The Teaching and Learning of Probabilistic Thinking: Heuristic, Informal and Fallacious Reasoning

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Abstract/Summary

In this chapter we examine the use of heuristic principles in mathematics by paying particular attention to the counterintuitive nature of problems in probability and ways in which heuristics can be used to overcome fallacious reasoning.

Introduction

The teaching and learning of the majority of topics found in mathematics classrooms concurrently provides an opportunity to teach (mathematical) heuristics, which, according to the eminent mathematician George Polya (1954), are the mental operations useful for understanding the process of solving problems. The teaching and learning of combinatorics, as an example, further lends itself to the "Do you know a related problem?" heuristic: individuals tasked with determining "How many ways can x individuals sit around a table?" rely on the result and, to an extent, the method for determining the related problem of "How many ways can x individuals sit in a row?" Stated in more general terms, the teaching and learning of the majority of topics in mathematics classrooms lends itself to the teaching and learning of thinking and, as a domain of research, is well established.

Polya (1954) observed that in "trying to solve a problem, we consider different aspects of it in turn, we roll it over and over in our minds; variation of the problem is essential to our work." (p.120). Polya emphasized the use of a variety of heuristics for solving mathematical problems of varying complexity. In examining the plausibility of a mathematical conjecture, mathematicians use a variety of strategies. In looking for conspicuous patterns, mathematicians use a variety of heuristics such as (1) verifying consequences, (2) successively verifying several consequences, (3) verifying an improbable consequence, (4) inferring from analogy, (5) deepening the analogy Polya (1954) also elucidates heuristics used to (a) examine a consequence, (b) examine possible hypotheses, and (c) examine a conflicting conjecture. These are presented in the table below. The purpose of the table is to illustrate the nuances of heuristic thinking which often gets confused with simplistic inductive reasoning.

	(1) Demonstrative	(2) Shaded Demonstrative	(3) Shaded Inductive	(4) Inductive
1. Examining a consequence	$A \Rightarrow B$ B False ——— A False	$A \Rightarrow B$ B less credible ——— A less credible	$A \Rightarrow B$ B more credible ——— A somewhat more credible	$A \Rightarrow B$ B true ——— A more credible
2. Examining a possible ground	$A \Leftarrow B$ B true ——— A true	$A \Leftarrow B$ B more credible ——— A more credible	$A \Leftarrow B$ B less credible ——— A somewhat less credible	$A \Leftarrow B$ B false ——— A less credible
3. Examining a conflicting conjecture	$A \mid B$ B true ——— A false	$A \mid B$ B more credible ——— A less credible	$A \mid B$ B less credible ——— A somewhat more credible	$A \mid B$ B false ——— A more credible

(Table I from *Patterns of Plausible Inference* (p.26))

Note: $A \mid B$ is read A incompatible with B

However, the sole use of heuristic reasoning processes is insufficient to fully characterize creative mathematical thinking. By creative mathematical thinking we mean thinking that does not rely on rote procedures. The topic of probability in particular offers insights into the use of heuristics as well as the opportunity to think creatively because of its counterintuitive nature. For instance, Individuals who, as an example, are shown both the result and a variety of methods for solving the (in)famous Monty Hall Problem still fall prey to the counterintuitive nature of closely related problems, such as The Two Child Problem and Bertrand's Box Paradox. The overarching counterintuitive nature of these probability problems — stemming from the root concept of conditional probability, which is key to the method of solving and correct response to all three problems presented (and others) — renders certain mathematical heuristics (e.g., "Do you know a related problem?") moot.

However, in attempts to account for counterintuitive, incorrect, inconsistent and incomprehensible responses to task such as those described above, researchers investigating probabilistic thinking utilize a different, psychological notion of heuristic. With the topic of probability now having emerged "as a mainstream strand in mathematics curricula "worldwide (Jones, Langrall & Mooney, 2007, p. 938), the teaching and learning of probability concurrently presents a new opportunity to teach the psychological notion of heuristic and, as a result, a different perspective to the teaching and learning of probabilistic thinking than is traditionally found in the mathematics classroom.

Brief historical overview

Eminent psychologists Amos Tversky and Daniel Kahneman (1974) demonstrated that "people rely on a limited number of heuristic principles which reduce the complex tasks of assessing probabilities and predicting values to simpler judgmental operations" (p. 1124). As an example, Kahneman and Tversky (1972) presented participants with a number of six children families (in a city) that had the "exact order of births" (using B for boys and G for girls) as GBGBBG (the number was 72). Participants were subsequently asked to estimate the number of families with birth order sequence BGBBBB and BBBGGG. By presenting sequences that are considered equally likely, but not equally representative, Kahneman and Tversky established that participants incorrectly found the sequence BGBBBB less likely than GBGBBG because the ratio of five boys to one girl does not reflect the ratio of boys to girls found in the larger population, that is, one to one. Further, the sequence BBBGGG was incorrectly deemed less likely than GBGBBG because it does not appear random. The similarity of a sample to its population and the appearance or reflection of randomness are two central determinants of the representativeness heuristic, which is where "an event A is judged more probably than an even B whenever A appears more representative than B" (p. 431). The three central heuristics associated with the original heuristics and biases program (see Kahneman, Slovic and Tversky, 1982), that is, representativeness, availability and anchoring (and the biases stemming from each heuristic) quickly permeated research in a variety of fields. One such field, with an inherent research focus on teaching and learning, was the field of mathematics education.

