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Commentary: Mathematical Creativity and Giftedness- A review of theory, new operational views and ways forward

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All the comments made in the word document have been addressed.

Commentary

Mathematical Creativity and Giftedness- A review of theory, new operational views and ways forward

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Abstract

In this commentary we synthesize and critique three articles in this special issue of ZDM (Leikin, & Lev; Kattou, Kontoyianni, Pitta-Pantazi, & Christou ; Pitta-Pantazi, Sophocleous, & Christou). In particular we address the theory that bridges the constructs of "mathematical creativity" and "mathematical giftedness" by reviewing the related literature. Finally, we discuss the need for a reliable metric to assess problem difficulty and problem sequencing in instruments that purport to measure mathematical creativity, as well as the need to situate mathematics education research within an existing canon of work in mainstream psychology.

Keywords: Creativity; Mathematical Creativity; Gestaltism; Problem sequencing and metrics; Psychology, mathematics education; Qualitative psychometrics

Introduction

Creativity is, according to Harpen & Sriraman (2011), a buzzword of the 21st century and seen as a major component of education. Yet research on the creativity specifically in mathematics is sparse (Leikin, Berman, Koichu, 2010). Further complicating the picture is the lack of a consistent definition of creativity (Mann, 2006). In this commentary, features of problem solving, problem posing and problem sequencing, as aspects of mathematical creativity are distilled. A discussion of creativity, giftedness and ability serves as a background for the synthesis and critique of three articles in this issue of ZDM (Leikin, R., & Lev, M; Kattou, M., Kontoyianni, K., Pitta-Pantazi., & Christou, C.; Pitta-Pantazi, D., Sophocleous, P.,& Christou. C). First, the need for a reliable and valid metric to assess problem difficulty and

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problem sequencing is discussed. Then the focus of the synthesis shifts primarily on to the conceptual relationship between creativity, ability and giftedness in mathematics. However, characteristics of the mathematically creative will also be extracted from the three articles and implications for mathematics teaching will be commented on. Last, the synthesis attempts to place the findings in the context of work in mainstream psychology.

Historical Origins of the research in Gestalt theory

While investigating the creative process of mathematicians Hadamard writes, in regards to Poincare's own personal research notes:

"...the very observations of Poincare show us three kinds of inventive work essentially different if considered from our standpoint,..

> *-fully conscious work*
>
> *-illumination preceded by incubation*
>
> *-the quite peculiar process of the sleepless night." (Hadamard, 1945, p.35)"*

Poincare describes here the importance of working hard on a problem, then taking a break and letting the mind be occupied by other problems. During this break the mind works subconsciously on the problem until a sudden burst, after a sleepless night (!), of insight into the problem presents itself. Later he then goes back to the problem and verifies the results by language or writing. Hadamard's investigation was inspired by Gestalt psychology and in particular Wallas (1926) four stage model of problem solving: preparation, incubation, illumination and verification. The first stage of preparation refers to an initial period of working on a problem using logic, different strategies and reasoning. If a solution is not reached the problem solver stops working on the problem. This marks the beginning of the incubation period. The incubation period can last from a few minutes to years. During this period the attention of the problem solver is diverted from the problem, either taking a break or focusing on other problems. The third stage, illumination, is when the solution to the problem suddenly appears, sometimes while the problem solver is engaged in unrelated activities. The final stage is when the problem solver goes back to the problem and verifies that the solution is correct.

The first and last stage in the four stage gestalt model, preparation and verification, has not been subject of much research as it has been thought to involve mostly regular reasoning

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Comment [R1]: It seems that some connectio to introduction paper (Leikin & Pitta-Pantazi, this issue) can be made processes (Wallas, 1926). Incubation and illumination, however, have been the focus of many studies. In recent reviews Dodds et al. (2003) and Sio & Ormerod (2009) showed that incubation has a significant effect on problem solving. In the former review, the authors found that problem solving performance is related to incubation length and cues presented during the preparatory phase. In the latter review, the authors investigated more specific moderators such as problem type. The authors concluded that divergent tasks benefited greater from incubation than linguistic and visual insight tasks. Furthermore, longer preparation periods gave a greater incubation effect. The authors go on to claim that the different incubation effects provide support "*for differential invocation of knowledge-based vs. strategic solution processes across different classes of problem."*

There are several theories proposed to explain incubation and illumination. Dodds et al. (2003) presents a short summary of the better known theories. *Unconscious work theory* proposes that the problem solver continues to work on the problem while the problem solver is occupied with other activities. A creative solution is developed unconsciously and then reaches consciousness as a whole. *Conscious work theory*, on the other hand, proposes that a creative solution is found by working intermittently on the problem while attending to cognitive less demanding tasks. Related to the conscious work theory is recovery from fatigue. Working on complicated problems can cognitively fatigue the problem solver. The incubation period is simply a respite period, which allows rejuvenation of problem solving skills.

