FORMULAS FOR CHARACTER VARIETIES OF 2-BRIDGE KNOTS

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Abstract. We introduce the first closed-form formula for the defining polynomials for the character varieties of an infinite collection of hyperbolic knot manifolds. In particular, these knots belong to a subfamily of 2-bridge knots.

1. Introduction

We study the relationship between knot manifolds and their $\mathrm{SL}_2\mathbb{C}$- and $\mathrm{PSL}_2\mathbb{C}$-character varieties. This relationship has been a very active research field for the last 30 years and has yielded many well-known and fruitful theorems and applications including proofs of the Cyclic Surgery Theorem [3] and the Smith Conjecture [11].

Computations of character varieties are difficult and time consuming. In the literature, there are explicit calculations only for the simplest manifolds. Here, we introduce the first closed-form formula for the defining polynomials for the character varieties of an infinite collection of hyperbolic knot manifolds. This formula reveals connections between Pascal's triangle and the coefficients of the defining polynomials for the character varieties of two-bridge knots.

1.1. Character varieties.

We discuss the $\mathrm{SL}_2\mathbb{C}$-character variety for the complement N of a 2-bridge knot in S^3. Let $\Gamma = \pi_1 N$ and denote the set of $\mathrm{SL}_2\mathbb{C}$-representations of Γ as $\mathsf{R}(N)$ and the set of characters of representations in $\mathsf{R}(N)$ as $\mathsf{X}(N)$. Let $t\colon \mathsf{R}(N) \to \mathsf{X}(N)$ be the map which takes representations to their characters. It is shown in [4] that $\mathsf{R}(N)$ and $\mathsf{X}(N)$ are affine algebraic sets defined over $\mathbb{C}$ and the map t is regular. Henceforth, we will refer to $\mathsf{R}(N)$ and $\mathsf{X}(N)$ as the representation variety and character variety for N.

Culler and Shalen have revealed deep connections between the topology of N and the character variety $\mathsf{X}(N)$. For more background, see [4], the survey article [12], or Chapter 1 of [3]. Using their theory, it is well known and not difficult to deduce that every algebraic component of $\mathsf{X}(N)$ has dimension one. See, for example, Theorem 8 of [2]. Also, every 2-bridge knot is either hyperbolic or a torus knot and in both of these cases it is known that $\mathsf{X}(N)$ contains irreducible characters.

Take $\mathsf{X} \subseteq \mathsf{X}(N)$ an algebraic curve. Results from classical algebraic geometry [8] guarantee a smooth projective curve $\widetilde{\mathsf{X}}$, unique up to isomorphism, and a birational map $\xi\colon \widetilde{\mathsf{X}} \to \mathsf{X}$. The map ξ is defined only on a Zariski open set of $\widetilde{\mathsf{X}}$. The complement of this set is finite and is referred to as the set of *ideal points* for X. Deep results of Culler and Shalen show how to use the ideal points of X to find essential surfaces in N.

There is a nearly identical theory [1] for the $\mathrm{PSL}_2\mathbb{C}$-character variety for N. We use $\mathsf{Y}(N)$ to denote the $\mathrm{PSL}_2\mathbb{C}$-character variety for N.

1.2. **The 2-bridge knots $\mathbf{J(k,l)}$.** There has been some recent progress in obtaining explicit information about $\mathsf{Y}(N)$ when N is the exterior of a certain class of 2-bridge knots $\{J(k,l)\,|\,k,l\in\mathbb{Z}\}$. Given k and l, one obtains a knot $J(k,l)$ by replacing the boxes in Figure 1 labeled k and l with a two strand alternating braid with $|k|$ and $|l|$ crossings, respectively. The signs of the braids are taken to match the signs of k and l. Define $N(k,l)$ to be the complement of $J(k,l)$ in S^3 and $\Gamma(k,l)=\pi_1 N(k,l)$. It is not hard to see that $J(k,l)$ is a knot if and only if kl is even. If $J(k,l)$ is a knot, one may apply symmetries of the knot to assume that k is positive and $l=2d$ is even.

