PRE-PRINT OF CHAPTER [B2] TO APPEAR IN

The forthcoming ***Third International Handbook of Mathematics Education***, Springer 2012.

Edited by

M. A. (Ken) Clements, Alan Bishop, Christine Keitel, Jeremy Kilpatrick, and Fred Leung.

Theories in Mathematics Education: some developments and ways forward[i]

Bharath Sriraman*

The University of Montana

Elena Nardi*

University of East Anglia

Abstract: *In this survey of the state of the art, roots of mathematics education are traced from Piaget onto the current work on theorizing which utilize sociological and commognitive frameworks. Attention is given to the critiques of "Theories of Mathematics Education" (Sriraman & English, 2010), and productive discussions from the reviews are unpacked. The notions of "operational" versus "functional", and "models" versus "theories" are also tackled by focusing on conceptual frameworks that harmonize the terms as opposed to exemplify their polarities.*

Keywords: Models of Mathematics Education; Theories; Stage theories; Discursive frameworks; Philosophy of Knowledge; Structuralism; Conceptual Frameworks in Mathematics Education; Local theories versus Global theories; Operational definitions

Introduction

This chapter flows from the extensive discussion of *Theories of Mathematics Education: Seeking New Frontiers* (Sriraman & English, 2010), in book reviews and critical notices in several major outlets (Artigue, 2011; Ely, 2010; Fried, 2011; Jankvist, 2011; Schoenfeld, 2010; Umland, 2011). The dialogic form in which the community of scholars has undertaken theoretical developments in the field of mathematics education has allowed for the necessary intellectual critique needed to advance the field.

Mathematics education as a research discipline occurs at the nexus of numerous other domains of inquiry (Sriraman, 2008; Sriraman & English, 2010). Unlike other fields such as the natural and physical sciences, hermeneutic continuity in theory development is by and large absent in the learning sciences. While psychological theories served as the theoretical underpinnings of the field in the 1950’s and 1960’s, methods from other domains of study such as sociology, anthropology, cultural historical studies, have swayed mathematics education into multidisciplinary and uncharted directions. This contributes to both the complexity inherent in a field that deals with cognizing and socially situated subjects within the larger contexts of institutions and culture, as well as cause to celebrate the multidisciplinary nature of mathematics education. The perceived objectivity of mathematics coupled with the perceived subjectivity of the social sciences makes theory development in the field of mathematics education a particularly nuanced and complex task. In general, theories have been a troubling issue for the field of mathematics education, namely the “bewildering array of theories, theoretical models, or theoretical frameworks” (Jablonka & Bergsten, 2011) abundantly found in the literature that characterizes research today.

In this chapter we discuss briefly critiques of *Theories of Mathematics Education* (Sriraman & English, 2010) as well as address both the psychological foundations and the 'social turn' (Lerman, 2000) (or 'social brand' ((Jablonka & Bergsten, 2011)) in mathematics education. We also discuss different schools of thought on what theory means in mathematics education, and our assessment of ways in which we are progressing. Our overall observations of the field indicate that there seems to be increasingly more effort to employ theoretical frameworks with a more pronounced focus on institutional and social dimensions (e.g., ATD, the Anthropological Theory of Didactics as derived from the work of Brousseau and Chevallard), and discursive practices (e.g. Sfard's commognitive framework). Piagetian foundations are also in need of being revisited and revitalized given the canon of experimental studies in the last two decades (e.g., the Rational Number Project; Models and Modeling) that have developed models that can be subsumed within the larger theory. We begin with a brief presentation of critiques of *Theories of Mathematics Education* (Sriraman & English, 2010).

Critiques of *Theories of Mathematics Education*

Theories of Mathematics Education (Sriraman & English, 2010) grew out of a research forum at the 29th meeting of the International Group on the Psychology of Mathematics Education in 2005 and of a selection of articles previously published in *ZDM - The International Journal on Mathematics Education*, between 1994 and 2008, many of which were in the two special issues of ZDM on theories published in 2005 and 2006 (Sriraman & English, 2005, 2006). Some of the early parts set the scene through reflection on the role of theory in mathematics education research, on theory pluralism etc.. In resonance with the volume's eclecticism most parts can be read independently. Through its structure, both across and within the 19 parts, the book invites the reader to engage in a dialogue about theoretical issues.

The book has been reviewed in several key mathematics and mathematics education publications. While reviews have acknowledged its considerable ambition and strengths (e.g.: Artigue, 2011; Ely, 2010; Fried, 2011; Jankvist, 2011; Schoenfeld, 2010; Umland, 2011), in this section we focus on those constructively critical parts of some of the reviews that have led partly to the shaping of this chapter.

A first concern (Artigue, 2011) regards the presentation in the book of the 'French tradition of didactique des mathématiques' (e.g. in Part I). As Artigue notes 'Seen from the outside the [French] theoretical landscape seems homogeneous' (p311). However 'those who know this culture are well aware that the situation is much more complex, that different local theoretical constructions have emerged from TDS [Theory of Didactical Situations] and ATD [Anthropological Theory of Didactics] or from the connection of these with other frameworks such as Vergnaud's theory of conceptual field, the theory of activity or cognitive ergonomy' (p 311-2). Furthermore, 'most researchers productively combine different theoretical approaches in their research work, for instance joint action theory (Sensevy & Mercier, 2007, Sensevy, 2009), the didactic-ergonomic approach of teachers' practices (Robert & Rogalski, 2005), or the instrumental approach of technological integration (Artigue, 2002; Guin, Ruthven & Trouche, 2004))' (p 312). To redress the balance a little, in this chapter we have attempted a, necessarily brief, abridged account of recent developments in the areas highlighted in Artigue's comment. This only partially addresses Artigue's comment that the educational discourse in the *Theories* volume is rather 'too culturally connoted' (p313) and that the contribution of other, non-Anglo Saxon approaches remain largely 'invisible' (p315) to the international community - a concern has been echoed in historical treatments of the development of mathematics education in non-Anglosaxon parts of the world (e.g. Germany, France and Italy in (Sriraman & Törner, 2008)).

Another concern, also in (Artigue 2011), regards an impression that a reader may garner from the book that ‘investigating the relationships between […] different theoretical approaches and developing networking activities’ (p 312) is somewhat a novel enterprise. As Artigue notes, the *didactique* community has been dedicating parts of its energy to this task since the early 1980s. Furthermore, in recent years, theory ‘networking’, substantially addressed in Parts XV and XVI of the book, has been the focus of several European projects and work within ICMI. We comment briefly on these consideration of these in the final section of the chapter where ‘networking’ is the main focus.

A third concern, again in (Artigue 2011), regards another impression that a reader may leave the book with: the somewhat light-touch approach to the pitfalls of excessive eclecticism. We agree with Artigue that ‘theoretical diversity can only be a richness if it is not synonymous to theoretical fragmentation or eclecticism’ (p312). We share with her the concern about ‘the theoretical explosion in mathematics education’ (p312) and we recognize the need to ‘reflect on the way we tacitly contribute to it as a community when we value much more the creation of new theoretical entities than the hard work often required in order to appropriate and adequately exploit what has been already built by other researchers and communities’ (p312). To this purpose we conclude this chapter with a call to mathematics education researchers for a more incisive approach to, and employment of, theory that goes beyond knowing the ‘grammar’ (Lerman, 2010, Part IV, p101) of a theory, a trend that Lerman and other contributors to the book identified as a symptom of careless eclecticism. As observed in Umland’s review (2011) ‘a well-articulated set of linked and nested theoretical structures’ (p.74) need not be perceived as an out-of-reach, or totally undesirable goal of mathematics education research.

