and classroom observation. The analysis of the data has been focused at a macro level that seeks to expose characteristics of teaching practices and examine this for evidence of change, and interpret this as extrapolation or expansion.

Analysis.

The illustrations used in this paper apply to the teacher team at one school. The first stage of the analysis was to identify all the events in which the school team was explicitly involved, first by inspecting the projects' annotated record of events, and then iteratively as the examination of one section of data opened up a trail to data of related antecedent and successive events. Each piece of relevant data was then summarised through data reduction, which identified key themes, issues and categories. The third stage is a meso-level analysis of each episode to expose characteristics of mathematics teaching practices within the school team. Fourthly episodes are compared across time to expose change in practice, which is then interpreted as extrapolation or expansion. The first three authors of this paper collaborated in all stages of this process, comparing, critiquing, questioning and challenging interpretations. Assertions about the characteristics of practice arising from one episode are examined alongside counter assertions based on other episodes until agreement is reached. The outcome from this process is a narrative sequence of events that provides the basis for our assertions of extrapolation or expansion.

Table 2 lists pseudonyms of all the people who form part of the case study, an abbreviated time line of events relating to the school (Austpark) upon which this paper focuses is provided in figure 1, brief details of the events noted on the time line are provided in table 3.

Austpark teachers	**Project participation**
Harald (promoted to school principal in 2007)	LCM, ICTML and LBM
Gunnar,	LCM, ICTML and LBM
Frode, Elise	LCM and LBM
Jakob (2 years temporary appointment from 2005)	LCM and ICTML
Helga, Gunn, Runar, Eivind, Sigurd, Ingunn, Arild	LBM
Didacticians involved in the Austpark story	
Eli (leader of LCM and didactician initiator of TBM/LBM)	
Aud (leader of ICTML and didactician leader of TBM/LBM), second author.	
Otto (a teacher who had a didactician role in ICTML)	
Kai (a didactician PhD fellow whose research focused on teachers' implementation and orchestration of dynamic geometry software)	
First author was researcher in TBM/LBM. The second author was researcher in all projects. The Fourth author joined the writing team after completion of the projects as an independent consultant. The first three authors were active in planning and implementing project events, but only the first author and Kai took part in school meetings and observation.	

Table 2. Key 'players' in the Austpark story. (All names are pseudonyms).

We continue by setting out two instantiations of development that emerged over the six years of the project. These developments have occurred slowly and the history of each is rooted in events that precede the projects' inceptions. Reporting these instantiations of development entails a narrative of key events. The first narrative describes the development of a community of inquiry within the school, which, using the criteria listed above bears the hallmark of expansion.

2005

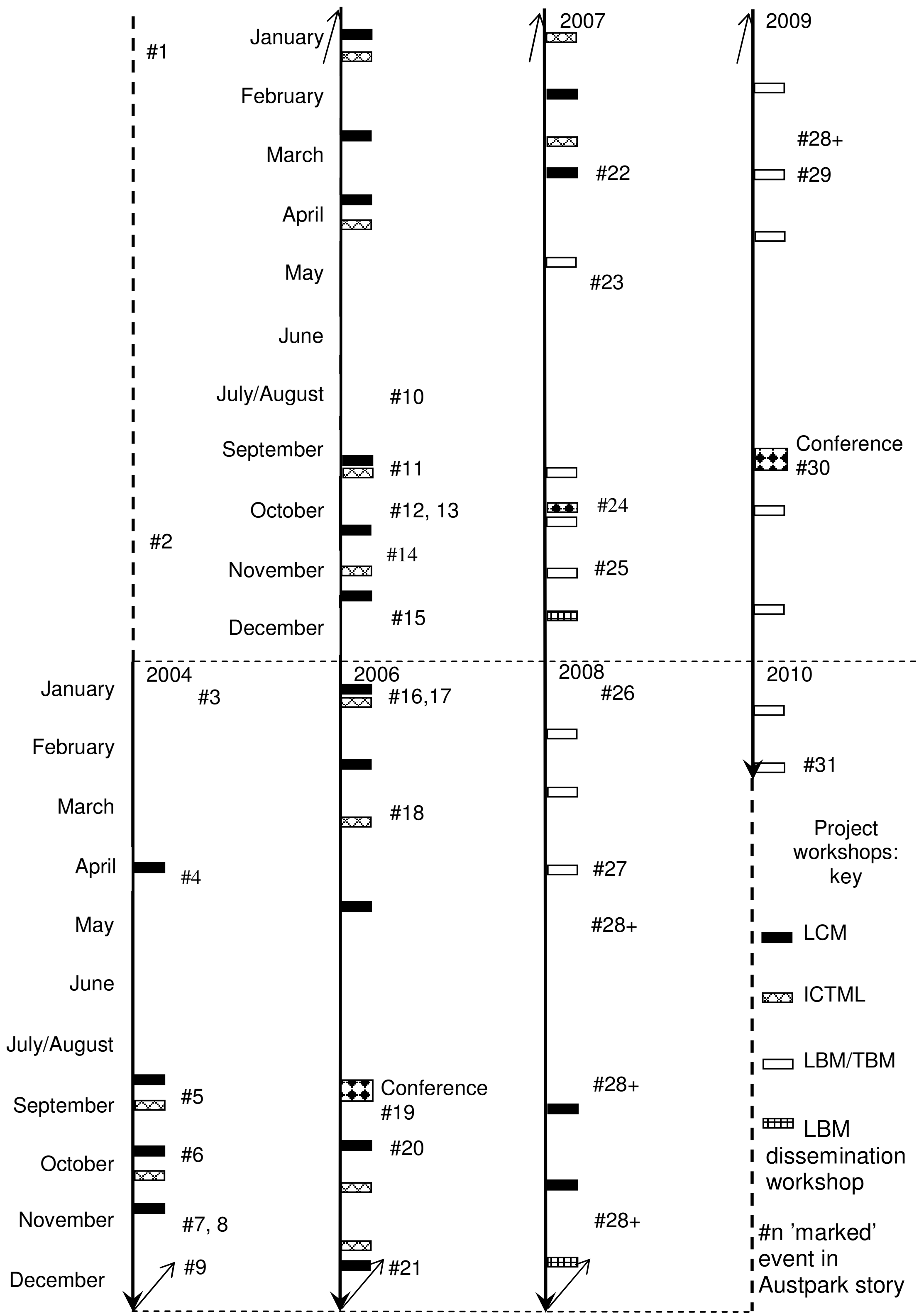


Figure 1. Time line: Workshops, conferences and other events in the Austpark story.

