Topological orderings of weighted directed acyclic graphs

Dániel Gerbner* Balázs Keszegh† Cory Palmer‡
Dömötör Pálvölgyi§

August 28, 2014

Abstract

We call a topological ordering of a weighted directed acyclic graph *non-negative* if the sum of weights on the vertices in any prefix of the ordering is non-negative. We investigate two processes for constructing non-negative topological orderings of weighted directed acyclic graphs. The first process is called a *mark sequence* and the second is a generalization called a *mark-unmark sequence*. We answer a question of Erickson by showing that every non-negative topological ordering that can be realized by a mark-unmark sequence can also be realized by a mark sequence. We also investigate the question of whether a given weighted directed acyclic graph has a non-negative topological ordering. We show that even in the simple case when every vertex is a source or a sink the question is NP-complete.

Keywords: topological ordering, directed acyclic graph

1 Introduction

A *directed acyclic graph* (or DAG) is a directed graph with no directed cycles. A subset M of vertices of G is *outdirected* if every edge between M and $V(G) \setminus M$ is directed towards $V(G) \setminus M$ (i.e., edges directed towards M are contained in M). A *prefix of length k* of a sequence s is the subsequence of the first k terms of s. A *topological ordering* of a DAG G

*Hungarian Academy of Sciences, Alfréd Rényi Institute of Mathematics, P.O.B. 127, Budapest H-1364, Hungary. Research supported by Hungarian National Science Fund (OTKA), grant PD 109537.

†Hungarian Academy of Sciences, Alfréd Rényi Institute of Mathematics, P.O.B. 127, Budapest H-1364, Hungary. Research supported by Hungarian National Science Fund (OTKA), grant PD 108406 and grant NN 102029 (EUROGIGA project GraDR 10-EuroGIGA-OP-003) and the János Bolyai Research Scholarship of the Hungarian Academy of Sciences

‡Department of Mathematical Sciences, University of Montana, Missoula, Montana 59801, USA. Research supported by Hungarian National Science Fund (OTKA), grant NK 78439.

§Eötvös Loránd University, Faculty of Science, Department of Computer Science, Pázmány Péter 1/C, Budapest H-1117, Hungary. Research supported by Hungarian National Science Fund (OTKA), grant PD 104386 and the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

is an ordering of the vertices of G such that every prefix of the ordering is outdirected. The following two processes yield topological orderings of a given DAG G with n vertices.

A *mark sequence* of G is a sequence $M_1, M_2, \ldots, M_n$ of subsets of $V(G)$ formed in the following way: first choose an arbitrary source v and put $M_1 = \{v\}$, i.e., *mark* v in step 1. For $i = 2, 3, 4, \ldots, n$, choose a vertex $u \notin M_{i-1}$ such that $\{u\} \cup M_{i-1}$ is outdirected and put $M_i = \{u\} \cup M_{i-1}$, i.e., *mark* u in step i.

A *mark-unmark sequence* of G is a sequence of subsets of $V(G)$ formed in the following way: first choose an arbitrary source v and put $M_1 = \{v\}$, i.e., *mark* v in step 1. For $i = 2, 3, 4, \ldots$ either choose a vertex $u \notin M_{i-1}$ such that $\{u\} \cup M_{i-1}$ is outdirected and put $M_i = \{u\} \cup M_{i-1}$, i.e., *mark* u in step i or choose a vertex $u \in M_{i-1}$ such that $M_{i-1} \setminus \{u\}$ is outdirected and put $M_i = M_{i-1} \setminus \{u\}$, i.e., *unmark* u in step i. This process stops when $M_i = V(G)$.

Clearly mark-unmark sequences are a generalization of mark sequences. Because we only mark a vertex if the new set M_i is outdirected, we get a topological ordering by arranging the vertices of G by the last step in which they were marked in the mark-unmark sequence. In particular, the ordering of elements given by a mark sequence is simply a topological ordering.

A DAG G is called *weighted* if there is an assignment of real numbers to each vertex of G. We call a topological ordering *non-negative* if the sum of the weights of the vertices in every prefix is non-negative. Similarly a mark-unmark (or mark) sequence is *non-negative* if at each step the sum of the weights in M_i is non-negative. (We use *negative* in place of "not non-negative".)