A valuable observation made in several reviews of the volume (e.g. Artigue, 2011; Umland, 2011)) has been that any enterprise with a focus on the generation and employment of theory in mathematics education must consider the interplay between general (and meta-) theoretical developments and theoretical developments within specific areas of research in mathematics education (a distinction made in (Artigue, 2011)). As Umland (2011) observed this enterprise often entails considering 'reflections on the philosophical foundations of mathematics education as a field' (meta-theory), 'theoretical perspectives from other disciplines that could be brought to bear on mathematics education research' and 'proto-theories'(p.73), descriptors of the processes underlying the teaching and learning of mathematics. In this chapter our focus is on the former part of Artigue's distinction (the second and third of Umland's distinction), even though the examples we mention originate in studies conducted within specific areas of research. Finally as several commentators have noted (e.g. Umland 2011) any enterprise with this focus needs to allocate some attention to what the authors mean by 'theory'. We do so, modestly, in the Introduction and other sections of this chapter.

Dynamic interactionism between models and theories: the case of Piaget's notion of 'operational'[ii]

Piaget's theory of cognitive development in children serves as one of the barometers through which we can analyze both model and theory development in mathematics education, particularly in ensuing experimental work in North America in various longitudinal projects funded by the National Science Foundation (see Lesh 1987, 1989, 1992, 2003; Lesh & Doerr, 2003). Simply put, the function of any theory is to explain phenomena. The natural and physical sciences have an established body of theories which have been validated over time through scientific experimentation and conforming data. Theory development has also been spurred by falsification (Popper, 1959). Newer theories are able to subsume older theories. Newtonian mechanics occurs as a special case in Hamiltonian mechanics. Euclidean geometry can be reduced to a special case of Riemannian geometry. Weyl's (1918) mathematical formulation of the general theory of relativity by using the parallel displacement of vectors to derive the Riemann tensor reveals the interplay between the experimental (inductive) and the deductive (the constructed object). The continued evolution of the notion of tensors in physics/Riemmanian geometry can be viewed as a culmination or a result of the flaws discovered in Euclidean geometry.

On the notion of Operational in Mathematics Education

Mathematics education, however, is neither the natural/physical sciences, nor is it mathematics per se. In mathematics education one often has to ask the question what are the phenomena that we are trying to explain (Schoenfeld, 2010). One answer to this question in relation to Piaget's body of foundational work is mathematical cognition. For instance mathematics education has developed empirically validated models within the Rational Number Project (see Lesh et al.,1987, 1989, 1992,2003) that explain how proportional reasoning develops in children that by and large cohere with Piaget's stage theory. There are models that also explain how combinatorial reasoning develops (e.g., Sriraman & English, 2004) and in this canon of studies some of the findings suggest that when Piaget's experiments are repeated with age appropriate materials, the stages proposed by him are not as discrete as they seem, but more porous with the possibility of children being able to reason at a more advanced level given contextual play materials. Dienes' six stage theory of learning mathematics bears resemblance to that of Piaget as well as Bruner, with a somewhat *different* conceptualization of what “operational” means (Dienes, 1960,1963, 1964,1971, 2000; Dienes & Jieves, 1965). Similarly the recent body of work by the models and modeling group build on operational definitions within the work of Dienes to develop models of student thinking in contextual problems. When viewed from the biological perspective of neural networks, many of the post-Piagetian body of experimental work in mathematics education can be made to cohere within his stage theory with a critical number of exceptions that warrant reconceptualization as a more dynamic theory with the possibility of the contexts allowing children to function at a higher stage. However a foundational point of continued argument remains the working definition of “operational”.

Biologists have found that methodological reductionism, i.e. going to the parts to understand the whole, which was central to the classical physical sciences, is less applicable when dealing with living systems. Analogously, the challenge confronting mathematics educators in the learning sciences who hope to create models (of the underlying conceptual systems) that students, teachers and researchers develop to make sense of complex systems occurring in their lives is: the mismatch between learning science theories based on mechanistic *information processing* metaphors and recent discoveries on how complex systems work. Not everything that students know can be methodologically reduced to a list of condition-action rules, given that characteristics of complex systems cannot be explained (or modeled) using only a single function - or even a list of functions. As physicists and biologists have proposed characteristics of complex systems arise from the <u>interactions</u> among lower-order/rule-governed agents – which function simultaneously and continuously, and which are not simply inert objects waiting to be activated by some external source (Hurford, 2010).

Given the paradigm shifts that have occurred in the physical and natural sciences, there have been proposals to view learning and the modeling of learning as analogous to the study of complex systems (Mousoulides, Sriraman, Lesh, 2008). Piaget studied children to fundamental questions about the nature and origins of knowledge. His focus was the child's understanding of space, time, and causality, and relations of invariance and change (Piaget, 1971, 1975). Trained as a biologist, he was also interested in exploring the "metaphors" from biology such as organization, development, and adaptation. Piaget's developmental theory consisted of sensorimotor, pre-operational, concrete operational and formal operational stages. Operations were defined as internalized actions, derived directly from the subject's physical actions as enacted in sensorimotor behavior. Consider the stage of generalized formal operations characterized by the organization of operations in a structural whole and the culmination of the sensorimotor, pre-operational, and concrete operational stages. Piaget (1958) suggests that at the stage of formal operations, there is a "structural mechanism" which enables students to compare various combinations of facts and decide which facts constitute necessary and sufficient conditions to ascertain truth. Those that are able to transform propositions about reality, such that the relevant variable can be isolated and relations deduced are said to have achieved "functionality" in their structural flexibilities. Another characteristic of the stage of generalized formal operations is the relative ease with which reversibility of thought operations occurs. Piaget essential claim was that there was a link between mathematics and biology. In other words operationality became functionality after reversibility in order to be able to generalize mathematical structures or subsume classes of counter examples into an existing structure. This was a highly unusual claim 60 years ago but one that he tried to substantiate with decades of research. Piaget's characterization of knowledge and cognition is therefore: forms of biological adaptation within which structures of *action*(evolving upwards from individual sensori-motor schemes) play a role.

An operational definition of mathematics (no pun intended) is that it is an intellectual activity concerned with the creation of structure, with new characterizations that emphasize embodiment and anchoring in culture. Piaget (1958, 1987) conceptualized the whole of mathematics in terms of creation of structures, not in a physical or literal sense but operations carried out in the idealized world of the mathematician. The relationship between the two worlds was explained as follows: the idealized constructions emerge as a result of a series of abstractions from their literal counterparts, which are the real actions and physiological movements human beings make in the world. Piaget's psycho-genetic account of mathematics retraces this descent from actions to formal thinking as one of increasing abstraction and generalization. Being enamored by the ongoing attempt of the Bourbaki[iii] at that time to fomalize all of mathematics, Piaget compared his operator structures of thinking to the structures espoused by the Bourbaki. The Bourbaki identified three fundamental structures on which mathematical knowledge rests. They are (1) algebraic structures; (2) structures of order; and (3) topological structures. Piaget claimed that there existed a correspondence between the mathematical structures of the Bourbaki and the operative structures of thought. He felt that in the teaching of mathematics a distinctive synthesis would occur between the psychologist's operative structures of thought and the mathematician's mathematical structures. Dubinsky (1991) used the Piagetian notion of *reflective abstraction,* to develop a model using "schemas" as a way of better understanding cognitive processes in advanced mathematics. The APOS theory developed in mathematics education to explain advanced mathematical thinking is another instance of theory development cohering within a larger theory. Within the realm of mathematical cognition there are micro-theories, or what should more accurately be labeled models that build on Piaget’s work and inform it sufficiently that it is possible to re-conceptualize the theory as being more dynamic.

The only contentious point in reconceptualizing Piaget is consensus on the definition of "operational" thinking and putting it on firm theoretical ground.