Other theories are related to more specific cognitive structures and processes. *Forgetting of inappropriate mental sets* refers to how false assumptions are made during the preparation phase. Forgetting these false assumptions during the incubation phase, opens up the possible solution space and allows the problem solver to find a solution. During the preparation period problem solvers sometimes wrongly assume constraints that are not part of the problem. Remote association theory is related to memory structure. When a new problem is encountered, solutions to old, similar problems could be retrieved, but also be inappropriate. Only when these familiar solutions have been discarded, can new creative solutions be discovered. There is nothing special about insight, other than the fact that a remote and unlikely association yielded a solution. Another theory that is related to memory structure is opportunistic assimilation. Unsolved problems are often stored in long term memory. The incubation period is the period when external cues are assimilated and help the problem solver find a solution to the problem.

Domain general views of creativity and giftedness

One of the main challenges in investigating mathematical creativity is the lack of a clear and accepted definition of the term mathematical creativity and creativity itself. Previous examinations of the literature have concluded that there is no universally accepted definition of either creativity or mathematical creativity (Sriraman 2005; Mann, 2005). Treffinger et al. (2002) writes for instance that there are more than 100 contemporary definitions of mathematical creativity. Nevertheless, there are certain parameters agreed upon in the literature that helps narrow down the concept of creativity. Most investigations of creativity take one of two directions: extraordinary creativity, known as big C, or everyday creativity, known as little c (Kaufman & Beghetto, 2009). Extraordinary creativity refers to exceptional knowledge or products that change our perception of the world. Leikin & Pitta-Pantazi relate these issues in their introductory paper to this volume. Feldman, Cziksentmihalyi & Gardner (1994) writes: *"the achievement of something remarkable and new, something which transforms and changes a field of endeavor in a significant way… the kinds of things that people do that change the world."* Ordinary, or everyday, creativity is more relevant in a regular school setting. Feldhusen (2006) describes little c as: *"Wherever there is a need to make, create, imagine, produce, or design anew what did not exist before – to innovate – there is adaptive or creative behavior, sometimes called 'small c."* Investigation into the concept of creativity also distinguishes between creativity as either domain specific or domain general (Kaufman & Beghetto, 2009).

Whether or not creativity is domain specific or domain general, or if you look at ordinary or extraordinary creativity, most definitions of creativity include some aspect of usefulness and novelty (Sternberg, 1999; Plucker & Beghetto, 2004; Mayer, 1999). What is useful and novel depends on the context of the creative process of an individual. The criteria for useful and novel in professional arts would differ significantly from what is deemed useful and novel in a mathematics class in lower secondary school. There is therefore a factor of relativeness to creativity. For a professional artist some new, groundbreaking technique, product or process that changes his or her field in some significant way would be creative, but for a mathematics student in lower secondary school an unusual solution to a problem could be creative. Mathematical creativity in a K-12 setting can as such be defined as the process that results in a novel solution or idea to a mathematical problem or the formulation of new questions (Sriraman, 2005).

Comment [R2]: Leikin & Pitta-Pantazi relate t this issue in their introductory paper as well as th reference and discussion is made in Leikin & Lev (this issue). The authors can probably make a reference to these papers

Giftedness

For decades giftedness was equated with concept of intelligence or IQ (Renzulli, 2005; Brown et al., 2005; Coleman & Cross, 2005). Terman (1925) claimed that gifted individuals are those who score at the top 1% of the population on the Stanford-Binet test. This understanding of giftedness has survived to this day in some conceptions. However, most researchers now view giftedness as a more multifaceted concept in which intelligence is but one of several aspects (Renzulli, 2005). One example is Renzulli's (1986) three ring model of giftedness. In an attempt to capture the many facets of giftedness, Renzulli presented giftedness as an interaction between above average ability, creativity and task commitment. He went on to separate giftedness into two categories: schoolhouse giftedness and creative productive giftedness. The former refers to the ease of acquiring knowledge and taking standardized tests. The latter involves creating new products and processes, which Renzulli thought was often overlooked in school settings. Many researchers support this notion that creativity should be included in the conception of giftedness in any area (Miller, 2012).