Clearly a non-negative mark sequence is equivalent to a non-negative topological ordering. However, a non-negative mark-unmark sequence may give a negative topological ordering. For example let G be a weighted DAG on four vertices $\{a, b, c, d\}$ with a single edge bc and weights $w(a) = w(c) = w(d) = 1$, $w(b) = -1$. Consider the following non-negative mark-unmark sequence: mark a, b, c, then unmark a, then mark d and a. This gives the topological ordering b, c, d, a which is negative. This suggests the following question of Erickson [4]: is there a weighted DAG G that has a non-negative mark-unmark sequence but no non-negative mark sequence?

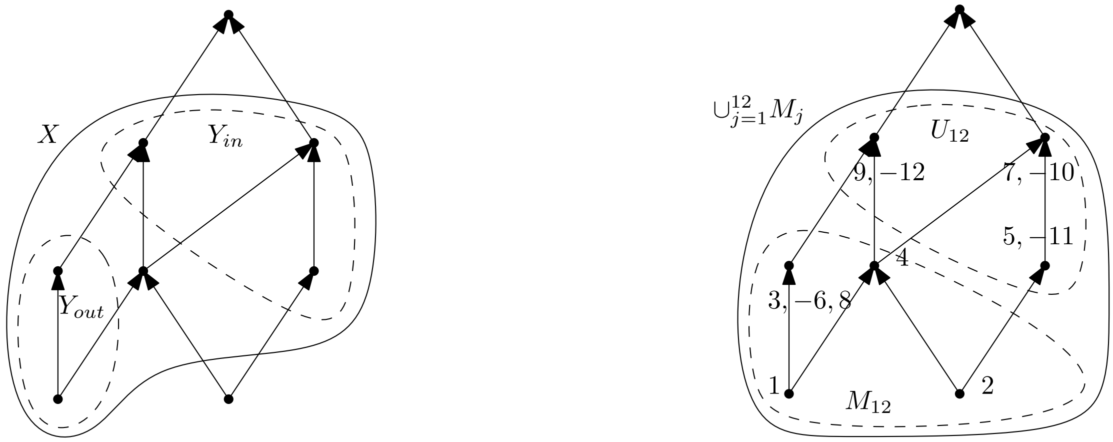
We answer this question in the negative with the following theorem.

Theorem 1. *If a weighted DAG G has a non-negative mark-unmark sequence, then G also has a non-negative mark sequence.*

A natural question is to determine the complexity to decide whether a weighted DAG G has a non-negative topological ordering. However, it is easy to show (see [3]) that this is a subproblem of the NP-complete problem SEQUENCING TO MINIMIZE MAXIMUM CUMULATIVE COST[1].

If we restrict the problem to weighted DAGs that consist of only sources and sinks we will prove that the problem is still NP-complete.

[1]This is problem SS7 in the famous book of Garey and Johnson [5]



(a) Y_{in} is an X-*indirected* set, Y_{out} is an X-*outdirected* set.

(b) 12 steps of a mark-unmark sequence, number j (resp. $-j$) denotes that the vertex was added (resp. removed) in step j.

Figure 1: Examples

Theorem 2. *Let G be a weighted DAG such that every vertex is either a source or a sink. Deciding whether G has a non-negative topological ordering is* NP*-complete.*

We will prove Theorem 1 in Section 2 and prove Theorem 2 using a series of reductions in Section 3.

2 Marking and Unmarking

In this section we prove Theorem 1[2]. In particular, given a weighted DAG G and a non-negative mark-unmark sequence, we will construct a non-negative mark sequence for G. We begin with some definitions. By $w(X)$ we denote the sum of the weights of the elements of a set of vertices X. We say that a set $Y \subset X$ is X-*indirected* if every edge between Y and $X \setminus Y$ is directed towards Y. Similarly, say that a set $Y \subset X$ is X-*outdirected* if every edge between Y and $X \setminus Y$ is directed towards $X \setminus Y$. For simplicity, we call a set of vertices Y of a DAG G *outdirected* (*indirected*) if Y is $V(G)$-outdirected ($V(G)$-indirected). Note that this definition corresponds to the definition of indirected given in the previous section. The first picture in Figure 1 give an example of X-indirected and X-outdirected sets.