In physics, one could say that theoretical terms are invariants of operations represented by physical measurement devices. Physicists have "learned" (pun-intended) that theoretical terms have to be defined operationally in terms of theories provided experimentation can back up notions occurring within the theories. (Dietrich, 2004). The question is how can this be adapted by researchers in mathematical cognition? That is, how can we operationally define observational terms (namely perceived regularities that we attempt to condense into theories, or – as Piaget attempted – to phylogenetically evolved mental cognitive operators) (Dietrich, 2004). The purpose of theoretical terms is to clarify the meaning of concepts. On the other hand the purpose of observational terms is to delineate how the concepts/constructs have been measured. Ideally there should be a perfect match between theoretical terms and observational terms, i.e., observations should confirm theory irrespective of when or where the observations are made as long as the initial conditions of an experiment are somewhat the same. For instance, within models and modeling research, if researchers consistently report similar observations of students modeling processes when confronted with the same complex situation across age groups and locations, then these observations can be used to develop a sound theoretical construct. Conversely, when the theoretical construct is tested (or subject to experiment) at a new location, researchers should be able to predict the types of behaviors that will be observed as long as the integrity of the experiment (starting conditions are replicated).

Having developed only slightly beyond the stage of continuous theory borrowing, the field of mathematics education currently is engaged in a period in its development replete with inquisitions aimed at purging those who don't vow allegiance to not always well-defined perspectives on mathematical learning (such as “constructivism” - which most modern theories of cognition claim to endorse, but which rarely generates testable hypotheses that distinguish one theory from another) – or who don’t pledge to conform to psychometric notions of “scientific research” (such as pretest/posttest designs with “control groups” in situations where nothing significant is being controlled, where the most significant achievements are not being tested, and where the teaching-to-the-test is itself the most powerful untested component of the “treatment.”). With the exception of small schools of mini-theory development that occasionally have sprung up around the work a few individuals, most research in mathematics education appears to be ideology-driven rather than theory-driven or model-driven (Lesh & Sriraman, 2005). As Artigue (2011) has pointed out there is also less acknowledgement of non-Anglo Saxon approaches to conceptualizing researchable phenomena in mathematics education.

Theories are cleaned up bodies of knowledge that are shared by a community. They are the kind of knowledge that gets embodied in textbooks. They emphasize formal/deductive logic, and they usually try to express ideas elegantly using a single language and notation system. The development of theory is absolutely essential in order for significant advances to be made in the thinking of communities (or individuals within them). But, theories have several shortcomings. Not everything we know can be collapsed into a single theory. For example, models of realistically complex situations typically draw on a variety of theories. Pragmatists (such as Dewey, James, Pierce, Meade, Holmes) argued that it is arrogant to assume that a single “grand theory” will provide an adequate basis for decision-making for most important issues that arise in life (Lesh & Sriraman, 2005). Models are purposeful/situated/easily-modifiable/sharable/re-useable/multi-disciplinary/multi-media chunks of knowledge. They often (usually) integrate ideas from a variety of theories. They are directed toward solving problems (or making decisions) which lie outside the theories themselves. That is, they are created for a specific purpose in a specific situation. Models are seldom worth developing unless they also are intended to be: (a) Sharable (with other people), (b) Re-useable (in other situations). So, one of the most important characteristics of an excellent model is that it should be easy to modify and adapt, and ideally cohere with a larger theory and in some instances develop it further.

From Social/Institutional Branding to Bernstein and Beyond

Theory is also important as it provides the lens through which to construct and look at data. Mathematics Education Research (MER) theories were originally largely psychological. But recent realisations of the importance of social etc. context have led to the emergence of sociological theories as well. Lerman (2010) draws on these recent traditions, largely Bernstein, to establish the claim that plurality of theories in MER not only is not a problem; it is necessary. Below we outline his argument roughly.

Despite external pressures on educational research, MER is thriving in quantity (journals, conferences etc.) as well as in terms of generated theories. Here Lerman looks at whether this is unusual and beneficial. He does so through an empirical meta-study that draws on the sociological theory of Basil Bernstein. Hierarchical discourses (vs horizontal discourses) are those that require apprenticeship and involve gradual distancing and abstraction (e.g. mathematics whether academic or in school). Horizontal discourses are acquired tacitly and concern specific contexts. Traditional pedagogies are explicit and are about performance. Authority is clearly located in the teacher. 'Reform' pedagogies are often implicit, sometimes invisible and privilege those with prior linguistic wealth (Bernstein's 'elaborated code').Verticality is a concept that describes how knowledge domains grow vertically (new theories replace old ones, as more or less in science) or horizontally (with adding new discourses and theories that develop in parallel and are often incommensurable with previous ones). MER, as any other social science, develops horizontally. Because, unlike mathematics, MER has a 'weak' grammar (but it does have a grammar, you need to know MER theories to do work in it), building theory across the boundaries of prior theories is possible.

Lerman's study was an analysis of papers published in *Educational Studies in Mathematics* (ESM), *Journal for Research in Mathematics Education* (JRME) and presented at conferences of the International Group for Psychology in Mathematics Education (PME). The majority uses theory, mostly traditional psychological learning theory; there are some, a minority, of theoretical papers. There was an increase with time towards more social theories (Vygotskian, ethnomathematical, social/critical, post-structuralistic: what Lerman terms 'the social turn') and the vast majority of authors were content with using the theory, not building upon it, refuting etc. A caveat was that only accepted, published research was looked at. Is there an issue of gatekeeping here? Lerman's proposition is that new theories are new voices and need to be heard. Theory should be used appropriately and do so in a way that also demonstrates concern for practice (unlike any other social science, a bit like Medicine or Computing). One also cannot ignore social and political implications.

In their Commentary to Lerman Jablonka & Bergsten (2010) overall agree with Lerman but also note the following:

1. There are examples of 'strong' grammar theories in MER: Anthropological Theory of Didactics and Embodied Cognition.
2. 'Use of theory' is weakly defined by Lerman. They looked at papers in ESM and JRME and many cite theory but do not use it in any substantial way; or offer common sense re-workings of data, not informed by theory at all!

3. Theoretical hybrids is one way of achieving some level of communication across theories.
4. There is a trend for selective reception of parts of a theory (e.g. accepting importance of bodily based metaphors but not the entire embodied cognition framework).
5. What Lerman calls a Social Turn, is more of a Social Brand.
6. MER is more focused on the ‘knower’ than ‘knowing’ and this can be dangerous.
7. Finally they propose looking at other fields too, moving away from the sometimes insular approach to theorising in MER.

Several treatments of the so-called state of the art of mathematics education in the past have been made by mathematicians and even mathematics educators who have done little or no research in the classroom or other learning settings to substantiate claims. This has created several canons of literature which are by no means easy to differentiate, and sieve out research that advances theory through empirical data, as opposed to theories advocated in the form of rhetoric in introspective articles and those that rely on quotes as a means of data presentation and analysis. This trend is even more evident when pieces of legislation or reports from advisory panels of government bodies become the basis of curricular changes and research programs. For instance the call of the National Mathematics Advisory Panel in the United States to use Algebra as a panacea for curricular ills as well as the stipulation of psychometric aptitude-treatment-interaction based clinical studies as the only genre of acceptable research worthy of funding has led many mathematics education researchers to a blind following of this mode of research without regard for the fundamental problems of the field (see Greer, 2008). This swerving of research focus based on political tides does not bode well for any field of research. One of the main points made by Jablonka & Bergsten (2011) is to carefully build theoretical bases for the field based on ongoing research as well as establish a rigorous framework for theorizing according to a specific research grammar such as Basil Bernstein's internal/external languages of description. Once this is achieved and a sort of coherence pervades the objects of theoretical discussion, there will be a natural and much needed end to cyclically justifying the existence of mathematics education as "research field" every decade or so! (e.g., Sierpinska & Kilpatrick, 1998; Sriraman & English, 2010).

