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PROBABILISTIC THINKING: PRESENTING PLURAL PERSPECTIVES

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From a historical perspective, probability, as we know it today, did not emerge until the 1650's (Hacking, 1975). Compared to other areas of mathematics (e.g., algebra), the study of probability is a relatively recent endeavor. Although the quantitative science of probability emerged in Pascal's time, mathematics historians (e.g., Franklin, 2001) have pointed out that qualitative probabilistic thinking has been a staple of human decision making well before the time of Pascal as seen in judiciary, theological and philosophical arguments in ancient cultures east and west. That is, although the mathematical science of probability only emerged since Pascal's time, probabilistic reasoning was always an aspect of our thinking. The birth of probability also played a role in the subsequent paradigm shift within physics in moving away from the deterministic mechanical universe of Newton onto the acceptance of relativity and probabilistic statements in quantum mechanics for the position of particles. The birth of probability also influenced Francis Bacon's re-conception of science, in which he took the radical leap of suggesting a "probabilistic or fallibilistic criterion of knowledge which went on par with the emergence of probability in Pascal's time"(Perez-Ramos, 1996, p.319).

In a similar vein, but in an entirely different field, the study of probabilistic thinking, from a historical perspective, began with Jean Piaget and Barb Inhelder's (1951) seminal book *The origin of the idea of chance in children.* This beginning point – which has essentially been determined through acclamation – means "Research into the development of probabilistic thinking and the teaching and learning of probability has occurred largely during the last 50 years" (Jones & Thornton, 2005, p. 65). Compared to other areas of mathematical thinking and learning (e.g., geometry), the study of probabilistic thinking is also a relatively recent endeavor.

Despite being a relatively recent undertaking, the last (approximately) 60 years of investigations into probabilistic thinking have resulted in a number of important – now famous – pieces of literature. For example, there are: dedicated yearbooks from the National Council of Teachers of Mathematics (e.g., Burrill & Elliot, 2006; Shulte & Smart, 1981); syntheses found in major mathematics education research handbooks (e.g., Borovcnik & Peard, 1996; Jones, Langrall & Mooney, 2007; Shaughnessy, 1992, 2003) and as independent articles (e.g., Garfield & Ahlgren, 1988; Hawkins & Kapadia, 1984); special issues of mathematics education research journals (e.g., Borovcnik & Kapadia, 2009); and, more recently, edited books dedicated to probabilistic thinking and the teaching and learning of probability (e.g., Kapadia & Borovcnik, 1991; Jones, 2005).

(Also) despite being a relatively recent undertaking, the history associated with the last 60 years of investigations into probabilistic thinking is starting to take shape. For example, Jones and Thornton (2005), in their "historical overview of research on the learning and teaching of probability" (p. 66), classified research in probabilistic thinking into three periods: the Piagetian Period, the Post-Piagetian Period, and the Contemporary Period. As expected, the Piagetian Period is clearly defined as the period of research dominated by Piaget and Inhelder's structure of probabilistic thinking of the 1950's and the 1960's. Also clearly defined, the Post-Piagetian Period, which refers to the period of research dominated by Fischbein's work on intuitions and Tverksy and Kahneman's heuristics and biases of the 1970's and 1980's. The Contemporary Period, which is implicitly presented as the research of the 1990's and the 2000's, is not as clearly defined as the previous two periods. Jones and Thornton account for this phenomena as follows: "Although it is premature to evaluate historically the significance of probability research in this third and contemporary phase, it is clear that the volume and diversity of the research is greater than in the previous two phases" (p. 83).

Given the historical context presented above, the book we propose, *Probabilistic Thinking: Presenting Plural Perspectives* (*PT: PPP*) will serve four main purposes. First, in 2010 research will have been in the Contemporary Period for nearly 20 years and, as such, we contend it is no longer premature to evaluate historically the significance of probability research in this period. Second, to address the volume and diversity (and the historical evaluation) of the research in the Contemporary Period, the structure (and the title) of this book are derived from *cubism*. Whereas a cubist artwork presents multiple perspectives, which represents the subject in a greater context, the different sections of this book will present a variety of perspectives, which will represent the subject (i.e., probabilistic thinking) in a greater context. Further, each of the sections of the book will be comprised of multiple chapters, which will also represent a variety of subjects (i.e., the different sections of the book) in greater context. Third, and within the forward looking spirit of the monograph series "Advances in Mathematics Education," the structure of *PT: PPP* will allow for an international group of scholars to begin to define the next Period of research in probabilistic thinking. Fourth, we plan, with this book, to contribute to the major pieces of literature on probabilistic thinking, as detailed above. To achieve the four main purposes, the content of the ten sections, consisting of anywhere between 5 to 10 chapters, has been strategically chosen and will now been commented on and justified in turn. The book is planned in two volumes given its substantial scope.

