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Heat Waves in a Supercritical Fluid

by

W. R. Derrick
Department of Mathematical Sciences
University of Montana, Missoula, MT 59802
derrick@mso.umt.edu

L. V. Kalachev
Department of Mathematical Sciences
University of Montana, Missoula, MT 59802
kalachev@mso.umt.edu

J. A. Cima
Department of Mathematics
University of North Carolina, Chapel Hill, NC 27599
cima@email.unc.edu

Abstract.

Experiments reveal that thermal energy transfers as an acoustic wave near the critical point of a classical fluid under microgravity. In this paper we use asymptotic methods and Fourier analysis to provide a complete mathematical proof of this piston effect. Our two-step method provides a "fast-time" solution describing the piston effect in detail, and a "slow-time" solution leading to solutions that agree in nature with those obtained by numerical methods applied to the original model system on "slow-time" scale.

1 Introduction

In recent years a number of authors have explored the effects of thermal energy transfer near the critical point of a classical fluid. Since the specific heat and the isothermal compressibility increase as the fluid temperature approaches criticality, it follows that the thermal diffusivity becomes very small. It had originally been believed that thermal energy could not be transferred near the critical point, because the diffusivity was so small. However, under microgravity as well as normal gravity conditions, experiments reveal that the temperature propagates as an acoustic wave. This problem has been analyzed thermodynamically by Onuki et al. [1] and Boukari et al. [2], as a thermofluid by Zappoli et al. [3] – [5], and also theoretically and experimentally by Ishii et al. [6] – [8]. In the latter papers the authors have developed the governing equations, done numerical simulations, and performed a linear analysis to determine the wave speed.

Ishii et al. [8] confine CO_2 in a cylindrical container and arrange two copper blocks with a 10 mm gap between them. The temperature is carefully controlled by Pertier elements and measured with thermisters in the blocks. The temperature field between the blocks is visualized by the shadowgraph method using a He-Ne laser. The gas is introduced into the cylinder with density set at the critical value, temperature is then adjusted just above the critical temperature (0.02 K), and finally the temperature of one of the blocks is raised by 0.122 K. The temperature propagates as acoustic waves: first running from the hotter block, bouncing back from the cooler block, then back again, repeatedly, until the average temperature is established over most of the gap, except for thermal boundary layers by each block. This is called a piston effect. Shortly after this average is established, thermal plumes are initiated from both boundary layers. This behavior is quite different from that of fluids under normal conditions. The onset of bouyancy convection has been studied by Maekawa et al. [9].

While several of these papers have determined the wave speed by a linear analysis, they have not provided a complete description of the acoustic wave propagation. It is the purpose of this paper to provide such a description. For simplicity, we will assume that the thermal process is happening under zero gravity.

In Section 2 we describe the governing equations present in the experiment. Since the acoustic waves bounce back and forth between the blocks with isotherms that are essentially parallel, we assume that the problem is one-dimensional. We simplify the governing equations, then assume that changes in density are extremely small, so that it does not change the problem significantly to assume that the density is constant. We also assume that the dynamic viscosity is very small. Finally we consider the boundary conditions that must apply in the problem, and show that a pulse occurs at $x = 0 = t$.

In Section 3 we use asymptotic methods and Fourier analysis to obtain the first two terms of the asymptotic expansion of the "fast-time" solution for the first part of the wave. The heat wave arises because this solution of the system becomes effectively a wave equation.

In Section 4 we describe the rest of the "fast-time" solution as it stabilizes into a heat profile midway between 0.02 K and 0.122 K. Thus, the "fast-time" solution describes the entire piston effect.

In Section 5 we study the "slow-time" solution that takes care of the discrepancies introduced into the boundary conditions by the "fast-time" solution.

2 Governing Equations

There are four governing equations for a critical thermodynamic fluid:

Continuity Equation:

$$\rho_t + \frac{\partial}{\partial x_j}(\rho v_j) = 0, \tag{1}$$

where ρ is the density and v_j is the fluid particle velocity in the j direction.

Momentum Equation:

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_i v_j)}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g k_3, \tag{2}$$

where P is the pressure, g is gravity, k_i is the direction vector, and σ_{ij} is the viscous stress tensor given by

$$\sigma_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k} \right),$$

where η is the dynamic viscosity (the volume viscosity is ignored).