Event	Description

Table 3. Summary of ‘marked’ events relating to Austpark School

#1	November 2002, Gunnar attends national teachers' conference, John Mason is keynote speaker.
#2	Autumn 2003 The Research Council of Norway announces support of the proposed LCM & ICTML projects. Didacticians decide to hold a series of introductory seminars to inform and discuss core concepts of the projects.
#3	Gunnar attends the second introductory seminar. Subsequently he contacts didacticians to signal wish for Austpark school to participate in the projects.
#4	Pilot workshop at the university, most participants are didacticians who will be active in the projects. Gunnar and two teachers from other schools participate. Gunnar sends an "insightful" reflection on the pilot workshop. He expresses enthusiasm towards the project describing it as "wonderful", "a gift from above".
#5	First project workshops. LCM: September 01; ICTML: September 15.
#6	Second LCM workshop.In plenary Gunnar talks about the impact of hearing John Mason speak at the conference in November 2002 and reading "Researching your own practice:The discipline of noticing" (Mason, 2002).
#7	School meeting: Gunnar, Harald, Frode, Elise, Principal, Aud, Eli.
#8	Incident resulting in major structural damage at Austpark school.
#9	Didacticians meeting: review of progress in project schools.
#10	Jakob, recently qualified with Masters in Mathematics Education joins Austpark.
#11	Concern about the lack of development in use of ICT at Austpark, Aud contacts the school to request a meeting.
#12	Eli and Aud combine to contact Austpark principal to expressing anxieties about school's commitment to the projects.
#13	Aud and Otto meet Austpark teachers and leaders to offer support.
#14	Eli and Aud visit Austpark school to discuss progress and commitment.
#15	Gunnar introduces class to dynamic geometry software (DGS).
#16	In group discussion in a workshop Gunnar tells of his experience using DGS.
#17	Otto provides school-based training sessions in use of DGS.
#18	Focus group: Eli, Aud, Kai, Jakob, Elise, Harald, Frode, and Vice Principal.
#19	LCM/ICTML Conference at the university. All teacher teams present their work in the projects. Conference draws participants from throughout Norway.
#20	In workshop groups teachers discuss goals for continued activity in the projects.
#21	Jakob presents in workshop plenary: a specially planned class activity was video recorded and used to stimulate reflection in a school team meeting.
#22	Workshop plenary, teachers talk about the impact of the projects and plans.
#23	New project Learning Better Mathematics (LBM). Project leaders from each school meet to report activity and discuss plans.
#24	National conference arranged by Research Council (RCN)
#25	Presentation of school goals for LBM.
#26	Schools' project leaders meeting.
#27	Austpark teachers tell about their approach to teaching linear functions in workshop plenary.
#28	Schools' project leaders meetings (07.05, 03.09, 19.11.2008, 11.03.2009)
#29	Focus group: Aud, Helga, Ingunn, Gunnar, Sigurd, Eivind, Harald, Elise.
#30	LBM/TBM Conference at the university. All teacher teams present their work in the project. Conference draws participants from throughout Norway.
#31	Final LBM workshop. Teachers report impact of project in their schools.

The second narrative describes the development of mathematics teaching at the school. This appears to be largely of the character of extrapolation, although there is some indication of creative innovation that might eventually result in expansion. The second narrative concludes with a consideration of why extrapolation might be more characteristic of teaching development than expansion. The two instantiations provide complementary accounts of mathematics teaching development through CoI. An important dimension of development that is not covered by these instantiations concerns changes in students' learning. However, as Jaworski and Fuglestad observe, "although the creation of better learning environments for students to learn mathematics in classrooms was central to both projects [LCM & ICTML], the main focus was on the teaching of mathematics and its development and on the use of technology as a part of this teaching" (Jaworski & Fuglestad, 2010, p. 81).

Instantiation of Development 1: Expansion

We observe at the outset that although set in the context of mathematics teaching developmental research this example of expansion does not foreground teaching and learning mathematics. Nevertheless, we believe it is important to include here, both as an example of 'expansion' and to set out the context within which the second example, of extrapolation, occurs. The narrative of this 'expansive cycle' (Engeström, 1999) extends over eight years; we begin the account in November 2002, almost two years before the projects started. The account is divided into three episodes, each mapped onto a CHAT framework of an expansive cycle, which Engeström describes as follows:

> [T]he expansive cycle of an activity system begins with an almost exclusive emphasis on internalization, on socializing and training the novices to become competent members of the activity as it is routinely carried out. Creative externalization occurs first in the form of discrete individual innovations. As the disruptions and contradictions of the activity become more demanding,

> internalization increasingly takes the form of critical self-reflection – and externalization, a search for solutions, increases. Externalization reaches its peak when a new model for the activity is designed and implemented. As the new model stabilizes itself, internalization of its inherent ways and means again becomes the dominant form of learning and development. (Engeström, 1999, pp. 33-34).

In Engeström's terms, the episodes might be labelled: first, internalisation; second, externalisation, becoming aware of the double binds and creative innovation; and third, completion of an expansive cycle and internalisation. Given the intention here is to present this as an example of 'expansion' the mapping onto CPT is not so easy. The first episode might be characterised as a period of engagement, alignment, and the exercise of imagination. The second episode as critical alignment and seeking new forms of participation; and the third reveals a transformed practice. The partition into episodes disguises the continuous flow and development of events, and it might suggest an unproblematic developmental process, we do not intend to claim that development is episodic in this sense, nor is it necessarily a smooth progression.

Episode 1 - Internalisation.