Jablonka & Bergsten (2011) critique the strengths and weaknesses of four ways of theorizing and illustrate what they mean by modes and qualities of theorizing. The four modes of theorizing selected from the literature and presented as examples are diverse enough to cover the spectrum of existing "theoretical" trends. The first example is the PISA framework in which the unclearly defined notion of "mathematization" has become a major constituent of mathematical literacy and despite the weak operationalization of basic notions and disregard and criticisms of the cross-cultural validity of PISA test items, the framework has dangerously mutated into a basis for curricular reform in many countries of the world, when there is generally lack of a theoretical foundation of it. The second example is the theory of authentic task situations taken from Sweden. The arbitrariness of categories (or aspects) chosen in the operational framework is evaluated and an argument made to show that relationships between categories appear vague as well as empirically tenuous. The APOS theory is next used as the third example of a theory that deals with conceptual development in mathematics. This neo-Piagetian theory is appraised as having a relatively strong internal consistency in its grammar and specific theorizing in terms of actions, processes and objects within schemas. Fourthly, finally, they tackle the French school which is the focus of the next section.

The ‘French’ Way

Jablonka and Bergsten (2011) use the Anthropological Theory of Didactics (ATD) to illustrate an example of a theory that uses a specialized language and develops hierarchical relationships between praxeologies. The last theory chosen by the authors, namely ATD, which is also elaborated more in their paper is of particular interest to the community given its ecological nature and the wideness of its applicability. Simply put ATD is the extension of Brousseau’s ideas from within the institutional setting to the wider “**Institutional**” setting [capital I is our choice]. Artigue (2002) clarifies this subtlety by saying that:

> The anthropological approach shares with “socio-cultural” approaches in the educational field (Sierpinska and Lerman, 1996) the vision that mathematics is seen as the product of a human activity. Mathematical productions and thinking modes are thus seen as dependent on the social and cultural contexts where they develop. As a consequence, mathematical objects are not absolute objects, but are entities which arise from the practices of given institutions. The word “institution” has to be understood in this theory in a very broad sense…[a]ny social or cultural practice takes place within an institution. Didactic institutions are those devoted to the intentional apprenticeship of specific contents of knowledge. As regards the objects of knowledge it takes in charge, any didactic institution develops specific practices, and this results in specific norms and visions as regards the meaning of knowing or understanding such or such object (p.245).

The motivation of Chevallard for proposing a theory much larger in scope than Theory of Didactical Situations (TDS) was to move beyond the cognitive program of mathematics education research, namely classical concerns (Gascon, 2003) such as the cognitive activity of an individual explained independently of the larger institutional mechanisms at work which affect the individual's learning. Chevallard's (1985,1992a,b; 1999a,b) writings essentially contend that a paradigm shift is necessary within mathematics education, one that begins within the assumptions of Brousseau's work, but shifts its focus on the very origins of mathematical activity occurring in schools, namely the institutions which produce the knowledge (K) in the first place. The notion of didactical transposition (Chevallard, 1985) is developed to study the changes that K goes through in its passage from scholars/mathematicians→curriculum/policymakers→teachers→students. In other words, Chevallard's ATD is an "epistemological program" which attempts to move away from the reductionism inherent in the cognitive program (Gascon, 2003). Bosch, Chevallard & Gascon (2005) clarify the desired outcomes of such a program of research:

> "ATD takes mathematical activity institutionally conceived as its primary object of research. It thus must explicitly specify what kind of general model is being used to describe mathematical knowledge and mathematical activities, including the production and diffusion of mathematical knowledge. The general epistemological model provided by the ATD proposes a description of mathematical knowledge in terms of mathematical praxeologies whose main components are types of tasks (or problems), techniques, technologies, and theories." (pp.4-5).

It is noteworthy that the use of ATD as a theoretical framework by a large body of researchers in Spain, France and South America resulted in the inception of an International Congress on the Anthropological Theory of Didactics, held biennially since 2005. The aim of this particular Congress and future congresses is to propose a cross-national research agenda and identify research questions which can be systematically investigated with the use of ATD as a framework.

In Sriraman & Toerner (2008) several focal points were isolated via historical analysis to suggest ways in which the theoretical differences between the German, French and Italian schools of thought can be bridged (or networked) and made to interact in the present and future. In outlining the differences and similarities between the various positions and schools of thought in these three countries it became apparent that researchers were often entrenched in "ideological" perspectives. Lerman (2000) explained that these ideological tendencies are a result of the field adopting theoretical frameworks via a process of recontextualization (Bernstein, 1996). In this process "different theories become adapted and applied, allowing space for the play of ideologies" (Lerman, 2000, p.19). However, Jablonka & Bergsten (2011) by using Bernstein's sociological framework, clarify and elaborate on different modes of classification, modeling and theorizing with respect to relational densities among basic concepts within a theory, as well as levels of discursive saturation (or lack of it) in the four examples of theorizing in mathematics education chosen. In mathematics education, there is the preponderance of homegrown theories, the lack of high relational density and intertextuality in the current modes of theorizing. More importantly there is sometimes a tendency of researchers in our field borrowing from terms and concepts fields such as sociology, social anthropology, linguistics etc, without committing to the deeper levels of theorizing that occurs in those fields. We close this chapter with a section on certain approaches to research in mathematics education that appear committed to such deeper level of theorizing: discursive approaches to research in mathematics education. In particular we focus mostly on Anna Sfard's commognitive framework.

Discursive approaches to research in mathematics education: the case of Sfard's 'commognitive' framework

Compared to, say, two decades ago, reports of research in mathematics education are substantially different: longer, often qualitative and examining learning no longer merely in terms of individual acquisition or construction but with ample reference to the context in which the learning occurs. Focus on environmental factors of learning has often meant shifting to a focus on communication and language; in other words our examination of learning has become *discursive*. In this section we outline this opening up of the field to discursive approaches – an opening up that was identified also quantitatively by Ryve (2011, p. 176)) in the increase of studies employing such approaches in recent years. We do so by tracing – be necessity rather selectively – some relevant developments and we focus on one framework that has been attracting increasing attention in recent years, Anna Sfard's (2008) *commognitive framework.*

Discursive approaches to research in mathematics education are one of recent steps in a trail of attempts to describe human thinking that goes back at least as far as behaviourism, the work of Jerome Bruner, artificial intelligence metaphors and models and Piaget's genetic epistemology. These works, despite often remarkable successes in advancing our understanding of how human beings think, did not always succeed in providing adequate explanations of persistent learning behaviours such as individual or collective failures in mathematics. Over the last 40 or so years these approaches to research, sometimes labelled en masse as ‘cognitivist’, have been met with increasing doubt both on methodological and epistemological grounds.

With regard to the former (methodological grounds) their clinical-experimental methods were deemed too remote from the milieu (of the classroom, the home, etc.) where mathematical learning typically occurs. And it is around that time that ethnographic approaches to examining learning were starting to emerge in the form of, for example, observation of ordinary practices (of learners, teachers etc.), gradually replacing laboratory-based data collection.

With regard to the latter (epistemological grounds) doubt was cast on the appropriateness of the ways in which these approaches view learning – namely that human thought somewhat mirrors nature, reflects external phenomena and therefore learning can be described as universal and context invariant acquisition of knowledge about these external phenomena. The sociocultural perspective, pioneered by Vygotsky, and with the work of Wittgenstein, Schutz and Mead providing its epistemological and philosophical grounds, largely opposes this view and emphasises learning as activity occurring in (and co-constituted by) a situational, cultural and historical milieu. As the editors of *Learning Discourse: Discursive approaches to research in mathematics education* (Kieran, Forman & Sfard, 2002) note, the discursive perspective espouses this socio-cultural tenet. Furthermore, its emphasis is firmly placed on thought as a type of communication:

> ‘within the discursive framework, thinking is conceptualised as a special case of the activity of communication and learning mathematics means becoming fluent in a discourse that would be recognised as mathematical by expert interlocutors’ (ibid, p. 5).