Shaughnessy (1977, 1981) was the first to resettle Tversky and Kahneman's heuristics and biases research from the field of psychology to the field of mathematics education. In the process, however, he was explicit to declare what he saw as the differences between researchers from the fields of psychology and mathematics education, which he, respectively, labelled as "observers or describers" and "interveners" (Shaughnessy, 1992, p. 469). He also declared amazement that "cognitive psychologists lament the depth and tenacity of certain nonnormative conceptions of probability … yet make little attempt to team up with mathematics educators to see if the misconceptions can be diminished under instruction" (ibid.). In essence, Shaughnessy was declaring that it would be the mathematics educators, that is, the natural interveners, whose "task is to improve students' knowledge of stochastics [and] wish to change students' conceptions of beliefs about probability and statistics" (ibid.), those from a field with an inherent research focus on teaching and learning that would lead to the remediation of the use of heuristics and biases associated with probabilistic thinking.

As the heuristics and biases program resettled in the field of mathematics education, the representativeness heuristic took centre stage for those investigating the teaching and learning of probabilistic thinking. Mathematics educators confirmed (Shaughnessy, 1977, 1981) and extended (Cox & Mouw, 1992; Hirsch & O'Donnell, 2001; Rubel, 2006, 2007) results established from the field of psychology, focused their research on perceptions of randomness (Batanero, Green & Serrano, 1998; Batanero & Serrano, 1999; Falk, 1981; Falk & Konold, 1997; Green, 1983, 1988; Lecoutre, 1992; Schilling, 1990; Toohey, 1995) and developed new theories, models and frameworks (Abrahamson, 2009; Chernoff, 2013, 2009).

The research of Konold et al. (1993), not only contributed to the resettlement process, but also contributed major findings "contrary to the results of Kahneman and Tversky (1972)" (p. 392). From analyzing response justifications, Konold et al. demonstrated that incorrect relative likelihood comparisons were approached differently when asked to determine the most likely sequence versus the least likely sequence. Although the majority of participants were answering the task correctly, "the majority of [their] subjects were not reasoning correctly" (p. 399). To account for this discrepancy, Konold et al. utilized Konold's (1989) outcome approach, an informal conception of probability where "the goal in dealing with uncertainty is to predict the outcome of a single trial" (p. 61). Applied to relative likelihood comparisons, "when asked about the most likely outcome, some believe they are being asked to predict what will happen and give the answer 'equally likely' to indicate that all the sequences are possible" (p. 399). Ultimately, the research of Konold et al. signified a shift in the research literature from heuristic reasoning to informal conceptions of probability, that is, from heuristic to informal reasoning. The heuristics and biases program provided the foundation for researchers investigating the teaching and learning of probabilistic thinking in the field of mathematics education; however, stemming from this brief historical overview, two issues, worthy of note, have emerged.

While the resettlement of the heuristics and biases program saw confirmation and extension of results, new research directions, new theories, models and frameworks and a shift from heuristic to informal reasoning, the (natural) interveners did not intervene — they, too, observed. Further, the mathematics education research literature has, until recently, ignored subsequent research results stemming from the field of cognitive psychology (Chernoff, 2012b). Current research on the teaching of probabilistic thinking is addressing the irony associated with the former and the latter.

Current state of the art

Current research on the teaching and learning of probabilistic thinking (presented in detail in what follows) has more recently witnessed a split. On the one hand, certain researchers are investigating a potential new shift from heuristic and informal reasoning to fallacious reasoning, which would address, among other issues, "the arrested development of the representativeness heuristic" (Chernoff, 2012b, p. 951) witnessed in the field of mathematics education. On the other hand, current research on the teaching and learning of probabilistic thinking is centered around (the established, controversial/contested areas of) differing philosophical interpretations of probability and differing interpretations of heuristic. The current state of the art for both groups of researchers is now commented on in turn.