The narrative begins in November 2002 (#1, numbers #1, #2, etc. refer to events listed in table 3) when one of the Austpark teachers, Gunnar, attended a professional development conference and heard John Mason speaking about 'the discipline of noticing' (Mason, 2002). The influence of this experience on Gunnar is evident from the repeated references he made to the event and Mason's book recorded in project data. During the ensuing year Gunnar and three colleagues worked on a proposal for a school based teaching development project that included observation of each other's practice followed by reflective discussions. Their intention, stated in the project proposal, was to 'up-date teachers' subject knowledge' (extract included in an e-mail Gunnar sent at the end of January 2004 expressing interest in the LCM project, #3). It is not clear whether this refers to knowledge of mathematics *for* teaching, specialised knowledge *of* mathematics, or pedagogical mathematics knowledge. At around the same time the school continued to support teachers to attend conferences in which research and development in mathematics teaching and learning was presented. The school was also engaged in a local authority led mathematics teaching project.

Gunnar learned about the planning of the LCM and ICTML projects through a personal contact and the university's web pages. He then attended a seminar in which the LCM director introduced colleagues to the projects' theoretical background, methodology and intended outcomes (#2). Immediately following the seminar Gunnar wrote to express an interest in his school being part of the projects, he explained that LCM and ICTML were very similar to the project he and his colleagues were already planning. Three months later Gunnar was invited to participate in a pilot workshop that was part of the preparation stage of the projects (#4), and following this he sent an enthusiastic reflection about the workshop and the implementation of the projects. Gunnar's reflection focused on a range of affective issues that would need to be considered when organising the projects' workshops to ensure that the teacher participants were to feel secure. In a didacticians' meeting about eight months later, the LCM project director recalled Gunnar's reflection and the expectations it raised about the engagement of Austpark teachers in the projects:

> I had high hopes for Austpark because of the early involvement of *Gunnar* and that really insightful reflection (#4) that we had from him after our first workshop planning meeting. (#9)

The projects commenced about nine months after Gunnar first expressed interest and Austpark school entered into a contract to participate in both projects. The contract set out the actions oriented towards development and research, which school and university each committed themselves. The contract also included an agreed payment to the school, which ostensibly was to enable the release of teachers to attend workshops. In the first LCM workshop (#5) participants were challenged to try out one of the inquiry based tasks that had been introduced in the workshop, and then return and report their activity in the next workshop. Gunnar was asked to report from his school team's activity in the second workshop (#6), he agreed to make a contribution but had to explain at the outset that he and his colleagues had been unable to make an occasion when they could prepare and observe each other. He then used his ten minute presentation to introduce all participants to Mason's book (2002) and outline some characteristics of observation.

Commentary.

Projects occur as relatively short periods with identifiable start and end points, but they take place in a seamless historical context, which must be considered in the analysis of project activity. The above narrative reveals one teacher and colleagues committed to and engaged in developmental activity prior to the projects. As we have analysed the participation of other school teams in the projects it has been noticeable that the schools that have shown the most encouraging signs of development are also the schools that were actively engaged in development before joining the projects. From the perspective of CPT it appears that listening to John Mason stimulated Gunnar's imagination and enabled him to produce 'images of (him) self and images of the world that transcend engagement' (Wenger, 1998, p. 177). However, the experience did not just lead to imagining, it stimulated Gunnar and his colleagues into goal directed (up-date teachers' subject knowledge) action (observation and reflective discussion). The proposed actions could be the beginnings of an expansive cycle. They are experimental, as they have not been tried before, they are innovative, because they will introduce a new dimension to practice, and they are risky because they entail the opening up of one's classroom activity and inviting the critical reflection of a colleague. The stated goals, however, appear to relate to individuals' participation in practice and thus their individual trajectories of identity negotiation (Wenger, 1998, p. 154). There is no evidence of critical alignment to the practice, in fact just the opposite; we interpret from Gunnar's enthusiastic response that his alignment and engagement in his school practice enables him to imagine himself aligned to the projects. Nevertheless, the final event included in this first episode suggests divergence between Gunnar (and colleagues') intentions and their actions; this becomes marked in the next episode of the narrative.

Episode 2 – Externalisation.

Two months into the projects, November 2004, the LCM project directors, Eli and Aud (#7) visited Austpark school and afterwards Eli reported her observations to fellow didacticians:

> I got the impression of three teachers who were all very interesting and positive but they were doing quite disparate things and they didn't seem to be talking [together] about what they were doing. (#9)

Nearly one year later (October 2005) didacticians expressed concern about the school's and teacher team's commitment to the project, and compliance to the contract of participation. Attendance at project workshops was sporadic and invitations for didacticians to attend project meetings and visit classes at Austpark had dried up. The directors of both projects visited the school to discuss the issues of concern with the school's leadership and teams of teachers in the projects (#13, #14). The meetings exposed a number of structural impediments experienced by the teachers: competition with other curriculum demands for the school's limited computing resources; problems with the installation of software on the school's computing network; patchy teacher knowledge because not all teachers in the school's grade level teacher teams were engaged in the projects; the school's requirement for all students to have common experiences and opportunities to learn; priority given to other meetings that coincided with project workshops, especially meetings called by the local education authority.

The discussion concluded with a recommitment of the school's leadership and teacher team to the projects and the agreed contract. In addition the school accepted the offer a didactician to hold two school-based workshops to introduce all the mathematics teachers to dynamic geometry software and share ideas for its use in class (#17).

In March 2006 group interviews were held in each of the project schools (#18). Essentially these had a research and evaluation purpose and in this respect the data is different because it is not naturally occurring within the developmental activity. However, as these group interviews stimulated reflection on action they did contribute to the developmental purpose of the project. The interview at Austpark exposed further obstacles in meeting the projects' intentions, but also possible signs of expansion in the form of 'discrete individual innovations' (Engeström, 1999, p. 33). Two problems were cited: once again the school's organisation of grade based teams was seen as an obstacle as these facilitated grade team meetings rather than across grade project team meetings. Second, a recently qualified mathematics teacher (Jakob) had joined the school and the project team but his teaching schedule made it impossible for him to meet with other project teachers. The early sign of expansion came from two of the teachers who described their joint lesson observation and reflective discussion.

Commentary.