In this paper we adopt the following definition of giftedness as be looked at in the domain of mathematics, as Csikszentmihalyi (2000) pointed out the field-dependent character of the concept of giftedness. Due to the lack of a conceptual clarity regarding giftedness and the heterogeneity of the gifted population, both in general and in mathematics, identification of gifted students has varied (Kontoyianni et al., 2011). Instead, prominent characteristics of giftedness in mathematics are found in the research literature. Krutetskii (1976) in his investigation of gifted students in mathematics found a number of characteristic features: ability for logical thought with respect to quantitative and spatial relationships, number and letter symbols; the ability for rapid and broad generalization of mathematical relations and operations, flexibility of mental processes and mathematical memory. Similar features of mathematical giftedness have been proposed by other researchers (see for instance Sriraman, 2005).

Ability

Comment [R3]: It could be good of these part of the paper would include connections to the papers from the special issue

Often, mathematical ability has been seen as equivalent to mathematical attainment and to some degree, there is some truth to that notion. There is a statistical relationship between academic attainment in mathematics and high mathematical ability (Benbow & Arjmand, 1990). However, Ching (1997) discovered that hidden talent go largely unnoticed in typical classrooms and Kim et al. (2003) state that traditional tests rarely identify mathematical creativity. Hong and Aqui (2004) compared cognitive and motivational characteristics of high school students who were academically gifted in math, creatively talented in math and non-gifted. The authors found that the creatively talented students used more cognitive strategies than the academically gifted students. These findings indicate that mathematical ability and mathematical attainment in a K-12 setting is not necessarily synonymous.

In the online thefreedictionary.com, ability is defines as *"the quality of being able to do something, especially the physical, mental, financial, or legal power to accomplish something."* Attainment is defined as "*Something, such as an accomplishment or achievement, that is attained*." The key difference is that ability points to a potential to do something, while attainment refers to something that has been accomplished. In the field of mathematics, mathematical ability then refers to the ability to do mathematics and not the ability to do well on mathematics attainment tests in school. In order to de facto define mathematical ability, mathematics itself has to be defined. It is beyond the scope of this essay to discuss what mathematics itself is (for a K-12 setting, see for instance NCTM, 2000). ~~or Niss, 1999), S~~so mathematical ability will simply be defined as the ability to do mathematics.

Studies that address problem sequencing in measuring creativity

In this section problem sequencing and how it may enhance or perturb aspects of mathematical creativity will be further explored. ~~A google search~~ A literature review on problem sequencing and mathematics yielded no clear definition. Problem sequencing will therefore be defined ad hoc for this particular essay. In mathematics education generalization is defined as the process in which one derives or concludes from particular cases (Davydov, 1990; Polya, 1954; Krutetskii, 1976). Problem sequencing is as such a sequence of mathematical problems carefully chosen and/or created in order to facilitate generalization of relational and structural mathematical properties and similarities[1]. However, research shows that students find it difficult to generalize problems as they tend to focus on superficial properties and not structural and relational properties. This is also seen in Lee & Sriraman (2011) where students focused primarily on superficial properties and similarities when making mathematical analogies.

Comment [R4]: Would be better "literature review"

Comment [R5]: It is not clear how creativity w substituted by generalisation and why

[1] In mathematics theorems are often the formal statements expressing invariant properties detected within a class of examples. According to Ervynck (1991) and Sriraman (2004) generalization is a form of mathematical creativity ~~(p.48)~~.

According to Piaget generalization is an evolving process of abstractive reflection and Skemp (1986) writes that abstraction is "*an activity by which we become aware of similarities… among our experiences*." Mathematical ideas and concepts are developed by abstracting common features in varying situations and learning to ignore the specifics. Mitchelmore (1993) distills this into a basic hypothesis: "*abstraction necessarily follows recognition of similarity*." A key distinction, made by Piaget (1977), here is between abstraction on the basis of superficial characteristics and abstraction on the basis of relationships and structural characteristics. In the setting of problem sequencing the goal becomes the abstraction and generalization of structural and relational characteristics and not superficial characteristics. That means, and for the purpose of this essay, abstraction is the activity by which the learner becomes aware of structural and relational characteristics in a particular problem and generalization is the activity of finding and exploiting common properties across a set of abstractions. In other words, generalizing across a set of problems, based on abstractions of relational and structural properties on individual problems. The desired generalization occurs when superficial differences are seen to be less important than the deep structure. That means the learner has to suppress superficial similarities. Problem sequencing must therefore first and foremost facilitate the abstraction and generalization of deeper structural and relational characteristics and not superficial characteristics.