Thermal Balance Equation:

$$(\rho T)_t + \frac{\partial(\rho v_j T)}{\partial x_j} + \rho\frac{\gamma - 1}{\alpha_P}\frac{\partial v_j}{\partial x_j} = \frac{1}{c_V}\left\{\frac{\partial}{\partial x_j}(\lambda\frac{\partial T}{\partial x_j}) + \sigma_{ij}\frac{\partial v_i}{\partial x_j}\right\}, \tag{3}$$

where T is the temperature, γ is the ratio of specific heats, α_P the temperature coefficient of volume expansion, and c_V the specific heat at constant volume, and λ is the thermal conductivity.

Equation of State:

$$dP = \frac{1}{\rho\kappa_T}d\rho + \frac{\alpha_P}{\kappa_T}dT, \tag{4}$$

where κ_T is the thermal compressibility. This is a variation of the standard van der Waals equation

$$\left(P + \frac{a\rho^2}{N^2}\right)\left(\frac{1}{\rho} - b\right) = RT, \tag{5}$$

where R is the universal gas constant, N is the number of moles, a is a measure of attraction between particles, and b is the volume excluded by a mole of particles.

Since we are assuming that our problem has one spatial direction, and since the experimental and computational problems exhibit isotherms that are essentially parallel, we may rewrite the first three equations as:

$$\rho_t + (\rho v)_x = 0, \tag{6}$$

$$(\rho v)_t + (\rho v^2)_x = -P_x + \frac{4\eta}{3} v_{xx} + \rho g, \tag{7}$$

$$(\rho T)_t + (\rho v T)_x + \rho \frac{\gamma - 1}{\alpha_P} v_x = \frac{1}{c_V} \Big\{ (\lambda T_x)_x + \frac{4\eta}{3} v_x^2 \Big\}. \tag{8}$$

Applying equation (6) to the left sides of (7) and (8) yields

$$\rho v_t + \rho v v_x = -P_x + \frac{4\eta}{3} v_{xx} + \rho g, \tag{9}$$

$$\rho T_t + \rho v T_x + \rho \frac{\gamma - 1}{\alpha_P} v_x = \frac{1}{c_V} \Big\{ (\lambda T_x)_x + \frac{4\eta}{3} v_x^2 \Big\}. \tag{10}$$

Applying the equation of state (4) in (9) and dividing both equations by ρ, we have

$$v_t + v v_x = -\Big(\frac{1}{\rho^2 \kappa_T} \rho_x + \frac{\alpha_P}{\rho \kappa_T} T_x \Big) + \frac{4\eta}{3\rho} v_{xx} + g, \tag{11}$$

$$T_t + v T_x + \frac{\gamma - 1}{\alpha_P} v_x = \frac{1}{\rho c_V} \Big\{ (\lambda T_x)_x + \frac{4\eta}{3} v_x^2 \Big\}. \tag{12}$$

We now assume that there is no gravity. Further, we assume that the density is essentially constant, since the volume is fixed and the number of moles is constant (even though there will be small density waves present), so that $\rho \equiv 1$. We obtain a system in the variables temperature T and the fluid particle velocity v:

$$v_t + v v_x + \frac{\alpha_P}{\kappa_T} T_x = \frac{4\eta}{3} v_{xx}, \tag{13}$$

$$T_t + v T_x + \frac{\gamma - 1}{\alpha_P} v_x = \frac{\lambda}{c_V} T_{xx} + \frac{4\eta}{3} v_x^2. \tag{14}$$

We must now consider the initial and boundary conditions that apply to this problem. We begin with the initial conditions:

$$T \equiv T_a \quad \text{and} \quad v \equiv 0, \qquad 0 \le x \le L, t = 0. \tag{15}$$

Assume that at $t > 0$, the temperature of the block at $x = 0$ is raised to T_b, while the temperature of the block at $x = L$ is maintained at T_a. Then a heat flux proportional to the temperature difference will exist at both boundaries, but because the boundaries do not allow the escape of particles, the particle velocity at the boundaries will be zero, so that we get the boundary conditions:

$$\frac{\lambda}{c_V}T_x = \beta(T - T_b), \quad v = 0, \qquad \text{at} \quad x = 0; \tag{16}$$

$$\frac{\lambda}{c_V}T_x = \beta(T_a - T), \quad v = 0, \qquad \text{at} \quad x = L. \tag{17}$$

The instantaneous increase in the temperature (from T_a to T_b) will produce a pulse in T_t at $x = 0$, $t = 0$ crucial in propagating the heat wave that is bouncing from one block to the other. (A similar pulse, which we will compute later, occurs in T_t when the wave reaches $x = L$.)