From the perspective of CPT the events outlined in this second episode suggest a marked lack of alignment between the school team and the projects. This required teachers to decide between which of the two practices they would align with. However, a CHAT perspective considers the 'contradictions' and double binds described above as having the potential to contribute to the expansion of practice, "such contradictions generate … innovative attempts to change the activity" (Engeström, 2001, p. 137). The raising of awareness of the structural entailments that impeded the implementation of the proposed actions contributed to the development of participants' critical alignment, and the double obligation to school and to the projects created a tension that had to be resolved. Harald and Gunnar, however, had managed to overcome the structural problems, possibly (we can produce no evidence for this proposal) for this pair of teachers the 'problems' existed in their imagination, from their extrapolating from the team's joint experience into the possibilities of their partnership. Their joint action *may* be an example of individual innovations that form part of an expansive cycle. At this stage of the narrative one cannot be more assertive than 'may be an example', it is only when seen within the context of a completed cycle that one might make the claim more assertively.

Episode 3 – Expansion.

In this episode we draw on data arising from the final year of the LCM and ICTML projects and the whole of the follow-on TBM project, our purpose is to make a case to support our claims that changes in practice occurred that are reasonably described as 'expansion'. At the beginning of the third year of the LCM project, (September 2006) school teams were asked to set out their development goals for the final year of the LCM project (#20). The Austpark team set their goals as: better dialogue between teachers and between students; mutual observation of lessons, with video recording and joint reflection on viewing the recordings. Three months later Jakob spoke at a project workshop (#21), he described collaborative activity at Austpark that included joint lesson preparation, implementation of the prepared lesson with video recording and subsequent reflective discussion involving the whole mathematics teacher team.

Funding for the LCM and ICTML projects came to an end in 2007 but was followed by funding for the new project, TBM, which commenced as the others concluded. Austpark mathematics teachers joined the new project, but this time all the mathematics teachers at the school participated. School teams were asked to set out their goals for their activity in the new project and the Austpark teachers presented the following:

- We want to have an even better understanding about how students learn mathematics and what we can do to adapt this knowledge to our teaching.
- We want to exploit teachers' resources such that they can be shared within a learning community.
- We want to benefit from the competence the university can offer. (#25, authors' translation).

When leaders of the school project teams came together to discuss progress and plans within the project, Harald, who by now had been promoted to Austpark school principal, described a community of himself and nine teachers who were engaged in school based project activity. The team had regular meetings, once each month and the teaching schedule had been adjusted to allow teachers to observe each other in class. These monthly meetings would include working on tasks that had been presented in project workshops, and teachers' reflecting on practice. Classroom observation was presented as mutually beneficial to the observer and observed, for reflection and feedback. In March 2009 a second round of group interviews was held (#29). On this occasion the teachers at Austpark described their collaboration in joint planning, observation and reflection. They talked about a transformation of teaching and how the mathematics team were having an influence across the whole school. Sigurd captures the mood of the teachers in the group interview when he explains:

> You [referring to Aud, second author] ask about what we have succeeded with. At any rate, I feel that whether it is you that has succeeded or it is us who succeed, I do not know, but we have at any rate succeeded at Austpark school with approaching mathematics teaching today in a different manner than three or four years ago. Very many of us now

> have opened up very (indistinct) and dare to challenge more of the students more than we did earlier. I do not believe boardwork has disappeared, but it is very much reduced. (Sigurd, focus group, #29, authors' translation).

A year later Harald, in an e-mail to Aud, described the prevailing culture of sharing that by then existed at Austpark, and the need to nurture this culture.

> We see that it takes time to incorporate this culture (of sharing) … I see that it is important that we set aside time to share with each other in continuation, otherwise the culture may quickly disappear. But it is good to see that even if we have lost several mathematics teachers during the projects, the culture continues to "live" with us. (e-mail April 27, 2010, authors' translation).

Commentary.

Episode 1 provided evidence of teachers' proposing actions that were consistent with the definition of expansion we proposed earlier. Episode 2 drew attention to the contradictions that the Austpark teachers were experiencing as they tried to implement the proposed actions. In the third episode, the report from Harald in the leaders' meeting, Sigurd's contribution to the group interview, and Harald's e-mail suggest that the teachers recognise that their practice has developed. Harald draws attention to working practices within the mathematics group and in the group interview it is claimed that the changes are being accepted by other subject groups within the school. Sigurd's contribution suggests that the changes are not just about the engagement between teachers, but also that there are changes of classroom practice. The teachers' mutual engagement in observation and shared reflection is probably enabled by a development of the joint enterprise that brought the whole mathematics teacher team into the TBM project. It is possible that the change in school leadership made a significant contribution; the new principal, Harald, had been committed to the projects' enterprise from the outset. We also draw attention to the change in the school community's repertoire, which is evident in the teachers' articulation of their developmental goals. At the outset the goal had been 'to up-date teachers' subject knowledge', whereas at the beginning of the TBM project the focus had changed to a better understanding of students' learning and the adaptation of teaching to learning. The developments relate not just to individuals' participation in an unchanged practice, but changes to the enterprise, modes of engagement and repertoire. The changes have taken place through goal directed action and creative innovation to overcome the experienced contradictions of practice, we thus claim this to be an example of expansion.

Instantiation of Development 2: Extrapolation

The second example of development focuses more sharply on classroom didactical practices of teaching mathematics at Austpark, and we will argue that the changes observed have the characteristics of extrapolation that were outlined earlier. We have characterised extrapolation as a slowly evolving incremental process that contrasts with the creative innovation of expansion. This could imply some underlying judgment that one form of development is superior to the other; we want to distance ourselves from any such implication. Our intent as researchers is to recognise qualitatively different forms of development without making any suggestion that one is better than the other. In the previous instantiation of expansion, teachers confronted and to a large extent overcame contradictions that arose within their school's internal structure, but in this second instantiation contradictions arise out of factors external to the school and beyond teachers' immediate control and scope for action. There is a nationally prescribed curriculum, students sit national examinations and tests that have a bearing on their progression to the next phase of education. The amount of time students spend in school is beyond teachers' control, and the time available has to be shared to facilitate coverage of the whole curriculum. It is a major and expensive process to drop one textbook scheme and introduce something new. Thus there are legal, educational, temporal and economic constraints. One of the defining characteristics that we argued contrasts expansion and extrapolation is that the former entails (greater) risk. There are formidable risks for teachers and their students if teachers challenge or ignore the external factors that influence didactical practices.