Obviously, sweeping discursive approaches under an apparently unified, umbrella term such as ‘the discursive framework’ does not do justice to the diversity of these approaches. This diversity is evident, for example, in the seven chapters of the 2002 book, as well as in more recent efforts to delineate the many and varied strands of discourse-oriented research such as Sfard’s entry on Discourse in the *Encyclopedia of the Sciences of Learning* (Seel, 2012); Jaworski & Coupland’s (2005) *The Discourse Reader*; and, within mathematics education Andreas Ryve’s (2011) formidable review of 108 papers that report research deploying a discursive approach. The 2001 *Educational Studies in Mathematics* Special Issue that the 2002 Kieran et al volume originates in had the subtitle ‘bridging the individual and the social’ and, indeed, the impression is that the editors extend an open invitation to the reader to reflect on ‘the social nature of the individual’ (p. 10). To do so, and to carry out research within this complex framework, is a formidable task. For the rest of this section we focus on the recent work to this purpose of one of that volume’s contributors, and her colleagues, Anna Sfard (Ben-Yehuda, Lavy, Linchevski, & Sfard, 2005; ibid, 2002; Sfard, 2007; Sfard, 2008; Sfard & McClain, 2002; Sfard & Prusak, 2005; Sfard, 1987; Sfard, 2002). On the way we aim to illustrate the potency and some limitations of the discursive perspective. Note: Parts of the next few paragraphs have been adopted from a review-essay on the 2002 volume by one of us (Nardi, 2005).

Beyond recognising communication in mathematical learning as an *aid to*, or a component of, thinking, Sfard’s take on communication is as ‘almost tantamount to the thinking itself’ (Kieran et al, 2002, p. 13). Sfard distances herself from a perspective on learning as acquisition of some entities (e.g. concept, schemes, etc.). Learning here is seen as change in one's participation in well-defined forms of activity. The shift to this perspective is necessary simply because the perspectives employed so far have failed to provide satisfactory accounts for problems of both theory and practice. Two examples from classroom data in (Sfard, 2002) aim to highlight two such problems. One concerns our limited ability to explain failure, and success, in mathematical learning, despite extensive work on students' perceptions of, and difficulties with, specific mathematical topics. The other concerns our limited ability to establish pedagogical practices that warrant understanding (as opposed to task completion). Both highlight our limited insight into what determines the ways in which interlocutors (in this case, students and teachers) choose to proceed in a mathematical conversation. We note that in more recent work, such as the 2008 book, Sfard has provided many more examples – see, for instance, the five quandaries (p. 263-275) she discusses in Chapter 9.

So what is it then that the learning-as-acquisition metaphor, the Piagetian account – the equation of understanding with 'perfecting mental representations' and learning-with-understanding with 'relating new knowledge to knowledge already possessed' (Sfard, 2002, p. 21) – that succeeded behaviourist ones has left unaccounted for? The shift from behaviourism to cognitivism was clearly useful, Sfard stresses, and the approach proposed here, originating in the work of Vygotsky, intends to complement the latter in order to enrich our understanding of the issues that cognitivism has left unresolved.

The basic difference between cognitivist and participationist perspectives on learning is that the latter dispenses with the belief in the existence of context-free cognitive invariants, which underlies the former; and, that this latter perspective emphasizes the social origins of human learning (Sfard's account (2002, p. 22-25) is lucid on the differences and the complementary nature of the two frameworks). Thinking in these terms is conceptualised as communicating with oneself, whereas communication may be diachronic or synchronic, with others or with oneself, predominantly verbal or with the help of any other symbolic system' (p. 28). If 'communicational' (a term which is explained here as epistemologically distinct but akin to 'discursive', and was later, e.g. in the 2008 volume, replaced by 'commognitive', see later in this section) psychology of human thinking posits that speech is no longer a window to thought but its determining element, then as long as thought is language, the two – the thought and the speech –are inseparable. The interlocutor is determined by and influences too the situation in which the communication takes place. In this sense learning is initiation to a discourse – where discourse is meant as a type of communication that characterizes a particular community. In this sense we can talk about the discourse of science, or the discourse of a professional group or a social class, Mathematical learning is then initiation into mathematical discourse.

Sfard introduces two factors that determine this initiation: ‘mediating tools’ and ‘metadiscursive rules’. Within mathematical discourse the former include the ‘artefacts’ used ‘as part and parcel of the act of communication’, e.g. the language and symbols of mathematics; the latter include the tacit ways that ‘guide the general course of communicational activities’ (p. 29), that determine what is ‘appropriate in a given context’ and what is ‘behaviour that would look out of place’. Within mathematical communication acts such as defining and proving in certain well-defined ways count as appropriate; making an unsubstantiated claim does not. Sfard clarifies that meta-discursive rules are not a novel concept: they are akin to Wittgenstein’s ‘language games’, Bourdieu’s dispositions’ (which taken together constitute the ‘habitus’ within which communication takes place) as well as other sociological and ethnomethodological concepts. Meta-discursive rules are thus to be seen as ‘an almost self-explanatory term supposed to encompass all the phenomena signalled by’ these concepts (p. 30).

Sfard closes her introduction to what at the time was called ‘the communicational framework’ with a cautionary methodological remark. Wittgenstein wisely attacked mentalism – any reference to ‘mental states’ and the inherently unobservable entities ‘in the mind’ (p. 32). However, as ‘experiences, feelings and intentions are central to all our decisions’ (p. 32), research must find ways of incorporating those in its accounts. Doing so is ‘safe’, Sfard suggests, ‘as long as it is understood that the status of any claim about other people’s intentions the researcher can make is *interpretive* [Sfard’s emphasis]’ (p. 32). All research can offer is compelling, cogent, trustworthy researchers’ *interpretations*.

A key contribution, in our view, of this approach has been in the research methods proposed as a means towards generating aforementioned interpretations. These include ‘focal analysis’ and ‘preoccupation analysis’:

- ‘Focal analysis’ involves an analysis of the ‘effectiveness of communication’ between interlocutors with regard to the degree of clarity of the ‘discursive focus’ – defined as ‘the expression used by an interlocutor to identify the object of her or his attention’ (p. 34). There are three ‘focal ingredients’ considered in this type of analysis: ‘pronounced’ (what one is attending to), ‘attended’ (how is one attending to what one is attending) and ‘intended’ (the collection of experiences evoked by the ‘pronounced focus’ and the ‘assortment of statements that the interlocutor is now able to make on the entity identified by the pronounced focus’, all the ‘discursive potentials’ born out of the ‘pronounced focus’). The constantly evolving nature of the ‘intended focus’ is ‘the crux of the matter’: successful communication often relies on the ‘attended focus’ being ‘used as a public exponent of the intended focus’.
- ‘Preoccupation analysis’ involves an analysis of the ‘two types of intentions which may be conveyed through communicative actions’ (p. 38): ‘object-level’ intentions, e.g. the intention to solve the mathematical problem in question, and ‘meta-discursive intentions’, which are ‘often less visible even if not less influential’, e.g. the ways in which the interaction is managed, the relationship between the interlocutors, etc. The former are

> often taken care of with the help of ‘focal analysis’. ‘Preoccupation analysis’ aims to explore the *interrelation* between these two types of intentions. Its principal tool is the ‘interactivity flowchart’, a diagram characterised by ‘proactive’ and ‘reactive arrows’ to reveal ‘initiating’ and ‘responsive’ attitudes, ‘discourse-spurring’ and ‘face-saving’ techniques used by the interlocutors.

Both methods have the potential to illustrate learning and teaching in useful ways. To us, a highlight is on the way in which the analyses generated through these methods, not only acknowledge that not any communication generates learning, but they offer ways of describing what *type* of communication does. This is a vital issue to which several other authors in (Kieran et al, 2002) return – including Kieran’s chapter in the volume.