Lemma 1.

$$T_t(x, 0) = \delta(x)\beta(T_b - T_a). \tag{18}$$

Proof. We begin by proving that for every function $g(x)$ in $\mathrm{C}^1([0,L])$

$$\lim_{t\to 0} \frac{\lambda}{c_V} \int_0^L g(x)T_{xx}(x,t)dx = \int_0^L g(x)T_t(x,0)dx. \tag{19}$$

Use equation (14) to replace the term $(\lambda/c_V)T_{xx}$ on the left hand side, so that we obtain

$$\lim_{t\to 0} \int_0^L g(x)[T_t(x,t) + v(x,t)T_x(x,t) + \frac{\gamma-1}{\alpha_P}v_x(x,t) - \frac{4\eta}{3}v_x^2(x,t)]\ dx$$

$$= \int_0^L g(x)T_t(x,0)dx + \int_0^L g(x)v(x,0)T_x(x,0)dx$$

$$+\frac{\gamma-1}{\alpha_P}\int_0^L g(x)v_x(x,0)dx - \frac{4\eta}{3}\int_0^L g(x)v_x^2(x,0)dx.$$

But $v(x,0) \equiv 0$, so that $v_x(x,0) \equiv 0$, and the last three integrals are zero.

On the other hand, if we integrate the left side of equation (19) by parts, and use the boundary conditions (16)-(17), we get

$$\lim_{t\to 0}\left(g(x)\frac{\lambda}{c_V}T_x|_0^L - \int_0^L g'(x)\frac{\lambda}{c_V}T_x(x,t)dx \right)$$

$$= \lim_{t\to 0}\left(g(L)\beta(T_a - T(L,t)) - g(0)\beta(T(0,t) - T_b) - \frac{\lambda}{c_V}\int_0^L g'(x)T_x(x,t)dx \right).$$

But by (15) since $T(x,0) \equiv T_a$, it follows that $T_x(x,0) \equiv 0$, so that when we pass to the limit, the last integral is zero and we get

$$\int_0^L g(x)T_t(x,0)dx = g(L)\beta(T_a - T(L,0)) - g(0)\beta(T(0,0) - T_b) = g(0)\beta(T_b - T_a).$$

Since $g(x)$ is any continuously differentiable function, $T_t(x,0)$ behaves like a multiple of a delta function $\delta(x)$ at the origin, as asserted.

3 Asymptotic Analysis: "Fast-Time" Solution

To simplify our notation as we do an asymptotic analysis, let $a = \alpha_P/\kappa_T$, $b = (\gamma - 1)/\alpha_P$, $D\epsilon^2 = \lambda/c_V$ (small diffusion for supercritical fluid), $\beta = \beta^*\epsilon$ (parameter β has to be small for the existence of "small amplitude" temperature waves), and $N\epsilon^2 = 4\eta/3$ so that the system of equations and conditions become in this notation

$$v_t + vv_x + aT_x = N\epsilon^2 v_{xx}, \tag{20}$$

$$T_t + vT_x + bv_x = D\epsilon^2 T_{xx} + N\epsilon^2 v_x^2, \tag{21}$$

$$T \equiv T_a \quad \text{and} \quad v \equiv 0, \qquad \text{for} \quad 0 \le x \le L, \quad t = 0, \tag{22}$$

$$D\epsilon T_x = \beta^*(T - T_b), \quad v = 0, \qquad \text{at} \quad x = 0, \tag{23}$$

$$D\epsilon T_x = \beta^*(T_a - T), \quad v = 0, \qquad \text{at} \quad x = L. \tag{24}$$

We use the result in Lemma 1, with $\beta = \beta^*\epsilon$, as a second initial condition that will be required in our analysis:

$$T_t(x, 0) = \delta(x)\beta^*\epsilon(T_b - T_a), \quad 0 \le x \le L. \tag{25}$$

Let "fast-time" asymptotic series have the form:

$$\begin{aligned} T(x,t) &= T_0(x,t) + \epsilon T_1(x,t) + \epsilon^2 T_2(x,t) + ..., \\ v(x,t) &= \epsilon v_1(x,t) + \epsilon^2 v_2(x,t) + ... \end{aligned} \tag{26}$$