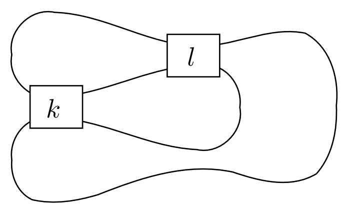

FIGURE 1. The 2-bridge knot $J(k,l)$

The papers [5], [6], and [7], which are based on work of Riley ([9] and [10]), give recursively defined equations for the character variety and A-polynomial for $J(k,2d)$. Further, [7] uses these formulas to compute the genus of the algebraic components of $\mathsf{Y}(N(k,2d))$.

Let
$$w_k = \begin{cases} (ab^{-1})^n(a^{-1}b)^n & \text{if } k=2n \\ (ab^{-1})^n(ab)(a^{-1}b)^n & \text{if } k=2n+1. \end{cases}$$
Then $\langle a,b\,|\,aw_k^d = w_k^d b\rangle$ is the standard two-bridge presentation for $\Gamma(k,2d)$. In particular, a and b represent the meridian slope. Let Y be the union of all algebraic components of $\mathsf{Y}(N(k,2d))$ excluding the abelian component Y_A. Let X be the corresponding algebraic set in $\mathsf{X}(N(k,l))$. Define the following elements of $T(\mathsf{X})$.
$$\begin{aligned} &x=I_a && z=x^2 && v=I_{a^2}=z-2 \\ &y=I_{ab} && r=I_{ab^{-1}}=z-y && t=I_{w_k}. \end{aligned}$$
The functions y and z generate the $\mathrm{PSL}_2\mathbb{C}$-trace ring $T(\mathsf{Y})$. It follows that Y is isomorphic to a plane curve in $\mathbb{C}^2$ with coordinates (z,y). This is the standard embedding of Y. Consider the regular map
$$(r,t)\colon\ \mathsf{Y}\to\mathbb{C}^2$$
and let Y' denote the closure of its image. Proposition 4.4 of [7] shows that (r,t) is birational onto Y'.

Let $\epsilon=\mathrm{sign}(d)$ and $n=\frac{k-1}{2}$ if k is odd and $n=\frac{k}{2}$ if k is even. Choose a branch of the square root function and define
$$\begin{aligned} &U=\sqrt{t^2-4} && V=\sqrt{r^2-4} \\ &A=(t+U)^{|d|}-(t-U)^{|d|} && B=(t+U)^{|d|}+(t-U)^{|d|} \\ &C=(r+V)^n-(r-V)^n && D=(r+V)^n+(r-V)^n \end{aligned}$$

Recall that $J(k,2d)$ is a hyperbolic knot if and only if $k \geq 2$, $d \neq 0$, and $(k,d) \neq (2,1)$.

Theorem 1. *Suppose that $J(k,2d)$ is hyperbolic. Then*

$$\phi(k,d) \;=\; \begin{cases} \frac{BU((r-2)C+DV))+\epsilon\cdot(2-t)((r+2)AC+ADV)}{UV\cdot 2^{n+|d|+2}} & \text{if } k \text{ is odd} \\[2ex] \frac{BCU(r-2)+\epsilon\cdot(2-t)ADV}{UV\cdot 2^{n+|d|+1}} & \text{if } k \text{ is even} \end{cases} \tag{1}$$

is an element of the polynomial ring $\mathbb{Z}[r,t]$. Moreover, the zero set in $\mathbb{C}^2$ of this polynomial is precisely Y'.

The proof of this theorem follows from the recursive formulas in [7]. We use induction to show that the coefficients of a defining polynomial for Y' satisfy some simple formulas given in terms of binomial coeficients involving k and d. Theorem 1 then follows from these formulas and the binomial theorem.

Aside from the obvious advantages of having a closed-form expression for this polynomial, notice also that this formula is a big step forward in computational efficiency. Using a standard laptop computer and the recursive formulas from [7], it took almost 5 minutes to compute the defining polynomial for Y' when J is a 60 crossing twist knot. Using formula (1), the same computer returned the polynomial in 0.008 seconds. The polynomial for a 5000 crossing twist knot was calculated in just over a minute. This calculation was previously untouchable. The polynomial for a 20,000 crossing twist knot was calculated in 1 hour 20 minutes and, in just over a day, formula (1) returned the polynomial corresponding to a knot with 50,000 crossings.