There is little doubt that the work summarised above is a well-thought through endeavour to bridge the individual with the social within research in mathematics education; any concerns for example that such an elaborate analysis of mathematical conversation may miss an emphasis on the development of individual interlocutors are largely assuaged by the painstaking detail with which ‘focal analysis’ and ‘interactivity flowcharts’ elaborate such developments (and also generate *mathematically* rich accounts of the data). A pragmatic concern about the employment of these methods is one of scale. To generate an account of data through any of the above approaches is a time-consuming enterprise that by definition needs to focus on fractions of the typically vast amount of data collected in naturalistic studies of learning and teaching. Furthermore focusing on fragments of the data may impact on the analyst’s capacity to observe learning over a substantial period of time. Finally, and this is an issue that Sfard elaborates on in later work (e.g. the 2008 book), there is the issue of ‘ecological validity’ (Seeger, 2002, p293) of findings generated by discourse-oriented analyses – Seeger employs this term to describe the degree to which findings ‘give a fairly comprehensive and typical account’, particularly in cases where ‘experimental conditions do not match conditions in the real world’ (ibid.): by focusing on collecting evidence on the observable elements of mathematical behaviour, we may miss implicit, unobservable but perhaps significant processes taking place. In later work Sfard elaborates this issue – see, for example, (Sfard, 2008, p. 278-280) – through making a strong call for the discursive researcher to ‘alternate between being an *insider* and an *outsider* to the discourse under study’ (p. 278).

The ambitious book that followed the 2002 volume a few years later (Sfard, 2008) presented Sfard’s proposition in fuller scope – and in a way that situates her perspective ‘at the intersection of Consciousness Studies, Linguistics, Philosophy and Mathematics Education’ (Sriraman, 2009, p541). The perspective is known as the ‘commognitive framework’, with the hybrid term ‘commognition’ emphasising the interrelatedness, almost inseparability, of ‘cognition’ and ‘communication’. In Sfard’s words (2008, p. 83) the term is meant to refer ‘to those phenomena that are traditionally included in the term cognition, as well as to those usually associated with interpersonal exchanges’.

Sfard conceptualizes thinking as a particular type of communication. In her participationist view of learning, mathematical learning is seen as initiation into the discursive practices of the mathematical community. The learning of mathematics therefore involves a change of discourse. Teaching mathematics, in this sense, involves the changing of students’ discourse. Forms of communication include communication through written language, spoken language, physical objects and artifacts deployed for discursive ends. Specifically, a discourse is made distinct by a community’s *word use*, *visual mediators*, *endorsed narratives* and *routines*:

- *Word use* (vocabularies, key-words and their use) includes the use of mathematical terminology as well as use of ordinary words with a specific meaning within mathematics (such as ‘limit’, ‘open’, ‘continuous’, ‘group’)
- *Visual mediators* include diagrammatic and symbolic mediators of mathematical meaning (graphs, diagrams, symbols etc.) as well as the physical objects we often employ in mathematics lessons. school settings in the context of mathematical activity.
- *Endorsed narratives* include definitions, theorems and proofs and generally text, either spoken or written, that describes objects and processes as well as relationships amongst those and is subject to rejection or endorsement according to rules defined by the community.
- *Routines* include regularly employed and well-defined practices that are employed by, and distinctly, characterise the community. Within mathematics *routines* include conjecturing, proving, estimating, generalising, abstracting etc..

Readers of Sfard (2008) might ask ‘what makes Sfard’s efforts to illuminate our understanding of thinking any different from previous efforts’ (Yackel, 2009, p. 90). Her effort to ‘go to great lengths to develop an approach that meets accepted standards of scientific rigour’ through providing ‘operational definitions of keywords, such as thinking, communication, discourse, and mathematical object’ is one answer to this question. (And we note that her approach to developing operational definitions is Wittgensteinian, as evident in her endorsement of his ‘the meaning of a word is its use in the language’ (p. 73)). Another answer to this question is that, while Sfard casts a critical eye on acquisitionist metaphors for learning, she is also ‘careful to point out the benefits of objectification, namely the ways it contributes to effective and efficient communication’ – thus ‘not decrying our propensity for objectification’ (ibid, p.

91), simply calling for a careful scrutiny of the assumptions we make when we employ ‘object’ metaphors.

The potency and longevity of Sfard’s proposition will depend on how comfortably, or not, it will sit alongside other approaches in what we earlier described – borrowing from Jablonka & Bergsten (2010) – as the ‘Social Brand’ of approaches to research in mathematics education. For example, could the radical constructivist position be ‘subsumed as an extreme case within the commognitive framework’ (Sriraman, 2009, p. 544)? For the moment it seems that the framework puts forward a ‘grammar by which communication can be better fostered between researchers analyzing the same discursive “mathematical” objects in teaching and learning situations.’ (ibid), seems to be accumulating several ‘theoretical and methodological elaborations’ (Ryve, 2011, p. 187) and also seems to attract numerous uses of its ‘grammar’. We close this section with a brief outline of a few examples of such use: one from secondary mathematics education research, the award-winning *Research in Mathematics Education* paper by Natalie Sinclair and Violeta Yurita (2008); and, three from research into the teaching and learning of post-compulsory mathematics presented at CERME7, the 7th Congress of European Research in Mathematics Education.

Sinclair and Yurita (2008) investigate the impact of the introduction of a DGE (dynamic geometry environment) on the mathematical thinking of students and teachers in a secondary Geometry class, by identifying changes in the discourse engendered by its introduction. The paper focuses on the teacher and it is to the credit of the authors’ (commognitive) analysis that they reveal substantial – but remaining implicit in the classroom – differences between static and dynamic geometry (e.g. in the ways ‘the teacher talks about geometric objects, makes use of visual artifacts and models geometric reasoning’, p135).

Nardi (2011, in press) examines data from interviews with university mathematicians (reported extensively in (Nardi, 2008)) in order to outline issues that concern university students' discursive shifts in the early periods of their arrival at university. The paper focuses on verbalization skills, namely skills in the employment of ordinary language to convey mathematical meaning. In the interviews the mathematicians highlight: the role of verbal expression to drive noticing to the key idea of a symbolically formulated mathematical sentence; the importance of good command of ordinary language; the role of verbalisation as a mediator between symbolic and visual mathematical expression; and, the precision proviso for the use of ordinary language in mathematics. The analyses reveal that the community's discourse on verbalisation in mathematics tends to be risk-averse; that word-less mathematics discourse remains alluring; and, that more explicit, and less potentially contradicting, pedagogical action is necessary in order to facilitate students' appreciation of verbal mathematical expression and acquisition of verbalisation skills.

Stadler (2011, in press) examines a particular case of the transition from secondary school to university mathematics (the mathematical context is solving a parametric system of simultaneous equations) in order to discuss students' experience of the transition as an often perplexing re-visiting of content and ways of working that seem simultaneously familiar and novel (in this case dealing with variables, parameters and unknowns when solving equations). The paper's commognitive analyses of the student observations and interviews allow the multi-faceted (individual, institutional, social) nature of the transition from school to university mathematical discourse to emerge. Throughout the impression is that Sfard's perspective is a good fit to studies of transition.

Finally, Viirman (2011, in press) traces the variation of three university lecturers' discourse as they introduce the concept of function. Through analysis of the observations that focuses particularly on the routines and endorsed narratives characterising the lecturers' discourse, the variation across the three comes to the fore (the variation concerns the lecturers' variable resort to definitions, examples, the 'why' or 'when' of endorsing certain narratives etc.).

