Rainbow copies of C_4 in edge-colored hypercubes

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Abstract

For positive integers k and d such that $4 \leq k < d$ and $k \neq 5$, we determine the maximum number of rainbow colored copies of C_4 in a k-edge-coloring of the d-dimensional hypercube $\mathcal{Q}_d$. Interestingly, the k-edge-colorings of $\mathcal{Q}_d$ yielding the maximum number of rainbow copies of C_4 also have the property that every copy of C_4 which is not rainbow is monochromatic.

1 Introduction

For a graph G, an edge-coloring $\varphi : E(G) \to \{1, 2, \ldots\}$ of G is *rainbow* if no two edges receive the same color. Throughout this note, we will denote the d-dimensional hypercube by $\mathcal{Q}_d$. A convenient way to consider Q_d is as a graph with vertices corresponding to binary sequences of length d and edges as pairs of vertices with corresponding binary sequences of Hamming distance 1.

Various problems concerning edge-colorings of hypercubes have been studied, see e.g. [1, 2, 3, 4]. In particular, Faudree, Gyárfás, Lesniak and Schelp [5] proved that there is a d-edge-coloring of $\mathcal{Q}_d$ such that every C_4 is rainbow for $d = 4$ or $d > 5$.

Our main result determines the maximum number of rainbow copies of C_4 in a k-edge-coloring of $\mathcal{Q}_d$ for any positive integers k and d such that $4 \leq k < d$ and $k \neq 5$. Note that when $k = d$, by [5], there is an edge-coloring of $\mathcal{Q}_d$ using d colors where *every* C_4 is rainbow.

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Theorem 1. *Fix integers k and d such that $4 \leq k < d$ and $k \neq 5$ and write $d = ka + b$ such that a is a non-negative integer and $b \in \{0, 1, 2, \ldots, k-1\}$. Then the maximum number of rainbow copies of C_4 in a k-edge-coloring of $\mathcal{Q}_d$ is*

$$2^{d-2}\left[\binom{d}{2} - k\binom{a}{2} - ba\right].$$

Interestingly, the k-edge-colorings of $\mathcal{Q}_d$ that yield the maximum number of rainbow copies of C_4 have the additional property that every non-rainbow C_4 is monochromatic.

2 Proof of Theorem 1

Proof. First we prove the upper bound. Assume that $\mathcal{Q}_d$ is k-edge-colored such that the number of rainbow copies of C_4 is maximized. At each vertex v there are $\binom{d}{2}$ incident copies of C_4. For a set of t edges of the same color incident to v, none of the $\binom{t}{2}$ pairs form a rainbow copy of C_4. If there are t_i edges of color $i \in [k]$ incident with v, then there are at most

$$\binom{d}{2} - \sum_{i\in[k]}\binom{t_i}{2} \leq \binom{d}{2} - (k-b)\binom{a}{2} - b\binom{a+1}{2} = \binom{d}{2} - k\binom{a}{2} - ba \qquad (1)$$

rainbow copies of C_4 at v. Summing up (1) for each of the 2^d vertices of $\mathcal{Q}_d$ counts each C_4 four times, which gives the desired upper bound.

Now we prove the lower bound. For each binary sequence coding a vertex of $\mathcal{Q}_d$, we partition the first $(k-b)a$ binary digits into $(k-b)$ blocks, each of length a, and the last $b(a+1)$ binary digits into b blocks, each of length $a+1$. This yields k blocks of consecutive binary digits each of length a or $a+1$. Computing the sum of the terms in each block modulo 2 yields a binary sequence of length k. Thus we have associated a binary sequence of length k with each vertex of $\mathcal{Q}_d$. This gives a map, h, of the vertices of $\mathcal{Q}_d$ to the vertices of $\mathcal{Q}_k$. Recall that the edges of $\mathcal{Q}_d$ are pairs of vertices such that their corresponding binary sequences of length d have Hamming distance 1. If $u, v \in V(\mathcal{Q}_d)$ have Hamming distance 1, then $h(u)$ and $h(v)$ also have Hamming distance 1 since they differ exactly in one block. Therefore, we can also consider h as a map from $E(\mathcal{Q}_d)$ to $E(\mathcal{Q}_k)$. By [5], there is an edge-coloring, say φ, of the edges of $\mathcal{Q}_k$ with k colors such that every C_4 is rainbow. Now let us color the edges of $\mathcal{Q}_d$ with the color of their image under h in $\mathcal{Q}_k$ i.e. the color of an edge e in $\mathcal{Q}_d$ is $\varphi(h(e))$.