Episode 1: Introducing Dynamic Geometry Software (DGS).

This episode begins with the ICTML workshop in which DGS was introduced; our purpose is to set out teachers' opportunity to learn about using DGS prior to using it with their classes. The use of DGS was introduced by Otto (a school teacher who also had a didactician role in the ICTML project) in a project workshop in late 2005, not long before the teachers at Austpark used it with their classes. In addition, prior to its introduction in class and as agreed when the project leaders visited the school in late 2005 (#13), Otto led two school based workshops at Austpark to familiarise all the mathematics teachers with the software (#17). The approach in the workshop, commenced with a short, 12 minute, exposition of a small number of the functions of the DGS relating to drawing points, lines and circles, and how the dimensions of constructed objects could be linked to a variable number or line segment, thus enabling dynamic control over the figure constructed. Otto started by describing how a student in one of his own classes had been able to transfer knowledge of a ruler and compasses construction of the perpendicular bisector of a given line segment into the DGS context, without teacher assistance. Otto's exposition then extended the student's solution to include a demonstration of how DGS could be used to verify results, using the 'perpendicular bisector' construction within the software. Also, the dynamic capability of DGS was used to relate the radii of the intersecting circles used in the construction to a variable number and line segment and thus demonstrate the loci property of the perpendicular bisector, Otto explained how this application of DGS could expose the fundamental mathematics entailed in the construction. Following the exposition participants were set a number of tasks to produce solutions in DGS similar to those that students in school would usually undertake with ruler and compasses. Most of the teacher participants in the workshop had no prior experience of using DGS. Within the workshop participants worked in pairs and used DGS to produce other ‘classical’ ruler and compasses constructions, such as the normal to a line from a point outside the line, and an angle of 60º, etc. In subsequent workshops (after the introduction of DGS at Austpark) DGS was used in more challenging open problem situations, for example, to find the largest rectangle that can be drawn within a given right angled triangle.

We offer only a very brief summary account of the subsequent implementation of DGS at Austpark, a full account is available (Erfjord, 2008). The Austpark teachers expressed a number of concerns about the introduction of DGS. They were concerned that those colleagues who were not participating in the project would be unable to use DGS with their classes, and argued that all pupils within the school, or grade level, should have similar learning experiences. This concern was addressed through the school based workshops led by Otto. The DGS licence permitted the use of the program only within the school and thus, teachers argued, students would not be able to use it at home. As noted in the first instantiation the teachers experienced difficulties in getting access to the computer room, and this was resolved through the intervention of the school’s leadership. There was a limited amount of time available to cover the geometry section of the mathematics curriculum, and pencil and paper methods were also required, especially for external examinations. The response to these constraints was to use some worksheets that had been prepared by a student colleague of the newly appointed teacher Jakob. Also, it was decided that DGS and ruler and compasses methods would be introduced in consecutive lessons. The worksheets were highly structured and provided illustrated step-wise procedures for producing constructed figures on the screen. The approach required students to read, imitate and reproduce. The teachers’ offered several reasons for using these worksheets rather than following the approach they had experienced in the workshop. They argued that the worksheets were closely matched to the curriculum statements, they were ready to use and suitable for the beginning phase of using DGS. The step-wise approach fitted with a pattern of work the teachers wanted in class; they believed the worksheets would thus enable them to avoid a situation in which many students sought help at the same time. In reflecting on their experiences in a later workshop the teachers admitted that the worksheets had not opened up students' activity to inquiry.

Commentary.

The workshop approach was first expository, providing basic information about the software and its potential to expose mathematical ideas, and then exploratory, as participants were invited to use the software to reproduce familiar constructions. DGS was presented as a tool to create images on the screen, and as a mediating artefact that would stimulate reflection on mathematical properties and processes of construction. Participants were arranged in pairs to stimulate discussion, reflection and communication of ideas. From a CPT perspective the workshop experience might provide participants with an opportunity extrapolate from workshop activity into their own classroom experience based on their image of Otto working with classes and students similar to their own.

As the episode moves into the school context it exposes many of the risks teachers face when challenged to be innovative in their classroom practice. The teachers wanted to introduce DGS with their classes; indeed, the curriculum requires the development of students' digital competencies in all subjects (Ministry of Education and Research, 2006). However, as they considered the goals for the implementation of DGS the teachers were aware of the constraints of curriculum, examinations, time, resources, student expectations and teachers' competence. The introduction of DGS did not replace anything done previously; it was introduced alongside the familiar methods. Additional resources were introduced that adapted the new tool to existing practices. The introduction of DGS using the worksheets was to a very large extent reproductive of practice (it fitted the regular work pattern), it was safe because the worksheets were matched to the curriculum, they would support teachers who lacked confidence in using the software, and they allowed the assimilation of DGS into regular practice. Nevertheless, it was a development of practice to introduce DGS to classes, and mathematics classes moved from the regular classroom to the computer room. However, it could also be argued that the highly structured worksheets diverted students' attention away from the mathematics and onto reproduction of procedures, and the move from the mathematics classroom also signalled a move from mathematics. The episode does not reveal the creative innovation that is characteristic of expansion; it has the progressive, incremental and conforming characteristics of extrapolation.

The above episode covers a period of about four months. The instantiation of expansion provided earlier extended over a period of eight years. It may be questioned whether expansion is merely an accumulation of smaller episodes of extrapolation. We argue that this is not the case because, as in the DGS episode above, there is no evidence of creative innovation or any goal directed intention to change practice; the new resource was inserted into a largely unchanged and unchallenged practice. However, we want to address this critique by developing an episode that extends over the whole six year period of the projects.

Episode 2: Adapting tasks.