Proof of Theorem 1. Let G be a weighted DAG with a non-negative mark-unmark sequence. Let $M_1, M_2, \ldots, M_t$ be a mark-unmark sequence with at least one unmark step (otherwise we are done) of minimum length. For $i \in [t]$, put $U_i = (\cup_{j=1}^{i-1} M_j) \setminus M_i$, i.e., the set of elements that were marked in the first $i-1$ steps, but are currently unmarked (in one of the first i

[2]An early version of our proof below was posted on the web by the fourth author. See http://cstheory.stackexchange.com/questions/1399/positive-topological-ordering-take-2

steps). Note that U_i is $(M_i \cup U_i)$-indirected. The second picture in Figure 1 gives an example of the steps of a mark-unmark sequence and of U_i.

Claim 3. *$w(U_i) > 0$ for all i.*

Proof. We prove the stronger statement that the weight of any U_i-indirected set is positive (U_i is clearly U_i-indirected). Suppose the statement is false and let X be a minimal counterexample, i.e., X is U_i-indirected and $w(X) \leq 0$. If Y is a non-empty X-outdirected set, then $X \setminus Y$ is U_i-indirected and a proper subset of X, hence $w(X \setminus Y) > 0$ (by the minimality of X).

If $w(Y) > 0$, then $w(X) = w(Y) + w(X \setminus Y) > 0$ which is a contradiction. Thus we can suppose that for every X-outdirected set Y we have $w(Y) < 0$. Now let $M'_1, M'_2, \ldots, M'_{t'}$ be the subsequence of $M_1, M_2, \ldots, M_t$ that remains after removing each $M_1, M_2, \ldots, M_{i-1}$ that involves marking or unmarking an element of X.

We claim that this new sequence is also a mark-unmark sequence. First note that the elements of X are in U_i and are therefore marked at some step after i, i.e., each element of X will eventually be marked in the new sequence. Now we show that every M'_j is outdirected. Indeed, if it is not outdirected, then there is an edge uv with $u \notin M'_j$ and $v \in M'_j$. The set M_j is outdirected and contains M'_j (thus contains v), therefore $u \in M_j$. Thus $u \in X$. Now, as X is U_i-indirected, either $v \in X$ (which contradicts $v \in M'_j$), or $v \notin U_i$. But $v \in M_j$ implies it is marked before the ith step in the original sequence, hence $v \notin U_i$ is possible only if it is never unmarked, i.e., $v \in M_i$. However, $u \in X \subset U_i$ implies $u \notin M_i$, which contradicts the outdirected property of M_i.

Now we show that the new mark-unmark sequence $M'_1, M'_2, \ldots, M'_{t'}$ is non-negative. Indeed, let $X'_j = X \cap M_j$, then X'_j is an X-outdirected set, thus $w(X'_j) \leq 0$. Now $w(M'_j) = w(M_j) - w(X'_j) \geq w(M_j) > 0$. This new non-negative mark-unmark sequence is shorter than a minimal sequence, a contradiction. This completes the proof of Claim 3. □

We now construct a new sequence by starting with the original mark-unmark sequence $M_1, M_2, \ldots, M_t$ and skipping every step where a vertex is unmarked or marked beyond the first marking. Let $M''_1, M''_2, \ldots, M''_t$ be the new sequence. We claim that $M''_1, M''_2, \ldots, M''_t$ is a mark sequence. Clearly every vertex will be marked at some point as every vertex will be marked at the end of the original sequence. Furthermore, every M''_i is outdirected. Indeed, if M''_i is not outdirected, then there is an edge uv with $u \notin M''_i$ and $v \in M''_i$. But for v to be marked, u had to have been marked in a previous step. This is a contradiction, thus M''_i must be outdirected. Finally, to show the sequence is non-negative we must prove that $w(M''_i)$ is non-negative for all i. For each i, there is some $j \geq i$ such that $M''_i = M_j \cup U_j$. The original sequence is non-negative and M_j and U_j are disjoint by definition thus by Claim 3 we have $w(M''_i) = w(M_j) + w(U_j) > w(M_j) > 0$ and therefore the mark sequence is non-negative. This completes the proof of Theorem 1. □

Note that with this method we prove a stronger statement. Suppose we want to mark only a certain subset of the vertices, and all the other vertices are used only to help achieve this. Consider, for example, the weighted DAG G mentioned in the introduction with four vertices $\{a, b, c, d\}$, a single edge bc and weights $w(a) = w(c) = w(d) = 1$, $w(b) = -1$. Suppose our goal is to mark the subset $\{b, c\}$. Clearly to mark the subset $\{b, c\}$ we must unmark vertices. For example we can mark a, mark b, mark c and unmark a to get the desired subset. We show that unmarking is needed only to reduce the set of marked vertices to the desired set, i.e., we never perform another mark after the first unmark.