Preface

The preface will consist of explaining the beginnings and the motivation behind *PT: PPP*. The majority of the section will be dedicated to describing, in detail, how the notion of cubism is used in both the title and as the underlying structure of the chapters and sections of the book (*note:* the use of cubism and how it structures *PT: PPP* is expanded upon the in the Commentaries and Discussion sections below).

Introduction

In the introduction we will present a history of the pertinent research and literature on probabilistic thinking. Adopting the three chronological periods of Jones and Thornton (2005), we will touch on the Piagetian and Post-piagetian Periods, but will focus the introduction on synthesizing the research from the Contemporary period (a task which has yet to be completed within the existing literature). In doing so, we will, through the themes that emerge from the research synthesis of the Contemporary Period, lay the foundation for the forward looking nature and the specific sections and chapters of the book. For example, while the Contemporary Period has begun to recognize three interpretations of probability (e.g., classical, frequentist, and subjective), the research literature, more specifically Jones et al. (2007) “were not able to locate cognitive research on the subjective approach to probability” (p. 925). Consequently, the philosophical section of the proposed book, dedicates a chapter to the subjective interpretation of probability. However, addressing all three different interpretations of probability within the chapter will, as we have argued, represent subjective probability in a greater context. Further, having commentaries from prominent researchers will also provide a scholarly “take” to the multiple perspectives presented in each of the chapters. (Other examples, for each of the ten sections will be discussed, individually, below.) The majority of the introduction will be dedicated to discussing, in a similar fashion to the example presented above, each of the 10 sections, which are comprised of multiple chapters, in the book.

The end of the introduction will detail the motivation behind asking five prominent researchers (Ramesh Kapadia, Mike Shaughnessy, Manfred Borovcnik, Jane Watson, and Graham Jones) to comment not only on the chapters of each section, but on the commentaries provided in each of the sections. We end the discussion by foreshadowing the task we (read: Egan and Bharath) have in discussing, at the end of the book, the multiple levels of cubism (e.g., the chapters of each section, the commentaries of each section, the final commentaries, and our discussion of the book) found throughout the entire book and the greater context which arises.

1. The philosophical perspective

The motivation for this chapter is based on two famous distinctions. First, as Gillies (2000) notes, “The theory of probability has a mathematical aspect and a foundational or philosophical aspect. There is a remarkable contrast between the two. While an almost complete consensus and agreement exists about the mathematics, there is a wide divergence of opinions about the philosophy” (p. 1). Despite this wide divergence, an second important distinction frames much of the opinions about the philosophy: “probability...is Janus-faced. On the one side it is statistical, concerning itself with stochastic laws of chance processes. On the other side it is epistemological, dedicated to assessing reasonable degrees of belief in propositions quite devoid of statistical background” (Hacking, 1975, p. 12).

This chapter will provide (what we argue is) a much-needed venue for researchers to discuss the (future of the) three dominant interpretations of probability found in research on probabilistic thinking. Separate chapters will be dedicated to discussing classical, frequentist, and subjective probability, whereas other chapters will be dedicated to discussing the relationships between each of the interpretations (i.e., classical and frequentist, classical and subjective, and frequentist and subjective). Alternative interpretations of probability (i.e., other than the “big-three”) and their place within the existing and future literature on probabilistic thinking will provide an important contribution also within the spirit of the “Advances in Mathematics Education” series.