Mitchelmore's (1993) model for conceptual development in mathematics can shed some light on some of the issues regarding generalization of structural and relational characteristics. Inspired by Skemp's (1986) relational and instrumental understanding, Mitchelmore and White (1995) makes a distinction between two types of abstraction: abstract-general and abstract-apart. In the latter abstractions are made from a few similar situations and there are no links to any base contexts. The underlying mathematical idea (idea is here a wide concept that includes objects, operations, relations etc) is neither at a higher or lower level of abstraction than the contexts in each given situation. Mitchelmore & White goes on to say that abstract-apart ideas are meaningless and useless and never becomes linked to any other situations than from those which it originated. Abstract-general, however, is the result of abstraction in a variety of different contexts and the learner is able to recognize the underlying idea both on its own and in each different individual context. The result is that the learner can apply the underlying idea in a variety of different situations, but also see the underlying idea in a particular situation.

On the basis of this theoretical distinction, it becomes clear that problem sequencing must consist of problems in a wide variety of different contexts, in order to facilitate abstract-general abstraction and generalization. Problem sequencing consisting of problems in a multitude of different contexts and situations does not guarantee abstract general abstraction and the identification of structural similarities, but it is a necessary premise if we accept Mitchelmore & White's model for abstraction and generalization. When problems are given in a wide variety of contexts and situations, the learner has to suppress superficial conditions and similarities and instead focus on deeper, structural similarities. This ability to identify intrinsic properties and generalize across mathematical situations is closely related to mathematical creativity. Krutetskii (1976) proposed that the ability to generalize was an essential characteristic of mathematical giftedness. He went on to write that "*students with different abilities are characterized by differences in degree of development of both the ability to generalize mathematical material and the ability to remember generalizations.*" (p. 84). Haylock (1985) lends support to this claim by saying that generalization in itself is creative by linking previously unrelated mathematical ideas into one, single idea. However, he warns that generalized thinking may be found to conflict with the thinking required to break from certain mental sets. In problem solving situations, the ability to overcome fixation of thought and display flexible reasoning, based on intrinsic mathematical properties, is a key aspect of mathematical creativity (Haylock, 1987; Lithner, 2008). As with problem solving, the challenge for the learner in problem sequencing situations, if mathematical creativity is to be enhanced, is to identify intrinsic, structural characteristics and suppress superficial characteristics.

Comment [R6]: Krutetskii connects ability to generalise to mathematical giftedness that he cal "high ability in mathematics" he does not explicit talk about creativity

The Gestalt model for problem solving can also provide a helpful framework for enhancing mathematical creativity through problem sequencing. Although more commonly applied to one single problem, it can also be seen in the context of a sequence of problems. As outlined earlier, the Gestalt model outlines four stages of problem solving: preparation, incubation, illumination and verification. That means students need the opportunity and flexibility required to not only spend some time working on a problem (i.e. preparation period), but also allow time for the incubation period in order to promote the chance for an aha moment of illumination; a situation analogous to the world of professional mathematics (Sriraman, 2009). Morgan and Forster (1999) reported that time constraints were one of the obstacles to creativity. When students have worked on a problem for some time, they should be allowed to set aside that problem and focus on other tasks and thus transition from conscious to unconscious work.

The two most important factors, in the author's opinion, for problem sequencing to be a fruitful endeavor in terms of developing mathematical creativity, are a flexible learning environment and superficially diverse problems that all have an underlying structural similarity. However, when a broader view of problem solving is taken, it is perceivable that other factors may enhance or perturb aspects of mathematical creativity in problem sequencing. Schoenfeld (1992) claims that there is a general agreement on the importance of the following five aspects of cognition in problem solving: knowledge base, problem solving strategies, monitoring and control, beliefs and affects and practices. The five aspects provide a framework for problem solving, but are also analogous to problem sequencing. First because learners work on individuals problems in a particular sequence of problems and second because the five aspects can be seen in light of each individual problem or how the learner thinks about a sequence of problems. Each of the five aspects comes with a plethora of empirical and theoretical research and it is well beyond the scope of this essay to ascertain how each of the five aspects relates to problem sequencing. Instead they are simply mentioned quickly as important aspects that influence problem sequencing.

According to Schoenfeld a mathematical knowledge base is the information relevant to a specific mathematical situation an individual possess and how that information is accessed and used. The notion of a mathematical situation is, however, does not just have to be limited to one, isolated problem. Several connected mathematical problems, such as a sequence of particular problems, is a mathematical situation. The difference is just in what perspective the individual has: is he looking for a particular solution in one particular problem or is he trying to generalize and find a "solution" across a range of linked problems. For the purpose of enhancing mathematical creativity through problem sequencing, the learners must possess the necessary knowledge to work on the problems. As Schoenfeld asks, "*Did they fail to pursue particular options because they overlooked them, or because they didn't know of their existence?"* The problems in a problem sequence must be within the learners' knowledge base or within proximity of their knowledge base.