We call a mark-unmark sequence *partial* if it is the same as a mark-unmark sequence except it stops after t steps and M_t may be a proper subset of $V(G)$.

Proposition 4. *Let G be a weighted DAG, and suppose $M_1, M_2, \ldots, M_t$ is a non-negative partial mark-unmark sequence which stops after t steps. Then there is a non-negative partial mark-unmark sequence $M'_1, M'_2, \ldots, M'_{j+k}$ such that the first j steps are markings, the last k steps are unmarkings, and $M'_{j+k} = M_t$.*

We omit some details from the proof of Proposition 4 as it is very similar to the proof of Theorem 1.

Proof sketch. For $i \in [t]$, put $U_i = \cup_{j=1}^{i-1} M_j \setminus M_i$, i.e., the set of elements that have been unmarked in any of the first i steps. As in the proof of Theorem 1 we have that the weight of any U_i-indirected set is positive (from the proof of Claim 3). Let U be the set of vertices that are unmarked at some step in the mark-unmark sequence $M_1, M_2, \ldots, M_t$.

We now construct a new sequence by starting with the original mark-unmark sequence $M_1, M_2, \ldots, M_t$ and skipping every step where a vertex is unmarked or marked beyond the first marking. Let $M'_1, M'_2, \ldots, M'_j$ be the new sequence. Then add $|U| = k$ steps to the resulting sequence where we unmark the vertices of U in the order in which their first unmarking occurred in the original sequence. In other words, when an element of U is unmarked first we move that step to the end of the sequence, and skip all other steps where that element is chosen to be marked or unmarked. The resulting sequence is $M'_1, M'_2, \ldots, M'_{j+k}$.

To finish the proof we need to show that $M'_1, M'_2, \ldots, M'_{j+k}$ satisfies the definition of a mark-unmark sequence (we omit the details) and that the sequence is non-negative. To see that it is non-negative, observe that after any step of the new sequence the set of marked elements is the same as the set of marked elements after some step in the original sequence, with the addition of an U_i-indirected set (which is non-negative by Claim 3). This completes the proof of Proposition 4. □

3 NP-completeness

In this section we will prove Theorem 2. The proof is by a series of reductions of the original problem to a known NP-complete problem. We restate the decision problem in Theorem 2 here.

Problem 5. *Given a weighted DAG G such that every vertex is either a source or a sink, determine whether G has a non-negative topological ordering.*

First we show that there is a reduction from the following problem posed by Rote (see [2]) to Problem 5.

Problem 6. *Given a balanced bipartite graph G, determine whether edges can be added to G to create a bipartite graph with a unique perfect matching.*

We will need the following easy observation.

Observation 7. *Let G be a balanced bipartite graph with classes A and B. The graph G contains a unique perfect matching M if and only if there is an ordering $A = a_1, a_2, \ldots, a_n$ and $B = b_1, b_2, \ldots, b_n$ such that for all $i \in [n]$ we have $a_i b_i \in M$ and $a_i b_j \notin E(G)$ if $1 \leq j < i \leq n$.*

We now transform a given bipartite graph G with classes A and B into a weighted DAG such that every vertex is a source or a sink. First orient all edges in G such that they are directed to B. Then assign weight -1 to every vertex of A and weight 1 to every vertex of B. Finally, add an isolated vertex, v, with weight 1 to G. If G is extendable to a bipartite graph with a unique perfect matching, then v together with the order given by Observation 7 gives a non-negative topological ordering of G. In particular, $v, a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ is a non-negative topological ordering of G. Furthermore, any non-negative topological ordering on A and B satisfies the requirements of Observation 7.

The following problem can be reduced to Problem 6 by adding $n - k$ isolated vertices to each class.

Problem 8. *Let G be a balanced bipartite graph with class sizes n and let k be a positive integer. Determine whether G has an induced subgraph H with k vertices in each class of G such that edges can be added to H to create a bipartite graph with a unique perfect matching.*

To complete the proof of Theorem 2 we must show Problem 8 is NP-complete. An equivalent problem was shown to be NP-complete by Dasgupta, Jiang, Kannan, Li and Sweedyk [1]. We include a new proof here that is shorter and less technical.