This section will provide a necessary and timely contribution to the foundation of research in probabilistic thinking and, more specifically, will provide a necessary opportunity to discuss the current dominance between the classical and frequentist interpretations of probability and the philosophical issues which still exist from Tversky and Kahneman's non-committal stance to any one particular interpretation of probability. With this chapter we provide a dedicated venue for addressing the historical, philosophical, and nomenclatural obstacles (Hawkins & Kapadia, 1984; Shaughnessy, 1992) associated with different interpretations of probability

2. The mathematical perspective

The second chapter of *PT: PPP* provides what we consider quite a unique take on the mathematical perspective of probabilistic thinking. The perils of probabilistic thinking (e.g., relying on intuition) are well documented. For example, Paul Erdos, after seeing the solution to the Monty Hall problem, is famous for saying "I understand it, I just don't like it." However, what we have noticed is that the problems themselves become more famous than the individuals known to be involved with the problem. For example, Bertrand's box paradox is often presented without any reference to Bertrand Russell. As such, and picking up on a recent theme seen in pop-culture mathematics books, we have assembled a cast of all starts to pick a famous problem in probability and to discuss not only the mathematics behind the solution, but the trials and tribulations of the individual(s) who are considered as responsible for first finding the solution to the problem. For example, Keith Devlin recently presented, in detail, the thought processes and different approaches Pascal and Fermat had in their solving of the Problem of Points. Similarly, Mlodinow talks about the trials and tribulations of individuals (e.g., Cardano) involved with the early working of probability.

Given the prominence of certain probability problems in our society (e.g., the Monty Hall Problem appearing in movies and books), the commentary from this section will include a discussion on how famous problems (e.g., the Monty Hall problem, the birthday problem) and paradoxes (e.g., Bertrand's box, boy-girl paradox) have weaved their way into the social fabric of our society (e.g., the story of Maryln vos Savant, the popularity of Martin Gardner).

3. The psychological perspective

Past, present, and future research on probabilistic thinking (i.e., Piagetian, Post-Piagetian, and Contemporary Periods and beyond) is influenced by psychological research. The work of Piaget and Inhelder, Fischbein, Tverksy and Kahneman and countless others (e.g., Falk, Cosmides & Tooby, etc.), have, in many ways, laid the foundation for research in probabilistic thinking. This fact is not lost on those conducting research in the field of mathematics education and other areas. In fact, all major research syntheses and seminal pieces of literature pay homage to the foundational influence psychology has played in research on probabilistic thinking and, more recently, the teaching and learning of probability. For but one of countless examples, "The research of psychologists Daniel Kahneman and Amos Tversky, and many of their colleagues, has provided mathematics educators with a theoretical framework for researching learning in probability and statistics" (Shaughnessy, 1992, p. 470).

The purpose of this chapter is to showcase some of the more famous psychological contributions to research on probabilistic thinking (e.g., Tversky and Kahneman's heuristics and biases). This chapter will also take into account the alternative view of heuristics and biases, which is purported by Gigerenzer.

We also argue that the psychological research utilized by researchers in the field of mathematics education is “stuck” in the 1970’s. Researchers in mathematics education continue to reference the representativeness heuristic in their research, yet the representativeness heuristic has evolved into the notion of attribute substitution over the past 40 years. This evolution of the heuristics has been lost on researchers in mathematics education. Further, the evolution to dual process theory, which has taken place in the field of psychology, has not taken hold in the field of mathematics education—beyond Leron & Hazzan (2006, 2009). Presenting developments in the field of psychology will add an important contribution and, hopefully, push researchers in matheamatics education to adopt timely and relevant developments in their future research.

4. The mathematics education perspective: theories, models and frameworks

Building upon the foundation of psychology, researchers in mathematics education have also developed a variety of theories and cognitive models to account for probabilistic thinking. As noted by Shaughnessy (1992), “the perspectives of the observer [i.e., the psychologists] and the intervener [i.e., the mathematics education researchers] provide an excellent basis for cooperation and cross fertilization of theoretical models and research methodologies” (p. 469). Shaughnessy continues, “However, so far most of the sharing has been in one direction” (p. 469).

The purpose of this chapter is to showcase the models, theories, and frameworks developed by researchers, past and present, in the field of mathematics education. Presenting these developments, in a venue such as the book we propose, will provide an opportunity, perhaps, for “the sharing” to start to occur in both directions. Beyond the hope of promoting a reciprocal relationship between psychologists and mathematics educators, the chapters in this section represent the (mathematics education) foundation for future theories, frameworks, and models. Further, this section, coupled with the previous section focused on current psychological research, will help shape the future cooperation and cross fertilization of theoretical models and research methodologies, which will, eventually, result in current, timely, and evolved theories of probabilistic thinking in both fields.

5. The mathematics education perspective: Probabilistic concepts

The purpose of this section is to provide an opportunity for individuals to present current research on individuals’ thinking with respect to particular probability concepts, which include: conditional probability, sample space, probability distributions, probabilistic reasoning, the law of large numbers, probability models, outcomes, events, and comparisons of relative likelihood. Given the recent (world-wide) focus on the teaching and learning of probability (Jones, Langrall & Mooney, 2007), this chapter will provide an important foundation for future investigations into particular topics found in probability.