Problem solving strategies, or heuristics, refers to experience based techniques for solving problems. The problem(!) is when heuristics are reduced to a set of procedures applied blindly and randomly. Heller and Hungate (1985; from Schoenfeld, 1992)makes the following recommendations for teaching usable problem solving strategies: *"(a) Make tacit processes explicit (b) get students talking about processes; (c) provide guided practice; (d) ensure that component procedures are well learned; and (e) emphasize both qualitative understanding and specific procedures."* Similar, implicit, recommendations are seen in Sriraman (2004) and Lee & Sriraman (2011), where the authors encourage and stimulate the students to talk and think explicitly about the problem solving process. Encouraging students to think and talk about the problem solving process also stimulates their metacognition; an important aspect of problem solving.

Student beliefs and classroom practices also influence problem solving behavior. One of the more common misconceptions that serve as an example here is the idea that all problems can be solved in a matter of minutes. This creates a problem vis-à-vis the gestalt model, where a flexible learning environment allows the students to work on problems for a prolonged period of time. A flexible learning environment, where students are given time and opportunity to work on problems, is fruitless unless the students are also willing to invest time and effort into the preparation phase of the gestalt model. The problems must therefore be both challenging and interesting for the students and the students must know that spending time and effort on a problem is an essential aspect of mathematics.

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Metric for gauging problem difficulty

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In the last section of the essay, the author will comment on ways in which problem difficulty in the form a metric can gauge problem sequencing and how it may be empirically validated in order to study mathematical creativity. According to thefreedictionary.com a metric is a standard of measurement and problem sequencing was defined in this essay as a sequence of mathematical problems carefully chosen and/or created in order to facilitate generalization of relational and structural mathematical properties and similarities. First a metric will be proposed and then the author will discuss how it may be empirically validated.

Problem difficulty as a concept is not easily defined and even harder to operationalize for several different reasons. Traditionally, the difficulty level of a problem is measured by a ratio called the item difficulty ratio, which is the ratio between the number of respondents who answer correctly and the total number of responses to the problem (Gronlund, 1981). However this ratio can only be calculated a posteriori, i.e. problem difficulty can only be calculated after students have attempted to solve said problem. Furthermore there are issues related to validity, as Mason, Zollman, Bramble, and O'Brien (1992) points out: "*This* *definition (difficulty in terms of item difficulty ratio) implies that an easy item also would be easy in terms of the cognitive challenge it presents to a respondent. Such a conclusion might be incorrect. Easy items might be answered correctly for the wrong reasons.*"

The term difficulty in this context points to a subject that is cognitively engaged in problem solving, but this also means that there is a subjective component to problem difficulty. Individual subjects will find the same problem varying in difficulty, based on their problem solving ability. Schoenfeld, as mentioned earlier, outlined five categories that influence problem solving abilities: knowledge base, heuristics, monitoring and control, beliefs and affects and practices. Furthermore, to generate a metric for gauging problem difficulty, the concept has to be operationalized. However, as individuals perceive the same problems differently and it is impossible to know every variable that influence each individual's perception of a particular problem, the best case scenario is a pragmatic approach; or in other words a metric that works and consists of certain characteristics based on a theoretical and empirical foundation.

In summarizing previous research, Caldwell & Goldin (1979) notes that numerous variables have been shown to significantly influence difficulty of word problems: context familiarity, number of words, sentence length, readability, vocabulary and verbal cues, magnitudes of numbers, the number and type of operations or steps and the sequence of operations. In a later study the authors conclude that concrete problems are less difficult than abstract problems and factual problems are less difficult than hypothetical problems. Developing a measure for problem difficulty in algebra, Lee & Heyworth (2000) proposed six factors affecting problem difficulty: number of difficult steps, number of steps in total, numerical complexity, number of occurrences of log, number of operations in the question and degree of familiarity to the students. These empirical findings, combined with a theoretical framework for problem solving (for instance Schoenfeld's framework) a metric for gauging problem difficulty could be developed.

Ervynck (1991) saw mathematical creativity in terms of three stages. The first level, stage 1, refers to a technical or practical application of some kind of mathematical rules or procedures, without any theoretical foundation. The second stage, level 2, consists of algorithmic activity. Primarily performing and applying mathematical techniques and modeling. The third stage, level 3, is considered the creative stage and it represents the true mathematical activity and makes use of non-algorithmic decision making. Problem difficulty could be classified according to Ervynck's levels of creativity. Where simpler problems only require technical work, while more difficult problems require insight and non-algorithmic decision making.