As pointed out in [7], the trace function t is a polynomial of the form $t(z,y) = p(y)+q(y)z$, where $p,q \in \mathbb{Z}[y]$. The authors prove this by giving a recursively defined formula for t. Similar to the formula in Theorem 1, we have closed-form formulas for p and q in terms of k. This in turn, gives a closed-form formula for $\mathsf{Y}(N(k,2d))$ in the usual coordinates (z,y). Given a solution (r_0,t_0) to $\phi(k,d)=0$, the formula for t may be used to find an explicit representation $\rho\colon \Gamma(k,2d) \to \mathrm{PSL}_2\mathbb{C}$ which represents this character. In particular, these formulas should be useful in understanding the geometric structure of $N(k,2d)$ and the geometric structure of all those hyperbolic manifolds obtained from $N(k,2d)$ by Dehn filling.

2. Basic formulas

For $n \in \mathbb{Z}$, define the polynomial $f_n \in \mathbb{Z}[z]$ as in Definition 3.1 of [7]. That is,

$$f_0 = 0, \qquad f_1 = 1, \qquad \text{and} \qquad f_{n+1}+f_{n-1}-zf_n = 0.$$

Lemma 2.1. *For $n \geq 1$, define a function $\varphi_n(z)\colon \mathbb{C} \to \mathbb{C}$ by*

$$\varphi_n(z) \;=\; \sum_{0 \leq j \leq \frac{n-1}{2}} (-1)^j \binom{n-j-1}{n-2j-1} z^{n-2j-1}.$$

Then, for $n \geq 2$,

$$\varphi_1 = 1, \qquad \varphi_2(z) = z, \qquad \text{and} \qquad \varphi_{n+1}(z)+\varphi_{n-1}(z)-z\varphi_n(z) = 0.$$

Proof. It is immediate that $\varphi_1 = 1$, $\varphi_2(z) = z$. We have

$$\varphi_{n+1}(z) \;=\; z^n + \sum_{0 \leq j \leq \frac{n-2}{2}} (-1)^{j+1} \binom{n-j-1}{n-2j-2} z^{n-2j-2}$$

and

$$\varphi_{n-1}(z) \;=\; \sum_{0\le j\le \frac{n-2}{2}} (-1)^j \binom{n-j-2}{n-2j-2} z^{n-2j-2}.$$

So $\varphi_{n+1}(z)+\varphi_{n-1}(z)$ can be written as

$$z^n + \sum_{0\le j\le \frac{n-2}{2}} (-1)^j \left(\binom{n-j-2}{n-2j-2} - \binom{n-j-1}{n-2j-2} \right) z^{n-2j-2}. \tag{2}$$

If $n=2m+1$, apply Pascal's rule to the terms in (2) to obtain

$$\begin{aligned}\varphi_{n+1}(z)+\varphi_{n-1}(z) &= z^n + \sum_{j=0}^{m-1} (-1)^{j+1} \binom{n-j-2}{n-2j-3} z^{n-2j-2} \\ &= \sum_{j=0}^{m} (-1)^j \binom{n-j-1}{n-2j-1} z^{n-2j} \\ &= z\varphi_n(z).\end{aligned}$$

If $n=2m$, note that the last term in (2) is

$$0 \;=\; (-1)^{m-1}\left(\binom{m-1}{0} - \binom{m}{0} \right).$$

Hence, we may drop this last term and use Pascal's rule as before.

$$\begin{aligned}\varphi_{n+1}(z)+\varphi_{n-1}(z) &= z^n + \sum_{j=0}^{m-2} (-1)^{j+1} \binom{n-j-2}{n-2j-3} z^{n-2j-2} \\ &= \sum_{j=0}^{m-1} (-1)^j \binom{n-j-1}{n-2j-1} z^{n-2j} \\ &= z\varphi_n(z).\end{aligned}$$

□

For $n \le 0$, define $\varphi_n(z)$ by the formula $\varphi_n(z) = -\varphi_{-n}(z)$. Using Lemma 2.1 and this definition, it is immediate to check that $\varphi_{n+1}(z)+\varphi_{n-1}(z) = z\varphi_n(z)$ for every $n\in\mathbb{Z}$. This shows that the function φ_n is given by the polynomial f_n.