The reduction is from the NP-complete problem LARGEST BALANCED INDEPENDENT SET[3]. We call an independent set in a bipartite graph *balanced* if each class of the bipartite graph contains exactly half of the vertices of the independent set. We first state the LARGEST BALANCED INDEPENDENT SET problem.

Problem 9. *Let G be a bipartite graph and let k be a positive integer. Determine whether G contains a balanced independent set on $2k$ vertices.*

Let G be a bipartite graph with classes A and B and let k be a positive integer. Construct a bipartite graph G' as follows. The vertex set of G' consists of $k+1$ copies of each vertex v in G, denoted by pairs $(v, 1), (v, 2), \ldots, (v, k+1)$. We connect two vertices (u, i) and (v, j) in G' by an edge if either of the following are satisfied:

(1) $u \in A$ and $v \in B$ and $i < j$.

(2) $uv \in E(G)$.

[3]In [5] the equivalent problem of finding the largest BALANCED COMPLETE BIPARTITE SUBGRAPH is shown to be NP-complete.

Claim 10. *The graph G' has a subgraph H on $2k^2 + 2k$ vertices such that edges can be added to H to create a bipartite graph with a unique perfect matching if and only if G has a balanced independent set with $2k$ vertices.*

Proof. If G has a balanced independent set with $2k$ vertices, then call H the induced subgraph of G' spanned by the $k+1$ copies of this independent set. Clearly H has $2k^2 + 2k$ vertices. Furthermore, it is easy to see that adding the edges of a matching to each copy of the independent set results in a bipartite graph with a unique perfect matching.

Now suppose that G' has a subgraph H on $2k^2+2k$ vertices such that edges can be added to H to create a bipartite graph with a unique perfect matching. Let A_H and B_H be the two classes of H defined by the partition of G. Now order the vertices of A_H and B_H by Observation 7 such that if $a < b$, then there is no edge between the ath vertex in A_H and the bth vertex in B_H.

Let (w, i) be the first vertex in the ordering of A_H such that among the vertices that appear earlier in the ordering there are $k-1$ different values in the first coordinate. Let a be the index of (w, i) in the ordering of A_H. Let m be the smallest value among the second coordinates of the vertices with index less than a in A_H. Thus we have $a \leq (k-1)(k+1-m+1)$. Therefore $a \geq k^2 + k - (k-1)(k+1-m+1) - 1 = m(k-1) + 1$.

Recall that if $b > a$, then there is no edge between the ath vertex in A_H and the bth vertex in B_H. Furthermore, by (1), for $i < j$ there is an edge between each $(u, i) \in A_H$ and $(v, j) \in B_H$. Thus every vertex in B_H with index $b > a$ has second coordinate at most m. There are at least $m(k-1)+1$ vertices in B_H with index $b > a$. Therefore, by the pigeonhole principle, there is a set of k of these vertices with the same second coordinate. In particular, these k vertices represent distinct vertices in the original graph G. By the definition of (w, i) there is a set of k vertices in A_H with index at most a and distinct first coordinates, i.e., distinct vertices in G. These two sets of vertices form an independent set of size $2k$ in G. This completes the proof of Claim 10. □

Thus we have established that Problem 8 is NP-complete and therefore we have proved Theorem 2.

We end this section with another problem posed by the fourth author [6] that is equivalent to Problem 6.

Problem 11. *Suppose M is an $n \times n$ matrix. Determine whether it is possible to reorder its rows and columns such that we get an upper-triangular matrix.*

Acknowledgments

We would like to thank A. Frank, T. Király and L. Végh for bringing Problem 5 to our attention and G. Rote for useful discussions about the equivalence of Problem 8 and Problem 6..

References

[1] B. Dasgupta, T. Jiang, S. Kannan, M. Li and E. Sweedyk, On the complexity and approximation of syntenic distance. *Discrete Applied Mathematics* **88** (1998), 59-82.

[2] Graphs extendable to a uniquely matchable bipartite graph, Egres Open collection, http://lemon.cs.elte.hu/egres/open

[3] D. Eppstein, Positive topological ordering. http://cstheory.stackexchange.com/questions/1346 (2010)

[4] J. Erickson, Positive topological ordering, take 2. http://cstheory.stackexchange.com/questions/1399 (2010)

[5] M. R. Garey and D. S. Johnson. *Computers and Intractability.* W.H.Freeman & Co. 1979.

[6] D. Pálvölgyi, Positive topological ordering, take 3. http://cstheory.stackexchange.com/questions/1842 (2010)