Previous studies and a theoretical framework for problem solving provide a starting point for developing a metric. Based on this, problem difficulty can be conceptualized and observable characteristics that make up the concept's dimensionality can be extracted. However, in operationalizing the concept of problem difficulty, measurable items need to be created. Items that are both reliable and valid. Earlier research indicates that context familiarity, number of words and operations in the question, number of operations and/or steps, numerical complexity and abstract vs. factual problems are all variables that influence problem difficulty and can be measured a priori. Ervynck (1991) classified creativity into three levels according to what kind of thinking and cognitive activity is involved and required in the problem solving process. These variables could form a starting point for developing a metric for gauging problem difficulty of mathematical problems. How each individual variable would be measure and the exact relationship between the variables in the metric (for instance a sum score) is beyond the scope of this essay.

To further strengthen validity of such a metric, problem difficulty could be compared with item difficulty. First, an a priori analysis of problem difficulty could be carried out by the teacher/researcher. Second, after the students have worked on the problem, the teacher/researcher can collect the students' answers and calculate the item difficulty. If the metric for gauging problem difficulty is valid (and reliable) there should be a strong correlation between it and item difficulty over time. However, correctness is just one of several possible assessment criteria for problem solving. Building on the work of Ervynck (1991) and others (Torrance, 1974; Krutetskii, 1976; silver, 1997), Leikin and Lev (in press) proposed evaluating mathematical creativity in terms of *Multiple Solution Tasks* (MST). A multiple solution task is an assignment where a student is explicitly asked to solve a mathematical problem in different ways. The solutions are then assessed according to three dimensions: fluency, flexibility and originality. These three dimensions have been commonly used to assess mathematical creativity in varying combinations (Balka, 1974; Haylock, 1987; Kattou et al., in press; Pitta et al., in press). Fluency is the number of solutions, flexibility is the number of different categories of solutions and originality is the relative unusualness of the solution. As with item difficulty, fluency, flexibility, originality or some combination of them, could be used to gauge problem difficulty a posteriori in terms of creativity.

Conceptual relationships in the ZDM papers

The three studies synthesized in this essay all focus on aspects of mathematical creativity, mathematical giftedness and mathematical ability. Here, conceptual relationships will first be investigated. In each of the three articles mathematical creativity is linked to other concepts, relevant to both ability and giftedness in mathematics. Second, each of the three articles looks closer at what characterizes mathematically creative people. Third, implications for teaching and learning mathematics in school will be discussed.

The main purpose of all three articles was to investigate the relationship between mathematical creativity and mathematical giftedness. In the first article by Kattou et al. (in press), the relationship between mathematical ability and mathematical creativity was investigated quantitatively with the use of a mathematical ability test and a mathematical creativity test. Data were collected by administering the two tests to 359 elementary school students. The authors concluded, using confirmatory factor analysis, that mathematical creativity is a subcomponent of mathematical ability. Mathematical ability was measured by 29 items in the following categories: quantitative ability, causal ability, spatial ability, qualitative ability and inductive/deductive ability. The operationalization of mathematical ability was based on the assumption that mathematical ability is a multidimensional construct and Krutetskii's (1976) classification of giftedness in mathematics. Mathematical creativity was measured with five open ended multiple solution tasks that were assessed on the basis of fluency, flexibility and originality (Leikin, 2007).

Pitta et al. (this issue) investigated the relationship between mathematical creativity and cognitive styles. Mathematical creativity was measured similarly by Leikin et al. (this issue). These studies seemingly use multiple solution tasks but use different scoring schemes. A mathematical creativity test consisting of five tasks was given to 96 prospective primary school teachers and was assessed on the basis of fluency, flexibility and originality. Cognitive style was measured with the *Object-Spatial Imagery and Verbal Questionnaire* (OSIVQ) with respect to three styles: spatial, object and verbal. Using multiple regression, the authors conclude that spatial and object styles were significant predictors of mathematical creativity, while verbal style was not significant. Spatial cognitive style was positively related to mathematical creativity, while object cognitive style was negatively related to mathematical creativity. Furthermore, spatial cognitive style was positively related to fluency, flexibility and originality, while object cognitive style was negatively related to originality and verbal cognitive style was negatively related to flexibility.

Comment [R7]: The studies simingly use multiple solution tasks but use different scoring schemes. Leikin & lev are aimed to demonstrate t power of the suggested scoring scheme

Comment [R8]: Does the author refer to Leiki & Lev? Or another article?