As established, research into the teaching and learning of probabilistic thinking has, in the past, seen a focus on normatively incorrect responses (e.g., declaring one sequence of coin flips less likely than another). (Worthy of note, the focus on normatively incorrect responses does not suggest a negative view of the mind [Kahneman, 2011]). The theories, models and frameworks associated with heuristic and informal reasoning – rooted in the differing notions of conceptual analysis (Thompson, 2008; Von Glaserveld, 1995), grounded theory (Strauss & Corbin, 1998) and abduction (Lipton, 1991; Peirce, 1931) – have, traditionally, accounted for normatively incorrect responses to probabilistic tasks. (See, for example, Chernoff [2012abcd] and Chernoff and Russell [in press, 2012ab, 2011ab] for further detailed accounts of [the history of] the theories, models and frameworks associated with heuristic and informal reasoning in the field of mathematics education.) More recently, however, a thread of investigations is moving away from utilizing the more traditional notions of heuristic and informal reasoning as the framework for analysis of incorrect responses associated with the teaching and learning of probabilistic thinking.

An emerging thread of research associated with the teaching and learning of probabilistic thinking suggests that logically fallacious reasoning, more specifically, the use logical fallacies, best (abductively speaking) accounts for certain normatively incorrect responses to probabilistic tasks. For example, Chernoff (2012a) and Chernoff and Russell (in press, 2011b) demonstrated that certain individuals (prospective mathematics teachers), when asked to identify which event (i.e., outcome or subset of the sample space) from five flips of a fair coin was least likely to occur, did not use the representativeness heuristic (Kahneman & Tversky, 1972), the outcome approach (Konold, 1989) or the equiprobability bias (Lecoutre, 1992). Instead, they utilized a particular logical fallacy, the fallacy of composition: when an individual infers something to be true about the whole based upon truths associated with parts of the whole (e.g., coins [the parts] are equiprobable; events [the whole] are comprised of coins; therefore, events are equiprobable, which is not necessarily true). Worthy of note, the fallacy of composition accounted for both normatively correct and incorrect responses to the new relative likelihood comparison task.

In subsequent research, Chernoff and Russell (2012a, 2011a) applied the fallacy of composition to a task more traditionally found in the research literature. Participants (again prospective mathematics teachers) were asked to determine which of five possible coin flip sequences – not events – were least likely to occur. As was the case in their prior research (e.g., Chernoff, 2012a; Chernoff & Russell, in press, 2011b), the fallacy of composition accounted for normatively incorrect responses to the task. More specifically, the researchers demonstrated that participants referenced the equiprobability of the coin, noted that the sequence is comprised of flips of a fair coin and, as such, fallaciously determine that the sequence of coin flips should also have a heads to tails ratio of one to one. In other words, the properties associated with the fair coin (the parts), which make up the sequence (the whole), are expected in the sequence. Once again, the fallacy of composition, in addition to the traditional theories, models and frameworks associated with heuristic and informal reasoning, accounted for certain normatively incorrect responses to a probabilistic task.

Based on the success associated with utilizing the fallacy of composition, research in the teaching and learning of probabilistic thinking is branching out and, currently, determining which other logical fallacies may be utilized in a similar fashion. Early success with certain other logical fallacies, such as the appeal to ignorance (Chernoff & Russell, 2012b), which can be added to the fallacy of composition, strengthens the case for the use of logical fallacies as a new area of investigation for future research on the teaching and learning of probabilistic thinking. Despite what can be considered as early success with the use of informal logical fallacies, more specifically, the fallacy of composition and an appeal to ignorance, further research will determine to what extent logical fallacies are involved in the teaching and learning of probabilistic thinking, which will see competition from revisitation of the term heuristic in the field of cognitive psychology.

Controversial/contested areas

Current research on the teaching and learning of probabilistic thinking is venturing further into two controversial/contested areas. The first area deals with differing interpretations of the probability. The second area is a contestation over the term heuristic, which stems from two established camps with differing views. Each of these areas is commented on in turn.

"The theory of probability has a mathematical aspect and a foundational or philosophical aspect. There is a remarkable contrast between the two. While an almost complete consensus and agreement exists about the mathematics, there is a wide divergence of opinions about the philosophy" (Gillies, 2000, p. 1). The divergence of opinions is found amongst researchers investigating the teaching and learning of probabilistic thinking (Chernoff, 2008). A close read of important pieces of literature from the field of mathematics education reveals individuals' support for particular interpretations of probability (e.g., Hawkins and Kapadia, 1984; Shaughnessy, 1992). Despite these declarations of affinity for one interpretation over another, the infamous feud between those who espouse different philosophical interpretations of probability, specifically Bayesians and frequentists (see, for example, McGrayne, 2011), does not (appear) to exist to the same extent for researchers investigating the teaching and learning of probabilistic thinking. Most likely, the reason this feud is more subdued is because the very same research that advocates one interpretation over another also champions an approach to the teaching and learning of probability that "utilize[s] subjective approaches in addition to the traditional 'a priori' and frequentist notions" (Hawkins and Kapadia, 1984) or, alternatively stated, "involves modeling several conceptions of probability" (Shaughnessy, 1992, p. 469). In more general terms, research investigating the teaching and learning of probability continues to advocate for "a more unified development of the classical, frequentist, and subjective approaches to probability" (Jones, Langrall & Mooney, 2007, p. 949). Nevertheless, the controversial nature of differing interpretations of probability will forever remain at the very core of research investigating the teaching and learning of probabilistic thinking. A unified approach is not being advocated for in the other contested area mentioned earlier: heuristics.