We require that the "fast-time" approximation for the temperature T satisfies Neumann boundary conditions (instead of Robin type conditions in (23) and (24); the effect of the boundary temperature is taken into account by the initial pulse in T_t given by (25)):

$$T_x = 0 \qquad \text{at} \quad x = 0, \tag{27}$$

$$T_x = 0 \qquad \text{at} \quad x = L. \tag{28}$$

Substituting (26) into problem (20), (22) and (25), with boundary conditions for the temperature given by (27), (28), and collecting like powers of ϵ, we obtain the systems of equations for the terms in the asymptotic expansion. The zeroth-order approximation is given by the equations:

$$aT_{0x} = 0, \qquad T_{0t} = 0, \qquad T_0(x,0) = T_a,$$

so that $T_0 \equiv T_a$. Then, using $T_0 \equiv T_a$, the first-order approximation is given by the system:

$$v_{1t} + aT_{1x} = 0, \tag{29}$$

$$T_{1t} + bv_{1x} = 0. \tag{30}$$

If we differentiate (29) with respect to x and (30) with respect to t, we get

$$T_{1tt} = \ abT_{1xx}, \tag{31}$$

with corresponding boundary and initial conditions:

$$\begin{aligned} &T_{1x}(0,t) = 0, \quad T_{1x}(L,t) = 0, \\ &T_1(x,0) = 0, \quad T_{1t}(x,0) = \delta(x)\beta^*(T_b - T_a). \end{aligned} \tag{32}$$

Observe that (31) is a wave equation with wave velocity

$$C = \sqrt{ab} = \sqrt{(\gamma - 1)/\kappa_T}. \tag{33}$$

The "fast-time" problem (31)-(32) for T_1 has, by separation of variables, the Fourier series solution

$$T_1(x,t) = \frac{\beta^*(T_b - T_a)}{2L}\left(t + \sum_{n=1}^{\infty} \frac{2\sin(\pi nCt/L)\cos(\pi nx/L)}{(\pi nC/L)}\right). \tag{34}$$

The sum in (34) can be simplified by using trigonometry and known Fourier series to produce

$$T_1(x,t) = \frac{\beta^*(T_b - T_a)}{2L}\left(t + \frac{f(x+Ct) - f(x-Ct)}{2C}\right), \tag{35}$$

where

$$f(x) = \begin{cases} -x - L & \text{for} \quad -L < x < 0 \\ -x + L & \text{for} \quad 0 < x < L \end{cases}$$

is an odd, period $2L$, sawtooth. We can rewrite $f(x)$ as $f(x) = -x - L + 2LH(x)$, where $H(x)$ is the Heaviside function. Expanding (35) in the form

$$T_1(x,t) = \frac{\beta^*(T_b - T_a)}{2C}(H(x+Ct) - H(x-Ct)), \tag{36}$$

we obtain the first wave across the interval $0 \leq x \leq L$ with a temperature increase of $\tau_1 = \beta^*(T_b - T_a)/(2C)$. Clearly, it is important that $\beta^*/(2C) << 1$, implying that the flux rate must be small compared to the wave velocity.

To calculate v_1 we use (29), (30), and (35) so that

$$v_{1t} = -aT_{1x} = -\frac{a\beta^*(T_b - T_a)}{2L}\left(\frac{f'(x+Ct) - f'(x-Ct)}{2C}\right),$$

and

$$v_{1x} = -\frac{1}{b}T_{1t} = -\frac{\beta^*(T_b - T_a)}{2Lb}\left(1 + \frac{f'(x+Ct) + f'(x-Ct)}{2}\right).$$

Integrating the first of these equations with respect to t and the second with respect to x, we obtain

$$v_1 = -\frac{a\beta^*(T_b - T_a)}{2L}\left(\frac{f(x+Ct) + f(x-Ct)}{2C^2}\right) + \phi(x),$$

and

$$v_1 = -\frac{\beta^*(T_b - T_a)}{2Lb}\left(x + \frac{f(x+Ct) + f(x-Ct)}{2}\right) + \psi(t).$$

Since $ab = C^2$, we get $\psi(t) = 0$ and $\phi(x) = -a\beta^*(T_b - T_a)/2LC^2$, so that v_1 becomes

$$v_1 = -\frac{a\beta^*(T_b - T_a)}{2LC^2}\left(x + \frac{f(x+Ct) + f(x-Ct)}{2}\right), \tag{37}$$

or

$$v_1 = \frac{a\beta^*(T_b - T_a)}{2LC^2}\Big(1 - H(x+Ct) - H(x-Ct)\Big), \tag{38}$$

implying that as the wave passes a point x, the gas particles return to criticality ($v_1 \equiv 0$).