Concluding remarks

We conclude this chapter with a call to mathematics education researchers for a more incisive approach to, and employment of, theory that goes beyond knowing the 'grammar' (Lerman, 2010, p. 101) of a theory, a trend that Lerman and other contributors to (Sriraman & English, 2010) identified as a symptom of careless eclecticism. As Artigue (2011) notes,

> 'This is indeed hard work. Theoretical frameworks and constructs being dynamic entities shape research practices and are shaped by these, one cannot make sense of them without considering their different components and the research practices (or research praxeologies following Chevallard and ATD) they make possible and those they result from. It is not enough to know the elements of its "grammar" […] for making sense of a theory and appreciating its potential. Any productive dialogue around theoretical issues cannot stay at the level of the theoretical objects themselves but needs to enable collaborative work around appropriate exemplars of research praxeologies, and this is also a real challenge for us.' (p. 312)

Orchestrated efforts in this direction have been evident in the field for some time, and all the more evident in the two parts of (Sriraman & English, 2010), Parts XV and XVI, dedicated to 'networking of theories'. The call for even more systematic communication between theories that goes beyond mere borrowing extends to developments in other fields too – neuroscience being prominent amongst those fields where recent developments have made this call even more topical (see Campbell's (2010) strong case for a Mathematics Education Neuroscience). The words of Kristin Umland (2011) serve as a good indication of a way forward:

> '… a systematic survey of the big questions in mathematics education that need to be addressed. This survey should include a discussion of research methods that might be appropriately used to investigate them and weaknesses in both the relevant empirical record and extant theories, many of which are still very immature and should necessarily be refined as time passes. The ultimate test of the value of the ideas […] is whether they or their progeny help solve the problems that teachers, administrators, and policymakers face as they work to improve mathematics teaching and learning.' (p. 74)

References

Artigue M. (2002). Learning mathematics in a CAS environment: the genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. International Journal of Computers for Mathematics Learning, n°7,245-274.

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work.. International Journal of Computers for Mathematical Learning 7: 245–274, 2002.

Artigue, M. (2011). Review of Bharath Sriraman & Lyn English: Theories of Mathematics Education- Seeking New Frontiers. Research in Mathematics Education, 13(3), 311-316.

Ben-Yehuda, M., Lavy, I., Linchevski, L., & Sfard, A. (2005). Doing wrong with words: What bars students' access to arithmetical discourses. Journal for Research in Mathematics Education, 36(3), 176-247.

Bernstein, B. (1996). Pedagogy, symbolic control and identity: Theory, research, critique. London: Taylor & Francis.

Campbell, S. (2010). Embodied minds and dancing brains: New opportunities for research in mathematics education. In In B. Sriraman & L. English (Eds). Theories of Mathematics Education: Seeking New Frontiers (pp. 309-332). Springer Science

Chevallard, Y. (1985). La transposition didactique. Du savoir savant au savoir enseigné, Grenoble: La Pensée Sauvage.

Chevallard, Y. (1992a). Fondamentals concepts of didactics: perspectives given by an anthropological approach, Recherches en Didactique des Mathématiques 12(1), 73–112.

Chevallard, Y. (1992b). A theoretical approach to curricula. Journal für Mathematik Didaktik, 2/3, 215-230.

Chevallard, Y.(1999a). L’analyse des pratiques enseignantes en théorie anthropologique du didactique. Recherches en Didactique des Mathématiques 19(2), 221–266.

Chevallard, Y. (1999b). Didactique? you must be joking! A critical comment on terminology. Instructional Science, 27 , 5-7.

Dietrich, O. (2004). Cognitive evolution. In F.M. Wuketits and C. Antweiler (Eds). Handbook of Evolution (pp. 25-77). Wiley-VCH Verlag GmbH & Co: Weinheim.

Dienes, Z. P. (1960). Building up mathematics. London: Hutchinson.

Dienes, Z. P. (1963). An experimental study of mathematics-learning. London: Hutchinson.

Dienes, Z. P. (1964). The power of mathematics. London: Hutchinson Educational.

Dienes, Z. P. (1971). An example of the passage from the concrete to the manipulation of formal systems. Educational Studies in Mathematics, 3, 337–352.

Dienes, Z. P. (2000). The theory of the six stages of learning with integers. Mathematics in School,29, 27–33.

Dienes, Z. P., & Jeeves, M. A. (1965). Thinking in structures. London: Hutchinson.

Doerr, H., & Lesh, R. (2003). A modeling perspective on teacher development. In R. Lesh & H. Doerr (Eds.), Beyond constructivism: A models & modeling perspective on mathematics teaching, learning,and problem solving. Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Dubinsky. E. (1991). Reflective Abstraction in Advanced Mathematical Thinking, in D. Tall (Ed.), Advanced Mathematical Thinking (pp.95-126), Dordrecht: Kluwer, 95-126

Ely, R. (2010). Book Review: Theories of Mathematics Education. Educational Studies in Mathematics 75(2), 235-240.

Fried, M. (2011). Theories for, in, and of Mathematics Education. Interchange, 42(1), 81-95.

Gascón, J.(2003) 'From the Cognitive Program to the Epistemological Program in didactics of mathematics. Two incommensurable scientific research programs?' For the Learning of Mathematics 23(2), 44–55.

Greer, B. (2008) (Ed). Critical notice on the National Mathematics Advisory Panel Report. The Montana Mathematics Enthusiast, 5(2&3), 365-428.

Guin D., Ruthven K., Trouche L. (eds.) (2004). The didactic challenge of symbolic calculators. Dordrecht : Kluwer Academic Publishers.

Hurford, A. (2010). Complexity Theories and Theories of Learning: Literature Reviews and Syntheses. In B. Sriraman & L. English (Eds). Theories of Mathematics Education: Seeking New Frontiers (pp. 567-589). Springer Science.

Jablonka, E., & Bergsten, C. (2010). Commentary on Theories of Mathematics Education- Is plurality a problem? In B. Sriraman & L. English (Eds). Theories of Mathematics Education: Seeking New Frontiers (pp. 111-117). Springer Science.

Jablonka, E., & Bergsten, C. (2011). Theorising in mathematics education research: differences in modes and quality. Nordic Studies in Mathematics Education, 15(1), 25-52.

Jankvist, U. T (2011). Theories of Mathematics Education, Edited by Bharath Sriraman and Lyn English: Common ground for scholars and scholars in the making. Mathematical Thinking and Learning, 13(4),247-257.

Jaworski, A., & Coupland, N. (Eds.). (2005). The discourse reader, 2nd ed. London: Routledge.

Kieran, C., Forman, E., & Sfard, A. (Eds.). (2002). Learning discourse: Discursive approach to research in mathematics education. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Lesh, R., Behr, M., Cramer, K., Harel, G., Orton, R., & Post, T. (1992). Five versions of the Multiple embodiment principle. ETS Technical Report. Retrieved May 28,2011from http://education.umn.edu/rationalnumberproject

Lesh, R., Cramer, K., Doerr, H., Post, T., & Zawojewski, J. (2003). Model development sequences perspectives. In R. Lesh & H. Doerr (Eds.), Beyond constructivism: A models & modeling perspective on mathematics teaching, learning, and problem solving. Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Lesh, R., Post, T., & Behr, M. (1987). Dienes revisited: Multiple embodiments in computer environments. In I. Wirszup & R. Streit (Eds.), Developments in school mathematics education around the world. Reston,VA: National Council of Teachers of Mathematics.

Lesh, R., Post, T., & Behr, M. (1989). Proportional reasoning. In M. Behr & J. Hiebert (Eds.), Number concepts and operations in the middle grades. Reston, VA: National Council of Teachers of Mathematics.

Lesh, R., & Doerr, H. (2003) (Eds). Beyond Constructivism. Lawrence Erlbaum & Associates: Mahwah, NJ.

Lesh, R., & Sriraman, B. (2005). Mathematics education as design science. Zentralblatt für Didaktik der Mathematik, 37(6), 490-505.

Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed), Multiple perspectives an mathematics teaching and learning. (pp. 19- 44). Westport, CT: Ablex Publishing.