6. The randomness perspective

Probability models are a way to mathematically represent random phenomena. As such, probability can be considered as the study of randomness or, alternatively stated, randomness is the foundation of probability. However, randomness is not easily defined and the term has different meanings in different fields. Nevertheless, it is agreed upon that randomness and probability are inextricably linked. While investigating probabilities based on randomness is the work of mathematicians and statisticians, researchers in psychology and mathematics education, taking a slightly different approach, investigate probabilities based on *perceptions* of randomness.

As is the case with randomness and probability, perceived randomness and probability are also inextricably linked. Research has demonstrated that probability judgments (especially in the case of relative likelihood comparisons between equally likely outcomes) are often based on perceptions of randomness. For example, certain individuals contend that the lottery ticket 1 2 3 4 5 6 is less likely to win than the (equally likely) lottery ticket 4 8 15 16 23 42 because the former ticket appears less random and, as such, is deemed less likely. Researchers, attempting to account for mathematically incorrect responses to tasks such as the example presented above, have developed numerous theories and cognitive models, which explain how perceptions of randomness influence probability judgments.

There are numerous influences on an individual's perception of randomness. As such, this section will also present unique perspectives on how individuals interpret randomness and how, if at all, notions such as aesthetics and determinism influence not only our perceptions of randomness, but also the impact on our probabilistic judgments.

7. The combinatorics perspective

Mathematics educators have long recognized the value of discrete mathematics and nurturing combinatorial thinking in the school curriculum. Discrete mathematics, in particular combinatorics, unlike continuous mathematics, is accessible to students starting at the elementary levels because it builds from simple enumerative techniques. In an often quoted survey article in the literature, Kapur (1970) argued for the inclusion of combinatorial mathematics in the school curriculum for the following reasons:

1. Combinatorial mathematics is independent of calculus.
2. Combinatorics is useful to teach "concepts of enumeration, making conjectures, generalizations, optimization...and <u>systematic thinking</u>." (p.114)
3. Combinatorics has numerous applications to the physical, natural and computing sciences, probability, number theory, and topology.
4. Combinatorial mathematics not only creates opportunities for using computing tools, but also illustrates the limitations of such tools. For instance, several families of numbers such as binomial coefficients, the Fibonacci numbers, partition numbers etc. are derived from recurrence relations. Conjectures and some plausible proofs of such conjectures can be constructed without advanced mathematical training.
5. Many combinatorial problems and their applications illustrate recent developments in mathematics, thereby allowing students to develop a feeling for how mathematics grows.

In a similar vein, the Curriculum and Evaluation Standards (NCTM 1989) called for increased emphasis on accessible topics in discrete mathematics in Standard 12 for the 9-12 school curriculum. The Standards identified topics in discrete mathematics that secondary teachers could integrate at appropriate stages of the school curriculum.

> *"Computers are essentially finite, discrete machines, and thus topics from discrete mathematics are essential to solving problems using computer methods. In light of these facts, it is crucial that all students have experiences with the concepts and*

methods of discrete mathematics. " (NCTM 1989; 9-12 Curriculum Standard 12).

The NCTM followed its call for increased emphasis in discrete mathematics with the timely release of the 1991 Yearbook. The Yearbook contains a variety of topics in discrete mathematics that are accessible to students from the elementary to the high school level, with classroom activities that practicing teachers can incorporate in the K-12 curriculum. A natural question to explore twelve years later is, what has been accomplished by the mathematics education community? In particular what types of mathematics education research have been conducted in the area of combinatorics? What do these research studies show about the capacity of students to understand combinatorics? What do studies show about incorporating combinatorics in the curriculum? Research on the learning of combinatorics can be traced back to Piaget's research on the evolution of the notion of chance (probability) in children (Piaget & Inhelder, 1951). Piaget and Inhelder viewed combinatorial thinking as an aspect of the stage of formal operations where combinatorial reasoning was characterized as the capacity to determine all the possible ways in which one could link a given set of base associations with each other. Batanero, Navarro-Pelayo, and Godino (1997) provided a simple and highly illustrative account of Piaget and Inhelder's thesis on combinatorial reasoning: Given a problem where a set of objects are required to be arranged in all possible ways, children at the pre-operational stage use random listing procedures, without having an explicit systematic strategy. At the concrete operational level, children use trial and error strategies and are capable of devising "empirical procedures with a few elements." Finally at the stage of formal operations "adolescents discover systematic procedures of combinatorial construction, although for permutations, it is necessary to wait until children are 15 years old" (Batanero et. al 1997, p.182).