We have shown that a heat wave of amplitude $\tau_1 = \beta^*(T_b - T_a)/(2C)$ and velocity C reaches the $x = L$ boundary at time $t = L/C = t_1$. Further, (38) suggests that as the heat wave reaches the $x = L$ boundary, the particles in the gas move away from that boundary, producing a pulse at $x = L$.

4 The Rest of the "Fast-Time" Solution

In computing the first part of the heat wave in Section 3, using the pulse in Lemma 1 produced by the Robin boundary conditions at $x = 0$ and at $x = L$, we have assumed a particular model for the action of the "fast-time" solution at the boundaries:

Each heat wave has no effect on the boundary layers until it reaches its end at $t = t_1, 2t_1, 3t_1, \ldots$.

This is a common approach in such asymptotic analyses, but other models are available. We believe our model will not significantly affect the outcome, and has the advantage of offering a reasonably simple solution.

If the block at $x = L$ was insulated, the heat wave would bounce back when $t > t_1$ with a total amplitude of $2\tau_1$. However, since $x = L$ is refrigerated to temperature T_a, some heat loss will be produced. This will also cause a pulse in $T_t(L, t_1)$. Recasting the initial conditions in system (13)-(17), at time $t = t_1$, to take account of the heat flux at the boundaries, we have

Lemma 2.

$$T_t(x, t_1) = \delta(L)\beta(T_b - T_a - 2\tau_1).$$

Proof. Now $T(x, t_1) \equiv T_a + \tau_1$ and $v(x, t_1) \equiv 0$ in $0 < x < L$, so that $v_x(x, t_1) \equiv 0$ and $T_x(x, t_1) \equiv 0$. Hence by (14), for every periodic $g(x)$ in $C^1([0, L])$ we again have

$$\int_0^L g(x)T_t(x, t_1)dx = \lim_{t \to t_1} (\lambda/c_V) \int_0^L g(x)T_{xx}(x, t_1)dx.$$

Integrating the right side by parts and taking the limit, we obtain using (16) and (17)

$$\int_0^L g(x)T_t(x, t_1)dx = g(L)(\lambda/c_V)T_x(L, t_1) - g(0)(\lambda/c_V)T_x(0, t_1)$$

$$= g(L)\beta(T_a - T_a - \tau_1) - g(0)\beta(T_a + \tau_1 - T_b)$$

$$= g(L)\beta(T_b - T_a - 2\tau_1).$$

The last term (twice the wave amplitude) is the heat flux loss at $x = L$ through refrigeration, as well as heat flux reduction at $x = 0$. Hence $T_t(x, t_1)$ behaves like a multiple of a delta function at $x = L$, as asserted.

Setting $z = L - x$ and $s = t - t_1$ to consider the reflected wave and the action of the pulse arising in Lemma 2, we again substitute (for $t_1 < t < 2t_1$)

$$\begin{aligned} T(z, s) &= T_0(z, s) + \epsilon T_1(z, s) + \epsilon^2 T_2(z, s) + ..., \\ v(z, s) &= \epsilon v_1(z, s) + \epsilon^2 v_2(z, s) + ... \end{aligned}$$

into the system (13)-(17). The zero-order approximation is

$$aT_{0z} = 0, \qquad T_{0s} = 0, \qquad T_0(z, 0) = T_a + \tau_1,$$

so that $T_0(z, s) \equiv T_a + \tau_1$ in $[0, L] \times [0, t_1]$. The first-order approximation changes (29)-(30) to

$$v_{1s} - aT_{1z} = 0,$$

$$T_{1s} - bv_{1z} = 0,$$

which does not change the wave equation (31)

$$T_{1ss} = bv_{1sz} = abT_{1zz}, \tag{39}$$

with the no-flux boundary and initial conditions

$$\begin{aligned} &T_{1z}(0,s)=0, \qquad T_{1z}(L,s)=0, \\ &T_1(z,0)=0, \qquad T_{1s}(z,0)=\delta(z)\beta^*(T_b-T_a-2\tau_1). \end{aligned} \tag{40}$$

The "fast-time" problem is exactly the same as in Section 3:

$$T_1(z,s)=\tau_2(H(z-Cs)-H(z-Cs))=\tau_2(H(Ct-x)-H(2L-x-Ct)),$$

where the heat wave amplitude for the bounced wave is

$$\tau_2=\frac{\beta^*}{2C}(T_b-T_a-2\tau_1)=\left(1-\frac{\beta^*}{C}\right)\tau_1.$$