The article, by Leikin and Lev (in press), explored the relationships between mathematical excellence, mathematical creativity and general giftedness. Three groups of 11^{th} and 12^{th} grade students participated in the study: G group, HL group and RL group. The G group consisted of generally gifted students (IQ>130) who excel in mathematics. The HL group consisted of students who learn mathematics at a high level of mathematics instruction, while the RL group consisted of students who learn mathematics at a regular level of mathematics instruction. Four problems, plus one bonus problem, was given to 51 students – 6 from the G group, 27 from the HL group and 18 from the RL group. The test was scored on the basis of fluency, flexibility, originality and creativity. In addition the correctness of the solutions was also evaluated. An obvious limitation of the study is the small sample even though the study reports on the solutions produced by 51 out of a total of 155 participants. Nevertheless, the authors do report several interesting findings. The G students had higher scores than HL students on all the criteria, and HL students had higher scores than the RL students on all criteria. Furthermore, the authors conclude that the differences are task dependent. The G factor had a significant effect on rich problems open for insight-based solutions, while it did not have an effect on more conventional, calculation problems. Based on the findings in the study, the authors propose that knowledge associated with solving conventional and algorithmic problems can be developed in all students. However, problems that require some form of insight, requires a more particular ability level, such as the G factor (IQ).

Comment [R9]: Leikin & Lev report on the solutions produced by 51 out of a total of 155

The three articles all focus on slightly different, but related characteristics of mathematical creativity. Nevertheless there are certain similarities that might be inferred on a structural level and the studies add significantly to our overall understanding of giftedness, creativity and ability in mathematics. Certain cognitive styles, mathematical ability and general giftedness are all found to predict and have a statistical relationship with mathematical creativity. High IQ, spatial cognitive style and general mathematical ability are all linked to mathematical creativity. As a concept, mathematical creativity does not exist in a vacuum. The literature synthesized in this essay suggests that certain features and factors are required for mathematical creativity to arise. Pitta et al. (in press) points out that previous research has found that spatial cognitive style can be beneficial for physics, mechanical engineering and mathematics tasks (see for instance Kozhevnikov et al., 2005). In Leikin and Lev (this issue) the gifted students (high IQ) scored higher than the other students on all measured criteria on all tasks in the mathematics creativity test. On the other hand, the authors conclude that the level of instruction (the L factor) only has a minor effect on students’ creativity. Kattou et al.

(this issue) found a strong correlational relationship between mathematical ability and mathematical creativity. If we assume that all three articles (Leikin and Lev; Kattou et al; Pitta et al.) operationalize mathematical creativity similarly, it becomes clear that general giftedness (high IQ), mathematical ability and spatial cognitive style are all linked with mathematical creativity.

Syntheses of the ZDM studies into the existing canon

Closely related to conceptual relationships between mathematical creativity and other concepts, is the question of "who are mathematically creative?" Can individuals be distinguished into separate groups according to their mathematical creativity and what characterizes these groups? Kattou et al. (this issue) clustered students into three subgroups: low, average and high mathematical ability. The high ability students were also highly creative students, the average ability students had an average performance on the mathematical creativity test, while low ability students have a low creative potential in mathematics. Pitta et al. (this issue) classified the prospective teachers as spatial visualizers, object visualizers or verbalizers. The spatial visualizers scored higher on the mathematical creativity test than both other groups. In the third article examined here, Leikin and Lev (this issue) categorize students into three distinct groups: gifted (G), high level instruction (HL) and regular level instruction (RL). The gifted students scored higher on all criteria on the mathematics creativity test than the other two groups. The gifted students scored in particular higher than the other two groups on problems that were open for insight-based solutions.

All three studies can be said to cluster individuals according to their level of mathematical creativity. In the three studies mathematically able students, students with a preference for spatial cognitive style and gifted students (high IQ students) were all mathematically creative. Or correspondingly mathematically creative students are characterized by high IQ, a preference for spatial cognitive style and a high mathematical ability. However, the observant reader will however perhaps notice the seeming inconsistency in the findings of Kattou et al. (this issue) and Leikin and Lev (this issue). In Leikin and Lev the high level instruction group of students can be classified as mathematically able students. The students in the high level instruction group are previous high achievers in mathematics. Kattou et al. report that mathematically able students are also mathematically creative students. Leikin and Lev, on the other hand, found that gifted (or high IQ) was a much stronger predictor of mathematical creativity than level of instruction. The gifted students performed in particular stronger on insight-relevant mathematical problems. Leikin and Lev points out that gifted students do not necessarily excel in mathematics and can learn both HL and RL mathematics. Nevertheless it would have been interesting to further examine the mathematically able students in the study by Kattou et al. Haylock (1997), for instance, claims that within the group of high achievers in mathematics there are both low-creative and high-creative students.