Lemma 2.2. *For $n\in\mathbb{Z}$, $\varphi_n(z) = f_n(z)$.*

Choose a branch of the square root function and define $\alpha_n\colon \mathbb{C}\to\mathbb{C}$ by

$$\alpha_n(z) \;=\; \left(z+\sqrt{z^2-4}\right)^n - \left(z-\sqrt{z^2-4}\right)^n$$

for $n\in\mathbb{Z}$.

Lemma 2.3. *For $n\in\mathbb{Z}$,*

$$\alpha_{n+1}(z) + 4\alpha_{n-1}(z) = 2z\cdot\alpha_n(z). \tag{3}$$

Proof. Let $y=\sqrt{z^2-4}$ and $x_\pm = z\pm y$. Then $x_\pm + \frac{4}{x_\pm} = 2z$ and

$$\begin{aligned}\alpha_{n+1}(z)+4\alpha_{n-1}(z) &= x_+^n\left(x_+ + \frac{4}{x_+}\right) - x_-^n\left(x_- + \frac{4}{x_-}\right) \\ &= 2z\cdot\alpha_n(z).\end{aligned}$$

□

Let

$$\phi_n(z) \;=\; \frac{\alpha_n(z)}{2^n\sqrt{z^2-4}}.$$

Lemma 2.4. *For $n \in \mathbb{Z}$, $\phi_n(z) = f_n(z)$.*

Proof. By a quick calculation, the lemma holds for $n = 0, 1$. To complete the proof, divide both sides of equation (3) by $2^{n+1}\sqrt{z^2-4}$ to obtain

$$\phi_{n+1}(z) + \phi_{n-1}(z) \;=\; z\phi_n(z).$$

□

Let $\beta_n = x_+^n + x_-^n$. Then, as in Lemma 2.3,

$$\beta_{n+1} + 4\beta_{n-1} = 2z\beta_n.$$

For $j \in \mathbb{Z}$ define

$$\Phi_{2j-1} = f_j - f_{j-1} \qquad \text{and} \qquad \Phi_{2j} = f_j.$$

So

$$\begin{aligned}\Phi_{2d-1} &= f_d - f_{d-1}\\ \Phi_{2d+1} &= f_{d+1} - f_d.\end{aligned}$$

If $k = 2n$ then

$$\begin{aligned}\Phi_{k+1} &= f_{n+1} - f_n\\ \Phi_{k-1} &= f_n - f_{n-1}.\end{aligned}$$

If $k = 2n + 1$ then

$$\begin{aligned}\Phi_{k+1} &= f_{n+1}\\ \Phi_{k-1} &= f_n.\end{aligned}$$

Note that this definition for Φ matches that in [7].

Let $r = I_{ab^{-1}}$ and $t = I_{w_k}$ and define

$$p_{k,d}(r,t) \;=\; \Phi_{k+1}(r)\Phi_{2d-1}(t) - \Phi_{k-1}(r)\Phi_{2d+1}(t).$$

By Proposition 4.4 of [7], $p_{k,d}(r,t)$ is the defining polynomial for Y' $(k \geq 1)$.

So if $k = 2n$ then

$$\begin{aligned}p_{k,d} = f_{n+1}(r)\cdot &\left(f_d(t) - f_{d-1}(t)\right)\\ +f_{n-1}(r)\cdot &\left(f_{d+1}(t) - f_d(t)\right)\\ -f_n(r)\cdot &\left(f_{d+1}(t) - f_{d-1}(t)\right)\end{aligned}$$

and if $k = 2n + 1$ then

$$\begin{aligned}p_{k,d} = f_{n+1}(r)\cdot &\left(f_d(t) - f_{d-1}(t)\right)\\ +f_n(r)\cdot &\left(f_{d+1}(t) - f_d(t)\right).\end{aligned}$$

If we use Lemma 2.4 to simplify these expressions, we obtain Theorem 1.

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