If we continue in this fashion we find that each wave has precisely the same solution except that the amplitudes τ_3, τ_4, ... , τ_N, ... decrease. Observe that

$$\tau_3=\frac{\beta^*}{2C}(T_b-T_a-2\tau_1-2\tau_2)=\tau_2-\frac{\beta^*}{2C}2\tau_2=\left(1-\frac{\beta^*}{C}\right)^2\tau_1.$$

In general

$$\tau_N=\left(1-\frac{\beta^*}{C}\right)^{N-1}\tau_1.$$

Thus, the general solution for $T_1(x,t)$ in $[0,L]\times[0,(N+1)t_1]$ is

$$\begin{aligned} T_1(x,t)=\sum_{k=1}^{N}\tau_k H(t-(k-1)t_1)\Bigg[H\left(Ct-(-1)^k x-2\left[\left[\frac{k-1}{2}\right]\right]K\right) \\ -H\left(2\left[\left[\frac{k}{2}\right]\right]L-(-1)^k x-Ct\right)\Bigg], \end{aligned}$$

where $[[y]]$ is integral part of y.

The process of bouncing heat waves with decreasing amplitudes will end when

$$T=T_a+\tau_1+\tau_2+...=\frac{1}{2}(T_b+T_a) \quad \text{or} \quad \tau_1+\tau_2+...=\frac{1}{2}(T_b-T_a), \tag{41}$$

since then the pulses would cease. But

$$\sum_{k=1}^{N}\tau_k=\tau_1\sum_{k=0}^{N}\left(1-\frac{\beta^*}{C}\right)^k=\frac{T_b-T_a}{2}\,\frac{\beta^*}{C}\left[\frac{1-(1-\beta^*/C)^{N+1}}{1-(1-\beta^*/C)}\right]$$

or

$$\sum_{k=1}^{N}\tau_k=\frac{T_b-T_a}{2}\left[1-(1-\beta^*/C)^{N+1}\right]. \tag{42}$$

Thus, the heat waves cease and the amplitudes of the waves tend to zero as $N \to \infty$. Hence, as time increases, the heat waves become imperceptible and the "fast-time" solution stabilizes to a temperature profile midway between T_b and T_a. Note that the particle velocity $v \approx \epsilon v_1 \approx 0$.

The "fast-time" solution is illustrated in Figure 1.

Insert Figure 1 here!

5 "Slow-Time" Solution

The "fast-time" temperature defined as the sum of bouncing waves, with the amplitude of each consecutive wave gradually decreasing, in the limit as $t \to \infty$ stabilizes at the value $T_a + (T_b - T_a)/2 = (T_b + T_a)/2$ obtained in (41). This value may be considered as the constant initial condition for the "slow-time" portion of the temperature solution $\tilde{T}(x, \eta)$, where $\eta = \epsilon^2 t$.

We note that, since $v \approx \epsilon v_1 \approx 0$, in the system of equations (20), (21) we may just set the "slow-time" portion of the particle velocity $\tilde{v} = 0$. Under the condition that $a = \alpha_P/\kappa_T$ in (20) is sufficiently small (suppose α_P is small; since $b = (\gamma - 1)/\alpha_P$, the product $ab = (\gamma - 1)/\kappa_T = C^2$, the wave speed, will still be either moderate or large), we may neglect the only remaining term in (20) which does not depend on velocity. Thus, we have to solve only the second equation (21) of the original system, where the "slow" re-scaled time variable $\eta = \epsilon^2 t$ is used:

$$\epsilon^2 \tilde{T}_\eta = D\epsilon^2 \tilde{T}_{xx}, \quad \text{or} \quad \tilde{T}_\eta = D\tilde{T}_{xx}. \tag{43}$$

This equation must be considered together with the following initial and boundary conditions (see (23), (24)):

$$\tilde{T}(x, 0) = (T_b + T_a)/2, \qquad \text{for} \quad 0 \le x \le L, \tag{44}$$

$$D\epsilon\tilde{T}_x = \beta^*(\tilde{T} - T_b), \qquad \text{at} \quad x = 0, \tag{45}$$

$$D\epsilon\tilde{T}_x = \beta^*(T_a - \tilde{T}), \qquad \text{at} \quad x = L. \tag{46}$$

Let the "slow-time" asymptotic series for temperature have the form:

$$\tilde{T}(x, \eta) \quad = \quad \tilde{T}_0(x, \eta) + \epsilon\tilde{T}_1(x, \eta) + \epsilon^2\tilde{T}_2(x, \eta) + ... \tag{47}$$