Lerman, S. (2010). Theories of Mathematics Education- Is plurality a problem? In B. Sriraman & L. English (Eds). Theories of Mathematics Education: Seeking New Frontiers (pp. 99-109). Springer Science

Moschkovich, J. N. (2004). Appropriating mathematical practices: a case study of learning to use and explore functions through interaction with a tutor. Educational Studies in Mathematics, 55, 49 - 80.

Mousoulides, N., Sriraman, B., & Lesh, R. (2008). The Philosophy and Practicality of Modeling Involving Complex Systems. The Philosophy of Mathematics Education Journal, 23, 134-157.

Nardi, E. (2005). ‘Beautiful minds’ in rich discourses: On the employment of discursive approaches to research in mathematics education. European Educational Research Journal, 4(2), 145-154.

Nardi, E. (2008). Amongst mathematicians: Teaching and learning mathematics at university level. New York: Springer.

Nardi, E. (2011, in press). ‘Driving noticing’ yet ‘risking precision’: University mathematicians’ pedagogical perspectives on verbalisation in mathematics. In E. Swoboda (Ed.), Proceedings of the 7th Conference on European Research in Mathematics Education (pp. tbc). Rzeszow, Poland. Also to be available at: http://www.erme.unito.it/

Seel, Norbert M. (Ed.) (2012, in press) Encyclopedia of the Sciences of Learning, New York: Springer

Piaget, J. (1958). The Growth of Logical Thinking From Childhood to Adolescence. Basic Books

Piaget, J. (1971). Biology and Knowledge. Edinburgh University Press.

Piaget, J. (1975). The child’s conception of the world. Totowa, NJ: Littlefield, Adams.

Piaget, J (1987). Possibility and Necessity: The Role of Necessity in Cognitive Development. Vols. 1&2. Minneapolis:University of Minnesota Press.

Popper, K. (1959). Logik der Forschung. Hutchinson, London.

Robert A, Rogalski J. (2005). A cross-Analysis of the Mathematics Teachers’ Activity. An Example in a French 10th-Grade Class. Educational Studies in Mathematics, 59/1-3, 269-298.

Ryve, A. (2011) Discourse Research in Mathematics Education: A Critical Evaluation of 108 Journal Articles, Journal for Research in Mathematics Education, 42(2), 167-198.

Schoenfeld, A. (2010). Review of Bharath Sriraman & Lyn English: Theories of Mathematics Education - Seeking New Frontiers. ZDM- The International Journal on Mathematics Education, 42(5), 503-506.

Sensevy G., Mercier A. (eds.) (2007). Agir ensemble. L'action didactique conjointe du professeur et des élèves dans la classe. Presses Universitaires de Rennes.

Sensevy G. (2009). Outline of a joint action theory in didactics. Six Conference on European Research on Mathematics Education.

Sfard, A. (2002). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. In C. Kieran, E. Forman & A. Sfard (Eds.), Learning Discourse: Discursive approaches to research in mathematics education (pp. 13-57). Dordrecht/Boston/London: Kluwer Academic Publishers.

Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. The Journal of the Learning Sciences, 16(4), 565-613.

Sfard, A. (2008). Thinking as communicating. Human development, the growth of discourse, and mathematizing. New York: Cambridge University Press.

Sfard, A., & McClain, K. (2002). Analyzing Tools: Perspectives on the Role of Designed Artifacts in Mathematics Learning. Journal of the Learning Sciences, 11(2/3), 153-161.

Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. Educational Researcher, 34(4), 14-22.

Sfard, A. (1987). Mathematical practices, anomalies and classroom communication problems. In P. Ernest (Ed.), Constructing mathematical knowledge: Epistemology and mathematics education (Studies in Mathematics Education Series Vol. 4). London: The Falmer press.

Sinclair, N., & Yurita, V. (2008). To be or to become: how dynamic geometry changes discourse. Research in Mathematics Education, 10(2), 135-150.

Sierpinska, A., and Kilpatrick, J. (Eds.) (1998). Mathematics Education as a Research Domain: An ICMI Study. Kluwer Academic Publishers.

Sierpinska, A and Lerman, S. (1996). Epistemologies of mathematics and mathematics education. In A. Bishop et al (Eds), Handbook of Research in Mathematics Education (pp. 827-876). Dodrecht: Kluwer Academic Publishers.

Sriraman, B. (2008). Let Lakatos Be. Interchange: A Quarterly Review of Education 39(4),483-492.

Sriraman, B. (2011). Commentary on Theorizing in mathematics education research: differences in modes and quality, Nordic Studies in Mathematics Education. 15(1),53-58.

Sriraman, B., & English, L. (2004). Combinatorial mathematics: Research into practice. Connecting research into teaching. The Mathematics Teacher, 98(3), 182–191.

Sriraman, B., & English, L. (2005). Theories of mathematics education: a global survey of theoretical frameworks/trends in mathematics education research. Zentralblatt für Didaktik der Mathematik, 37(6), 450-456.

Sriraman, B., & English, L. (2006). Theories of mathematics education: European perspectives,commentaries and viable research directions. Zentralblatt für Didaktik der Mathematik 38(1), 1-2.

Sriraman, B., & English, L. (2010). Theories of mathematics education: Seeking new frontiers. Springer Science, Berlin/Heidelberg.

Sriraman, B & Törner, G. (2008) Political Union/ Mathematical Education Disunion: Building Bridges in European Didactic Traditions. In L. English et al (Eds) The Handbook of International Research in Mathematics Education (2nd Edition) (pp.660-694). Routledge, Taylor & Francis, New York.

Sriraman, B., & English, L. (2010) (Eds). Theories of mathematics education: Seeking new frontiers. Springer, Berlin/Heidelberg.

Sriraman, B. (2009). What's all the commotion over Commognition? The Montana Mathematics Enthusiast, 6(3), 541- 544.

Stadler, E. (2011) The same but different - novice university students solve a textbook exercise. In E. Swoboda (Ed.), Proceedings of the 7th Conference on European Research in Mathematics Education (pp. tbc). Rzeszow, Poland. Also to be available at: http://www.erme.unito.it/

Seeger, F. (2002) Research on discourse in the mathematics classroom. In Kieran, C., Forman, E., & Sfard, A. (Eds.) Learning discourse: Discursive approaches to research in mathematics education, 287-297. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Tall, D. O. (1989). Concept images, generic organizers, computers and curriculum change. For the Learning of Mathematics, 9(3), 37-42.

Tall, D. O. (Ed.). (1991). Advanced mathematical thinking. Dordrecht/Boston/London: Kluwer Academic Publishers.

Umland, K. (2011). Review of Theories of Mathematics Education: Seeking New Frontiers. TheMathematical Intelligencer, 33(2), 73-74.

Viirman, O. (2011). Discourses of functions – university mathematics teaching through a commognitive lens. In E. Swoboda (Ed.), Proceedings of the 7th Conference on European Research in Mathematics Education (pp. tbc). Rzeszow, Poland. Also to be available at: http://www.erme.unito.it/

Weyl, H. (1918). Raum-Zeit-Materie. Vorlesungen überallgemeine Relativitätstheorie, Berlin: Springer Verlag

[i] *Note by the authors: The second author is mainly responsible for the sections entitled 'Critiques of *Theories of Mathematics Education*' and 'Discursive approaches to research in mathematics education: the case of Sfard's 'commognitive' framework'' and the first author is mainly responsible for all other sections.

[ii] The section on re-examining the notion of "operational" definitions have come to the foreground in mathematics education research in North America due to the stipulation of major funding bodies to only solicit proposals that have an experimental design or a quasi-experimental design harking back to the days of the 1960's when psychology was the early fore-runner for theory building in mathematics education.

[iii] The Bourbaki essentially aimed to write a body of work based on a rigorous and formal foundation, which could be used by mathematicians in the future. For more information see Bourbaki, N. (1970). *Théorie des Ensembles dela collection elements de Mathématique*, Hermann, Paris. The internet savvy can also refer to the Bourbaki website located at http://www.bourbaki.ens.fr/