Implications for teaching

Although only one of the articles (Kattou et al., this issue) makes explicit recommendations for mathematics teaching, the implications of the three articles are on some levels related. Kattou et al. concludes that the encouragement of mathematical creativity is important for further development of students' mathematical ability. More importantly, they write, teachers should not limit their teaching to spatial conception, arithmetic, proper use of methods and operations. Teachers should recognize the importance of creative thinking in the classroom. Leikin and Lev (this issue), on the other hand, write that a particular type of mathematical problems that require insight, require a particular ability level; ability refers here to general giftedness (high IQ). Nevertheless they also point out that *"Based on the findings of this study we argue that students' knowledge associated with solving relatively algorithmic problems can be developed in all groups of students and is related to a similar level of creativity on these types of mathematical problems."* It seems both studies conclude that mathematical creativity can be developed in all students.

The article (Pitta et al., this issue) did not make any explicit recommendations or implications for teaching mathematics. However, as they investigated prospective teachers' mathematical creativity, the results and conclusions may be relevant for teaching when seen in a broader perspective. Pitta et al. found that spatial visualizers had a statistical significant higher creative performance than other teachers. The observed differences were related to the different strategies employed by the spatial visualizers, object visualizers and verbal visualizers. The spatial visualizers employed more flexible and analytic strategies to tasks. This allowed them to be more creative and provide more, different and unique solutions. In light of Lev-Zamir and Leikin (this issue) and Pitta el al (this issue) conclusions, the question becomes whether a flexible and analytic approach to mathematics tasks translates into an analytic and flexible approach to mathematics teaching. If that is the case, then flexible and creative teaching is also related to spatial cognitive style. However, as Pitta et al. (this issue) asks, it is unknown whether having a spatial cognitive style is a result of experience or inborn abilities. They go on to recommend further investigation to see if prospective teachers can be trained to use their spatial visualization. It may lead to enhanced spatial imagery and consequently facilitate mathematical creativity, possibly also in their mathematics teaching.

Closing Remarks: Mathematics Education as a subdomain of Psychology in the ~~problem~~ area of mathematical creativity studies.

Giftedness and creativity in psychology

The research into the field of general creativity focus on four different variables: person, process, product and press. The person category highlights the internal cognitive characteristics of individuals. The process category looks at the internal process that takes place during a creative activity. Product focuses on the characteristics of products thought to be creative. Last, the press category explores the ways environmental factors can influence creativity (Taylor, 1988Mathematical creativity is linked to and influenced by ability, beliefs, intelligence, cognitive style and the classroom environment (Leikin and Lev, this issue; Lev-Zamir and Leikin, this issue; Pitta et al., this issue; Kattou et al., this issue). These findings are analogous to much of the research into general creativity and giftedness. The star model of Abraham Tannenbaum (2003) conceptualizes giftedness into five elements, some of which seen in the studies synthesized in this essay: a) superior general intellect, b) distinctive special aptitudes, c) nonintellective requisites, d) environmental supports and e) chance. Creativity is here included in the nonintellective requisites. Mathematical ability would be placed in the distinctive special aptitudes, as it portrays to domain specific abilities, giftedness (or high IQ) as superior general intellect while both beliefs and cognitive styles would be classified as nonintellective requisites. Flexible teaching that stimulates mathematical creativity fall under the category of environmental support. As such, the observations in the studies synthesized here are in many ways analogous to research into general creativity and giftedness.

Similarly, the dynamic theory of giftedness (Babaeva, 1999), which emphasizes the social aspects of the development in giftedness, can also provide a theoretical perspective on the observations synthesized in this essay. This theory consists of three principles that explain the development of giftedness: a) an obstacle for positive growth is introduced, b) a process to overcome the obstacle and c) alteration and incorporation of the experience (Miller, 2012). Kattou et al. (this issue) points out how mathematical creativity is essential for the growth of overall mathematical ability (or giftedness), while Lev-Zamir and Leikin (this issue) shows how challenging mathematical problems and flexible teaching can help development of

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mathematical creativity. Both studies show the dynamic aspect of mathematical creativity, in the sense that it evolves and is influenced by other, external factors. Leikin and Lev also concludes that fluency and flexibility can be developed for all students, pointing out the dynamic nature of mathematical creativity.

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