Broadly speaking, heuristic research falls into one of two camps: the research of Daniel Kahneman (and the late) Amos Tversky and colleagues, and the research of Gerd Gigerenzer and colleagues. As mentioned, the original heuristics and biases program of Daniel Kahneman and Amos Tversky (e.g., Kahneman, Slovic & Tversky, 1982) is seminal to those investigating the teaching and learning of probabilistic thinking. However, developments associated with the original heuristics and biases program (e.g., Gilovich, Griffin and Kahneman, 2002; Kahneman, 2011) are not – despite particular exceptions (e.g., Chernoff, 2012b; Leron & Hazzan, 2006, 2009; Tzur, 2011) – found in mathematics education literature investigating probabilistic thinking. In particular, Chernoff highlights an "arrested development of the representativeness heuristic" (p. 951) in the field of mathematics education, which, if not for recent research (e.g., Chernoff, 2012ab, 2011) and the previously mentioned exceptions, may have been extended to heuristics, in general, in mathematics education. Research utilizing Gigerenzer's notion of heuristics (e.g., Martignon, in press; Meder & Gigerenzer, in press), further thwart continuation of this arrested development of heuristics in the field of mathematics education. Worthy of note, the research of Gerd Gigerenzer and colleagues has until recently been largely absent in the mathematics education literature. However, there are signs that the research of Gigerezner and colleagues is forging its way into mathematics education and signals the dawn of a new era of research for those investigating the teaching and learning of probabilistic thinking in the field of mathematics education. As mentioned, not only the research of Gigerenzer and colleagues, but also the renewed heuristics research of Kahneman and colleagues adopted by those investigating the teaching and learning of probabilistic thinking may define a renaissance period for psychological research in mathematics education. In doing so, may address the aforementioned irony and, finally, pave the way "for theories about mathematics education and cognitive psychology to recognize and incorporate achievements from the other domain of research" (Gillard, Dooren, Schaeken & Verschaffel, 2009, p. 13).

For further background on the larger developments associated with the heuristics and biases program, which are "now widely embraced under the general label of dual-process theories" (Kahneman and Frederick 2002 p. 49), and associated criticisms, Gilovich, Griffin and Kahneman 2002 and Kahneman 2011, and Gilovich and Griffin 2002 are recommended, respectively.

Idiomatically, research investigating the teaching and learning of probabilistic thinking has come full circle. The transition from the original heuristics and biases to informal conceptions of probability and, more recently, to fallacious reasoning, has, even more recently, witnessed a revisitation of varying notions of heuristics from different camps. As this current cycle continues to complete, new opportunities for research into the teaching and learning of probabilistic thinking emerge.

The Future State of the Art

The future state of the art, that is, future research into the teaching and learning of probabilistic thinking, will have new areas open once the newer notions of heuristics become a mainstream component of research. We explore, as one example, potential differences associated with utilizing Kahneman's and Frederick's (2002) notion of attribute substitution as opposed to the more traditional utilization of Tversky and Kahneman's (1972) representativeness heuristic.

Kahneman and Frederick (2002) recognized an underlying notion to the, then, disparate heuristics: "Early research on the representativeness and availability heuristics was guided by a simple and general hypothesis: when confronted with a difficult question people often answer an easier one instead, usually without being aware of the substitution" (p. 53). Elaborating on the common underlying process of answering a difficult question with an answer to an easier question that was, perhaps, not asked, they elaborated and contended that "heuristics share a common process of attribute substitution and are not limited to questions about uncertain events" (Kahneman and Frederick 2002 p. 81). Kahneman and Frederick (2002) define attribute substitution as follows: "We will say that judgment is mediated by a heuristic when an individual assesses a specified target attribute of a judgment object by substituting another property of that object — the heuristic attribute — which comes more readily to mind. Many judgments are made by this process of attribute substitution" (p. 53).

According to Kahneman (2002) "The essence of attribute substitution is that respondents offer a reasonable answer to a question that they have not been asked" (p.

469); how- ever, “An alternative interpretation that must be considered is that the respondents’ judgments reflect their under- standing of the question they were asked” (p. 469). Utilizing this alternative interpretation, researchers (Abrahamson, 2009; Chernoff, 2012b) have been able to gain some semblance of the legitimate reconstructed tasks (unknowingly) being answered by certain individuals.

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