Inquiry in teaching and learning mathematics was the common thread through the projects, and the importance of inquiry in learning mathematics was outlined at the beginning of this paper. We have found no satisfactory translation of the word 'inquiry' into the Norwegian language, there are several Norwegian words that might be used, but each carries unwanted connotations, such as a 'criminal investigation'. This has proved useful because it has provided an excuse to use the English word 'inquiry' and then offer a definition wanted within the project. In the projects 'inquiry' is presented as being about questioning, exploring, seeking (solutions), creating (knowledge), inventing (strategies and techniques), looking (critically), discussing, reflecting, imagining, and wondering. At the outset 'inquiry tasks' were presented to reify these inquiry actions in mathematical activity. Two examples of such tasks are:

> Any number greater than two can be partitioned as a sum of terms in several ways, e.g. 8=4+4=5+3=4+2+2, etc. Work out the product of each set of terms. Which set of terms produces the largest product?
>
> How tall must a mirror be in order to see a reflection of one's complete body?

Such tasks generated a great deal of mathematical activity in the workshops, and they well well received (according to feedback from the teachers), especially by the elementary school teachers. However, many of the secondary school teachers, whilst recognising the value of the tasks in generating mathematical activity, argued that they were not sufficiently aligned to the curriculum and the examinations students would take. In response to this criticism considerable effort was made to adapt the types of tasks teachers regularly use so that they motivate inquiry into the underlying mathematical ideas. Approaches for adapting and extending tasks presented by Prestage and Perks (2001) provided a valuable resource. Throughout the projects efforts were taken to maintain a balance between the so called 'inquiry tasks' and the adapted textbook tasks.

The sustained focus on inquiry, inquiry tasks and the adaptation of tasks to generate inquiry activity in class provides a useful setting to explore change for evidence of expansion or extrapolation. We draw on data arising from three events that occurred during the course of the latest project, a group interview with the mathematics teachers at Austpark, a workshop presentation by one member of the Austpark team, and project conference presentation, which was planned and shared by the Austpark teachers. These events may show the extent to which teachers have aligned to the presented notion of inquiry to which they were exposed over the six year period.

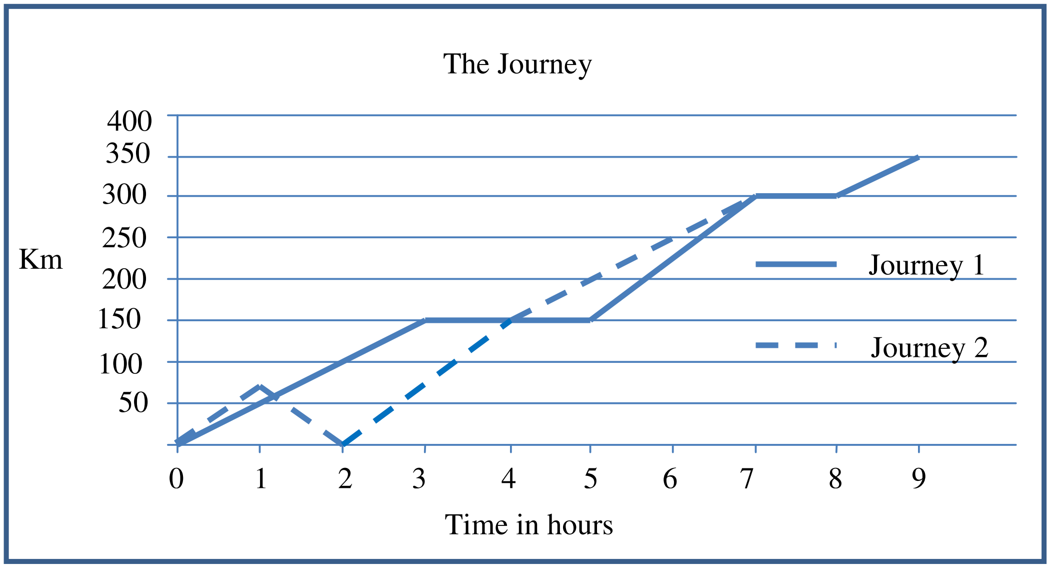
In the group interview that took place at the end of March 2009 (#29), Austpark teachers offered their account about how teaching and learning mathematics at the school had developed. At first when they were asked by Aud (second author) about the rationale behind the choice of tasks used the teachers were almost apologetic in their answers, perhaps thinking that they did not have much to tell, they emphasised "change (in the use of tasks) happens slowly", "change does not happen quickly … when you change tasks you add a little to experience". Nevertheless they appeared convinced that changes had happened, "in school we (teachers) talk much more about mathematics than we did, let us say ten years ago", "I believe it is quite different now than it was some years ago." "At Austpark school we engage in mathematics in a different way from three or four years ago … Many of us have opened up … and dare to challenge more students more than we did earlier. I believe that board(work) has not disappeared, but it is much reduced." At the same time the teachers gave some reasons for the slow pace of change and principally these related to students' experiences in elementary school and their expectations and learning goals and willingness to engage in more open (public) engagement in mathematical activity. Also noted was the unavoidable link between what happens in class and the way students are assessed. During the discussion that took place in the interview one teacher suggested that the projects might have been more forthright in requiring teachers to try out the tasks presented in workshops, but another disagreed, arguing that teachers will evaluate tasks and use a task if it fits with the curriculum and the teaching plan, and fundamentally if the task is considered worth using.

The Austpark teachers described changes in their practice; However, it has been recognised that self-reports of change may not be entirely accurate (e.g. Cohen, 1990). Accounts of their practice are also available from contributions to workshops and the project conference.[5] In a workshop that took place in April 2008 (#27) the Austpark teachers were requested to make a presentation of the approach used to teach about graphs of linear functions. Runar explained how the topic was approached in grade ten at Austpark. He first described the progression from grade nine when students start with the expression of a linear function, and use this to complete a table with coordinate pairs, which are then plotted against rectangular axes. In grade ten students were provided with less information and guidance (one of the approaches advocated by Prestage & Perks, 2001). Runar continued to explain that students were led through the relationship between the algebraic expression of a function and characteristics of the graph (gradient, y-intercept) and to be able to obtain the algebraic expression by inspection of the graph. The foregoing is a fairly standard approach found in the textbooks used in Norway. Runar continued:

> We have taken this a little further; it is possibly a type of inquiry. Can we find the function expression directly from two coordinate pairs without drawing the graph first? Without having the graph visually before one … [and he works through an example using the two points (0, -1) and (2,3)] … and so they (students) shall find the gradient … The function expression is then y=2x-1 … A little more difficult, if you are not given the y-intercept, coordinates (-2, -1) and (1, 2). (Author's translation)

In his exposition Runar explained all the calculations students were expected to do because, it will be recalled, workshops included both elementary school and kindergarten teachers.