Substituting (47) into (43) – (46) and using the standard asymptotic procedure we arrive the the following problem for the leading order approximation $\tilde{T}_0$:

$$(\tilde{T}_0)_\eta = D(\tilde{T}_0)_{xx}, \tag{48}$$

together with the following initial and boundary conditions:

$$\tilde{T}_0(x, 0+) = (T_b + T_a)/2, \qquad \text{for} \quad 0 \leq x \leq L, \tag{49}$$

$$\tilde{T}_0(0, \eta) = T_b, \tag{50}$$

$$\tilde{T}_0(L, \eta) = T_a. \tag{51}$$

The solution $\tilde{T}_0$ can be easily constructed in the form of Fourier series (this solution is illustrated in Figure 2):

$$\tilde{T}_0(x, \eta) = T_b - \frac{T_b - T_a}{L} x + \sum_{n=1}^{\infty} \phi_n \exp(-D(\pi n/L)^2 \eta) \sin(\pi n x/L),$$

where

$$\phi_n = \left(\frac{T_a - T_b}{\pi n} \right) [1 + (-1)^n].$$

Insert Figure 2 here!

6 Brief Conclusion

The "fast-time" and the "slow-time" solutions constructed in this paper allow one to better understand the heat transfer process near the critical point of a classical fluid. The temperature behaviors in the model obtained using perturbation approach mimic those observed experimentally and obtained using numerical computation for the original model formulation.

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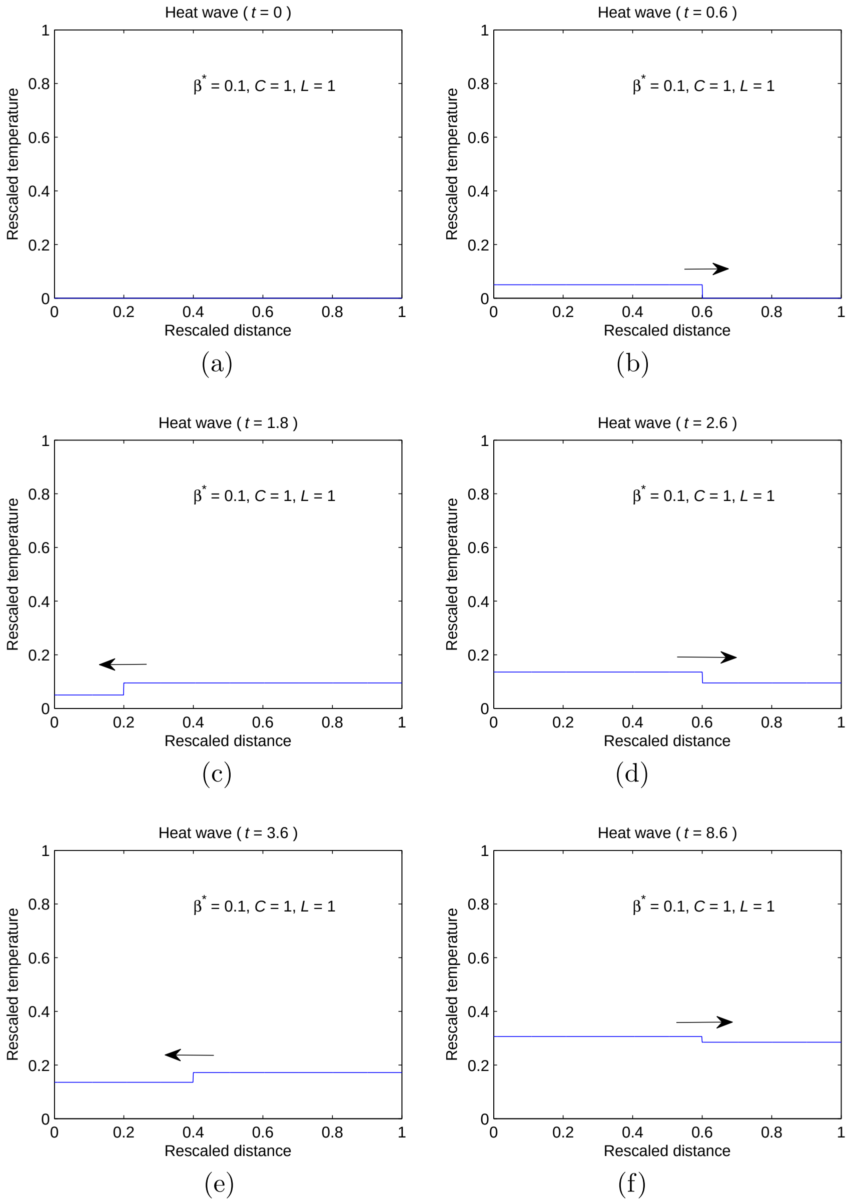