The findings reported by Maher and her colleagues validate the Piagetian notion of how combinatorial reasoning evolves in problems requiring a set of objects to be arranged in all possible ways. These longitudinal studies revealed that students' strategies evolved from random listing strategies and other trial and error or "empirical procedures" as fourth graders, to systematic counting strategies as tenth graders. This compares with the findings of Lyn English, except in her studies, the children developed sophisticated strategies across a set of tasks within the period of task administration. Increasing notational sophistication, a disposition to think abstractly, the ability to generalize and an affinity for constructing proofs characterized the evolving strategies of the students in Maher's studies (e.g., Maher 1993; Maher & Martino 1996a, 1996b; Maher & Speiser 1997), These findings do not simply confirm the findings of Piaget but in a sense reveal that the development of combinatorial reasoning "accelerates" from grades four to ten. The Piagetian model spans an eleven-year time period, whereas the findings of the longitudinal studies conducted by Maher and her colleagues indicate that with appropriate instruction, students' combinatorial thinking can evolve into sophisticated structures in only seven to eight years. It should be noted that this increased development is dependent on the use of appropriate tasks in order to facilitate this development in a much shorter time span. Maher indicates that the longitudinal study is now in its 17th year, and her research group's willingness to contribute and substantiate this particular section of the book (personal communication).

This section will include the most recent and summative findings from researchers in the domain of combinatorial problem solving and reasoning as it relates to probability topics in mathematics.

8. Other

Embracing the spirit of (1) diverse perspectives inherent and important to the structure of this book and (2) the forward looking nature of the series "Advances in Mathematics Education," this section provides an opportunity for individuals to present, what we consider, unique and timely research on probabilistic thinking. Through our examination of research during the Contemporary phase, the connection to burgeoning ideas in the field of mathematics education were underrepresented. As such, this sections provides an opportunity for individuals to connect recent developments (in areas such as: modeling, neuroscience, metaphor, complexity, gesture, embodiment) to the domain of probabilistic thinking.

9. Revisiting influential pieces

As presented in the introduction, despite having a relatively short history, there are a number of seminal pieces of literature on research involving probabilistic thinking. The purpose of this section is to provide an opportunity for both original authors and certain other individuals to, depending on the author, revisit or comment on the impact these pieces have had on the field and probabilistic thinking. The purpose of this section is to ensure that knowledge accumulation occurs in the book to ensure continuity in the lines of thought developed in the book. Copyright warranty permissions will need to be secured, but can be accomplished by enlisting the very authors that wrote the original pieces and providing them an opportunity to recast their work to the newer generations of researchers.

10. Wish list section

A variety of research (e.g., Jones et al., 2007; Kapadia, 2009; Shaughnessy, 1992, etc.) involves a "wish list." In these wish lists authors are able to set the future research agendas for the field. However, items in the wish list are not always addressed in subsequent year. For example, in the first *Handbook of Research on Mathematics Teaching and Learning* (Grouws, 1992), Shaughnessy's seminal (1992) review of "research in learning and teaching stochastics" (p. 466) concluded with "a wish list" (p. 488) for future research. Included in the list, a call for investigation into teachers' conceptions of probability. Fifteen years later, in the *Second Handbook of Research on Mathematics Teaching and Learning* (Lester, 2007), Jones, Langrall, and Mooney (2007), who were given the "main task of reviewing and analyzing research in probability education during the period since Shaughnessy's (1992) review" (p. 910), included "Stohl's (2005) review[, which] concluded that there had been limited response to Shaughnessy's call for research on teachers' knowledge and beliefs about probability" (p. 945), in their research synthesis. Given the example presented above, and others, this section is dedicated addressing previous wish list items. Having examined wish lists from existing literature, we have chosen a number of the topics (e.g., language and chance, teachers' probabilistic knowledge, etc.), which have either flown under the wish list radar or have not been sufficiently addressed since the list was created.