Clearly, each vertex in $\mathcal{Q}_d$ is incident to a edges of each of $k-b$ colors and it is also incident to $a+1$ edges of each of the remaining b colors. To complete the proof, we need to check that each pair of edges of different color incident to the same vertex is contained in a rainbow C_4. Among the four vertices in any C_4 the maximum Hamming distance is 2. Thus all differences among the length d binary sequences of the four vertices of the C_4 occur in at most 2 blocks. If all the differences occur in the same block, then the four edges of the C_4 are mapped to the same edge in $\mathcal{Q}_k$, and thus, the C_4 is monochromatic. If the differences occur in 2 distinct blocks, then the four edges of the C_4 are mapped to a C_4 in $\mathcal{Q}_k$ and thus receive different colors in the coloring of $\mathcal{Q}_d$. □

3 Remarks

Theorem 1 omits the case $k = 5$. This is because there is no 5-edge-coloring of $\mathcal{Q}_5$ where every copy of C_4 is rainbow, which was proved in [5]. Using a computer, we showed that the maximum number of rainbow copies of C_4 in a 5-edge-coloring of $\mathcal{Q}_5$ is 73 (there are 80 copies of C_4 in $\mathcal{Q}_5$). Of course, our blow-up method can be applied on a 5-edge-coloring of $\mathcal{Q}_5$ with 73 rainbow copies of C_4. However, the resulting bound does not match the upper bound. Moreover, it is even worse than a bound for 4-edge-coloring for large d. Our attempt to apply the flag algebra framework on 5-edge-colored hypercubes gave an upper bound that matched the trivial upper bound. We suspect that the trivial upper bound might be the correct order of magnitude for $d \to \infty$. More precisely, if $q_5(d)$ is the maximum number of rainbow copies of C_4 in a 5-edge-coloring of $\mathcal{Q}_d$, then

$$\lim_{d\to\infty} \frac{q_5(d)}{\binom{d}{2}2^{d-2}} = \frac{4}{5}.$$

A related question is to determine the number of colors needed to edge-color a graph so that at least some fixed number of colors appear in each copy of a specified subgraph. For graphs G and H and integer $q \leq |E(H)|$, denote by $f(G, H, q)$ the minimum number of colors required to edge-color G such that the edge set of every copy of H in G receive at least q colors. Using this notation, it was shown in [5] that $f(\mathcal{Q}_d, C_4, |E(C_4)|) = f(\mathcal{Q}_d, C_4, 4) = d$, for $d = 4$ or $d > 5$. Mubayi and Stading [6] proved that if $k \equiv 0 \pmod 4$, then there are positive constants, c_1 and c_2, depending only on k such that

$$c_1 d^{k/4} < f(\mathcal{Q}_d, C_k, k) < c_2 d^{k/4}.$$

They also showed that $f(\mathcal{Q}_d, C_6, 6) = f(\mathcal{Q}_d, \mathcal{Q}_3, 12) = f(\mathcal{Q}_d, \mathcal{Q}_3, |E(\mathcal{Q}_3)|)$, and that for every $\varepsilon > 0$, there exists d_0 such that for $d > d_0$

$$d \leq f(\mathcal{Q}_d, \mathcal{Q}_3, 12) \leq d^{1+\varepsilon}.$$

It would be interesting to determine the value of $f(\mathcal{Q}_d, \mathcal{Q}_\ell, |E(\mathcal{Q}_\ell)|)$ for $\ell \geq 3$. Combined with a generalization of our blow-up technique it may allow us to determine the maximum number of rainbow copies of $\mathcal{Q}_\ell$ in a k-edge-coloring of $\mathcal{Q}_d$ in general.

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