Figure 1. "Fast-time" solution: snapshots at times $t = 0$, 0.6, 1.8, 2.6, 3.6, 8.6. Here $T_a = 0$, $T_b = 1$, $t_1 = L/C = 1$; $\tau_1 = \beta^*(T_b - T_a)/(2C) = 0.05$; $\tau_N = (1 - \beta^*/C)^{N-1}\,\tau_1 = 0.9^{N-1} \cdot 0.05$.

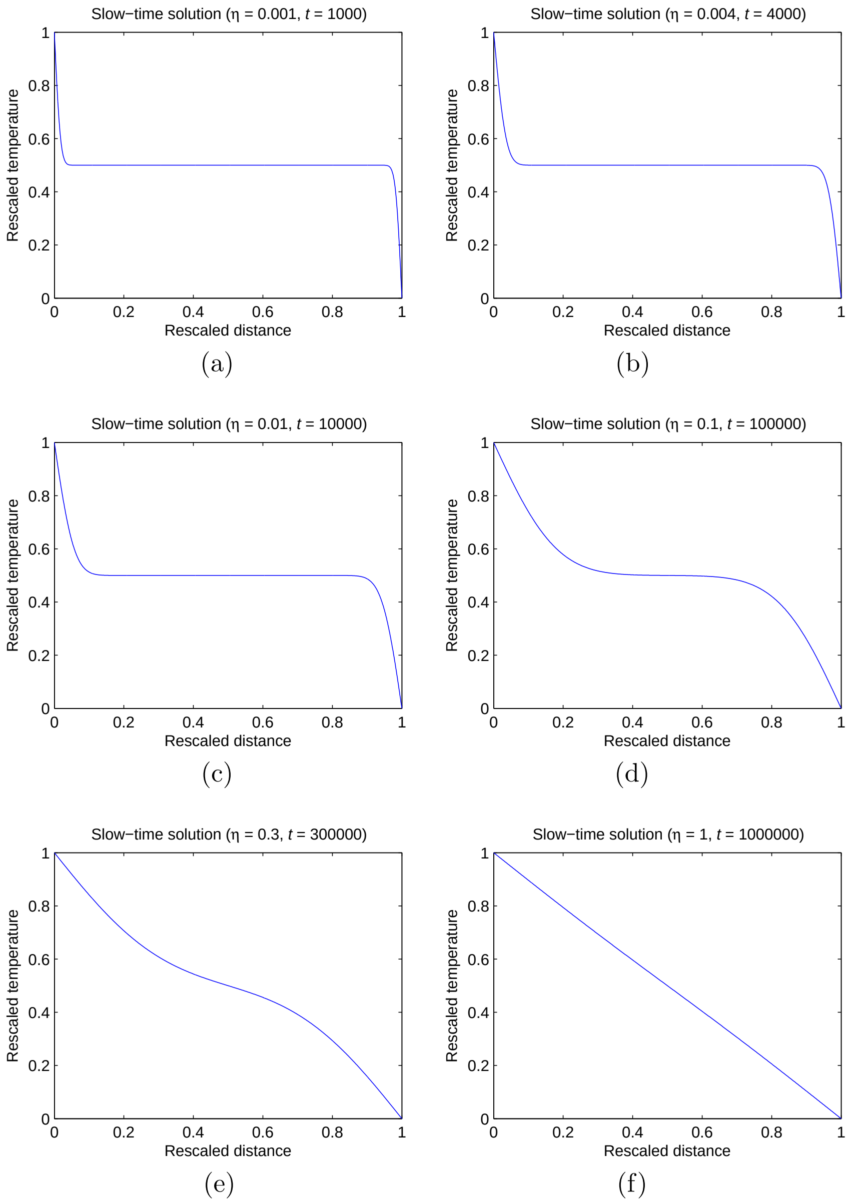

Figure 2. "Slow-time" solution: here $\epsilon = 0.001$; snapshots at times $\eta = 0$, 0.6, 1.8, 2.6, 3.6, 8.6; $t = \eta/\epsilon^2$.