Commentaries

The commentaries (and the discussion) section demonstrates the strength of the book's structure (i.e., the cubist framework, which presents diverse perspectives to represent the subject in a greater context). As presented, each of the ten sections of *PT: PPP* begin with a preface, consist of a number of diverse chapters, and conclude with a commentary from a prominent individual within the mathematics education community (e.g., Uri Leron, Pat Thompson, Rina Zazkis). The decision to conclude each section of the *PT: PPP* with a commentary was made to provide certain individuals, those who are well versed in mathematics education research, an opportunity, upon their reading of the chapters of their section, to present (i.e., give a commentary) their (learned) interpretation of the diverse perspectives presented throughout the section. Alternatively stated, while reading each of the chapters in each section *implicitly* provides an opportunity to represent the subject (i.e., the section) in a greater context, the commentary at the end of each section *explicitly* represents the section in a greater context. However, an explicit expression of greater context is not the only advantage to the ten commentaries, which conclude each section.

Similar to the argument presented above, reading each of the 10 commentaries (from each of the 10 sections) implicitly provides an opportunity to represent the subject (i.e., probabilistic thinking) in a greater context. Strategically, the commentaries at the end of each section of *PT: PPP* have a dual purpose. First, they are an explicit representation of each section in a greater context. Second, the commentaries themselves represent diverse perspectives, which, when read together, represent our subject, that is, probabilistic thinking, in a greater context. However, reading each of the ten commentaries implicitly provides an opportunity to represent probabilistic thinking in a greater context. As such, the five different commentaries in this section – from prominent individuals from the history of research into probabilistic thinking (e.g., Watson, Shaughnessy, Borovcnik, Kapadia, and Jones) – explicitly represent probabilistic thinking in a greater context.

Discussion

Continuing, once again, the argument presented above, the five commentaries of the Commentaries section have a dual purpose. First, they are an explicit representation, from a prominent individual involved in probability research, of probabilistic thinking in a greater context, which is derived from the reading (each chapter and) the commentaries at the end of each section. Second, the five commentaries themselves represent diverse perspectives, which, when read together, will once again represent probabilistic thinking in a greater context. However, when read together they implicitly provide an opportunity to represent probabilistic thinking in a greater context. As such, the purpose of the Discussion section will be for Bharath and myself to explicitly provide a greater context of probabilistic thinking from the five commentaries found in the previous section. Discussing the five commentaries in the previous section will not, however, provide justice to the unique structure of the book or our subject. The nested structure created with our chapters, section commentaries, final commentaries, and, ultimately, our final discussion, will require us to consider (separately and collectively, implicitly and explicitly) all possible perspectives. Through our commenting on these plural perspectives, we contend that our discussion will represent probabilistic thinking in a greater context.

Given the size of the proposed book, the structure we propose has the unique ability for sections to be able to stand alone and, simultaneously, exist as part of a larger picture. It works best when divided into two volumes.

Competing Books

Currently there is a paucity in the literature on probabilistic thinking in mathematics and mathematics education in general since the release of Jones (2005) book – the book was already outdated at its time of release because of the significant amount of research it overlooked that was developing concurrently within the timeline of that particular book. While some research has "st(r)ayed" in the realm of stochastical thinking and statistical thinking and reasoning, this book deliberately takes the path less chosen, that is, explores the roots and different facets of probabilistic thinking and fertile directions in which the research can be pushed. The book we propose is both anthological and future oriented, with the explicit purpose of becoming *the* standard reference book for mathematics education and mathematics. The cast of proposed authors include luminaries from the field of mathematics, philosophy of science and logic, psychology, in addition to the significant researchers from mathematics education working at the nexus of probability and other topics within mathematics education. Another redeeming feature of the book is the section in which the classical/influential pieces from the existent literature is visited anew by the various authors.

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The Falk Phenomenon
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Heuristics: A Different Perspective
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The Origin of the Idea of Chance in Children (1951/1975)
Piaget and Inhelder

The Intuitive Sources of Probabilistic Thinking in Children (1975)
Efraim Fischbein

Children's Conceptions of Probability: A Psychological and Pedagogical Review (1984)
Anne Hawkins and Ramesh Kapadia

Difficulties in learning basic concepts of probability and statistics: Implications for research (1988)
Joan Garfield (on behalf of Ahlgren)

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Kapadia & Borovcnik

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J. M. Shaughnessy

Probability (1996)

Borovcnik and Peard

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Graham Jones

Research in Probability: Responding to Classroom Realities (2007)
Graham Jones, Cynthia Langrall and Edward Mooney

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