In September 2009 the project's activities and outcomes were presented in a conference that invited national participation of teachers and teacher educators (#30). Each school team prepared a presentation based on the development of their teaching within the project, the intention was for conference participants to be offered an idea of how the project was realised in school practice. The Austpark school team decided they would offer participants an experience of learning mathematics in their 'inquiry' classrooms. The presentation focused on opening up tasks for students' inquiry. A rationale for the approach was offered, this focused especially on students' engagement and motivation, as illustrated by the following example and excerpt from the conference session. Austpark teacher Sigurd started by handing out a worksheet on which two journey graphs are illustrated against the same time-distance axes, and the single question “ What can you say about the journey to the cabin?” (Figure 2) and asking participants to work in pairs.



What can you say about the journey up to the cabin?

Figure 2 Worksheet used by Austpark teachers at the TBM Conference September 2009 (re-presented and translated by author).

After about four minutes Sigurd interrupted the discussions and asked several pairs of participants “what was the first thing you noticed?” He went on to explain that taking such an approach in class meant that students would enter the task at their own point of departure, what the students could relate to. Sigurd also drew attention to the difference between the approach he was suggesting and that found more commonly in school textbooks:

> Sigurd: The purpose of setting this task in this way is so that students can choose for themselves which theme they will work on. … Most (of you, referring to participants in the room) have seen this task before, but then, as a rule, there were five or six questions

> attached: How long did the journey take? How far to the cabin? When did they take the first break? How long was the first break? How fast did they drive before they took the first break? Do you recognise these (types of questions)? Why then? Why should we set such tasks? Why cannot the students come to these things themeslves? Why cannot the students take what they are secure in as their starting point? Because that is what the students start with; they begin with, 'this I know something about!' And so they get the feeling 'yes, I get it! I can succeed.' And when they succeed, so they are motivated, and when they are motivated, so they want to go further. And so they can develop. And there lies our opportunity as teachers, we get more motivated students, and that is a fact. I have tried this task in classes which are highly motivated. I have tried it in classes that are absolutely unmotivated. It works well in both classes. OK. Such open tasks as this, work well also when we have students who possibly are not so well motivated.

At the conclusion participants were given a document with all the tasks used in the session and the Austpark teachers had added some additional notes that set out the advantages of using open tasks, and the possibilities and challenges such tasks opened up for teachers. These notes drew attention to motivation, meeting students' individual needs, and challenging students; they did not make any mention about how such tasks engage students with fundamental mathematical ideas.

Commentary.

The Austpark teachers make the point that development in teaching is a slow and gradual process and this is true of both expansion and extrapolation. Our argument that the events described in instantiation 2 are characteristic of extrapolation is based on other characteristics. First we will assert that there are signs of changes in the teachers' practice. The teachers claim this, and their assertions about what they have gained from the projects in respect of opening up tasks have a ring of truth. There is, however, little evidence of any fundamental changes to classroom practice, the teacher remains firmly at the front controlling events and students' participation. Mediating artefacts, such as the introduction of DGS, or worksheets with open tasks, are adapted to regular didactical approaches and students' expectations. It will be recalled from the 'instantiation of expansion' above that in 2006 the Austpark teachers had declared one of their goals to be to gain a better understanding of how pupils learn. Possibly the teachers have set themselves on a trajectory of participation that will, eventually reach this goal, but the evidence points to development that has focused on didactical issues – managing, motivating and challenging students. There are constraints that limit teachers' scope for innovation: from outside the classroom, the curriculum, high stakes tests, and economic conditions; and from within, students' expectations. We also want to suggest that teachers' caution and their reluctance to take risks with didactical innovation might be crucial for continued development. We have, in the course of project activity, observed teachers (not at Austpark) implementing project inspired inquiry tasks in their classrooms with less than satisfactory engagement by students, which might lead the teacher to abandon any further effort to develop teaching and learning.

Discussion

The foregoing narratives reveal similarities and differences that appear to transcend the categories of expansion and extrapolation. Both instantiations of development contain evidence of teachers' engagement in their school practice and a trajectory of participation in the projects' practice. Alignment to the school practice is revealed by their attention to the curriculum, examinations, students' expectations, and observance of the school's teaching schedule and need to collaborate with colleagues in sharing limited resources. In both cases the teachers accept new ideas supported by the projects, including team work and the introduction of new resources (DGS) and tasks, thus showing their alignment to the project practice. The first instantiation, especially, reveals that when the two practices are experienced as contradictory the teachers' alignment with their school practice is strongest. In both instantiations, change has been prompted by experiences that have inspired teachers' imagination. In the first instance it was Gunnar's experience of John Mason's presentation of noticing; in the second we believe it was experiences within project workshops. Both instantiations describe developments that occurred over a period of several years. In the second instantiation the first episode about the introduction of DGS took place within the first eighteen months and the second episode considered events in the final year of the six years of the combined life of the projects. Inner tensions and contradictions are exposed in both instances; these arise from internal school structures and student expectations and from external constitutional demands of the curriculum and examinations. The first instantiation of change reveals that the tensions and contradictions were addressed by making changes to the school schedule and participation within the project, in the second there is no evidence of attempts to resolve the tensions.

The first instantiation relates to changes in practices within the school structure, administration and modes of participation within the projects. The second relates to changes in didactical practices and the relations between teacher, students and mathematics. In the first, Gunnar (and colleagues) came to the project inspired to work on their mode of participation in the school, and Gunnar perceived the project as a 'tool' to support them in achieving their ends. Gunnar and his colleagues had set their own goal for change independent of the projects' activity. In the second, the goals for changes in the teaching and learning of mathematics were initiated by the projects; we have no evidence that these goals were consistent with the teachers' goals prior to participation, but we have no evidence to the contrary either. The first instance concerns changes in which teachers exercised their agency to work on internal school structures to facilitate collaboration, joint observation and mutual reflection on practice. The second episode exposes tensions of practice that arise outside the range of teachers' agency. The differences noted here relate to contextual factors: constitutional order, institutional structures, arena (whole school or classroom), and level of participation. However, our claim is that the two instantiations are essentially different in nature, that the first exemplifies expansion and the second extrapolation, and that the difference transcends the context.

In the first case the teachers were goal directed from the outset, Gunnar expressed a clear intention that he and his colleagues wanted to observe each other in classrooms, discuss and reflect and up-date their subject knowledge. To achieve this goal it was necessary to challenge the rules and structures of their practice. The proposed development entailed risks as teachers opened up their classrooms to the critical gaze of their colleagues, but they were prepared to take these risks in being innovative and creative and overcoming the obstacles. Harald (Austpark principal) writes about a culture change at the school, and thus there is some evidence of the change being 'transformative', but the changes in structure suggest the development of a learning community and a context for continual development. In the second case it appears that the constraints experienced by the teachers have limited the scope for change. The projects' focus on inquiry in the teaching and learning of mathematics intended to engage participants in mathematical activity in which, as in Otto's introduction of DGS, fundamental mathematics would be exposed. The projects sought to shift the balance of relationships so that students engaged directly with mathematics, mediated by the teacher rather than the teacher dispensing mathematical knowledge to the students. Students' inquiry in mathematics is not intended to make the teacher redundant but rather that students would be equipped to meet challenges in the mathematics without the support of didactical (non-mathematical) cues and markers. The presentations described above, Runar about teaching graphs of linear functions, and Sigurd about travel graphs reveal changes in approaches to activity but there is insufficient to suggest that the goal is to engage students in meeting mathematical challenges through inquiry. Teachers' responses have been reactive to project expectations and their actions have been contingent upon project activities rather than explicitly directed by their own developmental goals. Changes in classroom practices have been progressive and incremental because they are 'safe'. The changes have not departed far from regular practice and have ensured the curriculum and examination goals remain in focus and not threatened by experimental intervention. The introduction of DGS was supported by the use of worksheets that were accepted because they were aligned with (reproductive of) regular practice. Teachers' changes in practice have not challenged students' expectations; indeed Sigurd's claim for opening up tasks was explicitly based on students' increased motivation.

Conclusion

As we review the Austpark story and the participation of Austpark teachers through six years of mathematics teaching development projects it is possible to see signs of development. We do not claim that the projects have been the cause of the development, rather that they have served the teachers' attempts to work on their practice. The development of collaboration of the teacher team has occurred through the transformation in the working practices, the rules, division of labour and community. The account provided above, we argue, is consistent with Engeström's description of expansion. Changes in teaching mathematics, however, appear modest and better characterised, for the most part, as extrapolation. We have suggested that the nature of the constraints experienced by teachers restrict their opportunities for action and thus teaching development. However, the circumstances for continued extrapolation of practice have been created. Harald described the changes in the working practices at Austpark as a 'change in culture', and the transformation provides the opportunity, motivation and favourable conditions for continued extrapolation of teaching practice. We reflect on Stigler and Hiebert's (1999) suggestion that the superiority of

Japanese students in international comparisons of mathematics performance might be, in part, the consequence of decades of teachers' commitment to lesson study. The expansion of the mathematics teachers' activity system at Austpark has led to an approximation of lesson study and it is hoped that it will contribute to the continual improvement of students' experience of, and performance in mathematics.

The analysis of the two instantiations of development at Austpark demonstrates the value of adopting both community of practice theory and cultural historical activity theory as complementary stances to explore mathematics teaching development. The use of the CHAT draws attention to development that appears to represent a radical change to practices because it is accompanied by changes to the rules, community and division of labour in the activity system; it represents a change in the culture. This is expansive development and appears consistent with the 'self-sustaining generative change' described by Franke, Carpenter, Fennema, Ansell and Behrend (1998). In the development of mathematics teaching extrapolation as a mode of change is more likely because of the constraints of curriculum and high stakes examination. We are led to inquire into fresh or modified approaches that might facilitate teachers' creative innovation in their classroom practice. This inquiry is informed by the analytic categories of expansion and extrapolation, and the issues facilitating and constraining change.

Finally, to produce this paper we have been confronted with the challenges common to many large teams working in teaching development projects. Teaching development is a slow process; substantial quantities of qualitative data accumulate over the period of many years. The data must be analysed, synthesised and presented in a manageable form. The temptation is to 'cherry pick' the data and write research reports on events or brief episodes, each interesting and informative but nevertheless do not address a fundamental requirement and wish to report on how the global developmental aims and research questions have been met.

Our engagement with the data to produce the case history of the Austpark teachers has opened a way for us to meet this challenge.

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Appendix: Glossary of Abbreviations Used

CHAT: Cultural historical activity theory

CPT: Community of practice theory

CoI: Community of inquiry

CoP: Community of practice

ICTML: ICT and mathematics learning (project 2004-2007)

TBM/LBM: Learning better mathematics/teaching better mathematics (project 2007-2010)

LCM: Learning communities in mathematics (project 2004-2007)

DGS: Dynamic geometry software

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[1] The authors are conscious of the density of abbreviations used in this paper and have decided to include a glossary in appendix 1.

[2] Participants in the projects based at the university are referred to as didacticians rather than the more usual researchers because within the projects teachers are also regarded as having an active role in researching teaching and learning.

[3] LCM, ICTML and TBM were supported by The Research Council of Norway. Additional support for TBM and support for LBM came from the Competence Development Fund of Southern Norway.

[4] The two upper secondary schools that had participated in the LCM project were combined into one school sharing the same campus in 2007, thus explaining why two and two only results in three schools.

[5] Ideally we would use data from classroom observations, but resources in the later project were insufficient to engage in systematic observation of practice in school.