Mathematical Pathologies as Pathways into Creativity

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Abstract
In this paper, the role of mathematical pathologies as a means of fostering creativity in the classroom is discussed. In particular, it delves into what constitutes a mathematical pathology, examines historical mathematical pathologies as well as pathologies in contemporary classrooms, and indicates how the Lakatosian heuristic can be used to formulate problems that illustrate mathematical pathologies. The paper concludes with remarks on mathematical pathologies from the perspective of creativity studies at large.

Keywords: creativity; convergent thinking; divergent thinking; heuristics; Lakatosian heuristics; pathologies; mathematical pathologies; problem-posing

1. Introduction

According to Webster's dictionary, the etymology of the term 'pathology' can be traced to Ancient Greek roots of 'pathos' (πάθος) meaning *experience* or *suffering*, and '–logia' (-λογία) meaning *an account of*. Thus, the term originally meant *an account of suffering* or *an experience of suffering*. Today the field of medicine views it as a study of disease. In mathematics, unlike medicine, the term 'pathological' is used to refer to examples that are specifically designed to violate properties that are perceived as valid. The term 'pathological' is also specifically used in mathematics to refer to objects "cooked up" to "provide interesting examples of counterintuitive behavior."[1] In each of these two senses there is necessarily a human element: the *perception* of validity that is violated, and the *intuition* to which the pathological runs counter are each located within a person, and both are with respect to the norms of a domain (here, mathematics). Moreover, we note that, whereas some pathologies are found in counterexamples, the two concepts should not be conflated. For example, the formula $P(n) = n^2 + n + 41$ may be considered pathological insofar as it appears to generate only primes[2]; indeed, such an assertion holds for the first forty natural numbers, namely, $n = 0, 1, \ldots, 39$, yet it produces a composite value at $n = 40$, at $n = 41$, and infinitely often thereafter. Strictly speaking, though, there is no counterexample afoot, for the formula P(n) does not itself contain an explicit proposition about primes; rather, we have an object cooked up to provide counterintuitive behavior, as tinkerers may mistakenly believe after trying a few cases that the behavior (i.e., prime output) holds over the entire domain of natural numbers (cf. Buchbinder & Zaslavsky, 2013). On the other hand, individual pathologies may be provided in the specific form of counterexamples; a common instantiation arises in courses on the Calculus, when students who harbor the (mis)conception that continuity of a real-valued function necessarily implies differentiability are presented with Weierstrass' continuous-everywhere-but-differentiable-nowhere function.

Although such examples may seem few and far between, positioned as isolated markers of dead ends within an otherwise traversable mathematical terrain, a central argument of this paper is that pathologies abound in mathematics as evidenced in the history of the field, and that this has ramifications for the creative development of mathematics learners. For mathematicians, it is well known that many significant advances have occurred as a result of the discovery of pathological properties in mathematical objects that called for the necessity to go beyond domain

[1] http://mathworld.wolfram.com/Pathological.html
[2] More generally, see http://mathworld.wolfram.com/Prime-GeneratingPolynomial.html on Prime-Generating Polynomials.

limiting tools and required formulation of a better theory. An alternative view of the history of mathematics can easily be provided through such classical pathological examples as well as the evolution of argumentation in the discipline (Umland & Sriraman, 2014).

Another important nuance to be conveyed to the reader about our discussion of pathologies as a pathway to creativity is the fact that it is a matter of degree of mathematical sophistication. In Sriraman's (2005) definition of mathematical creativity, a distinction is drawn between the sophistication of a professional mathematician and a K-12 learner of mathematics. For instance, a statement such as '$y = x^2$ represents a line' might appear as an untrue or even fantastical statement to a secondary student of mathematics, whereas a professional mathematician would be able to state that the truth of that statement depends on the conditions. It is certainly untrue in the Cartesian co-ordinate system that operates as the frame of reference for secondary students, but in the focus-directrix coordinate system it becomes a true statement. Arguably many mathematical results can then be viewed as pathologies depending on the degree of the sophistication of the individual, which is precisely an implication of the definition of creativity offered by Sriraman (2005). In other words, examples that are "cooked up" to knowingly violate valid properties in the frame of reference of a learner create domain limiting barriers that professional mathematicians encounter in their work, and overcome through astonishing acts of creativity. Haught and Stokes (in press) argue that domain constraints call for developing competencies to overcome the basic constraints in the domain, which can take the form of novel problem solving. This idea can be further stretched to what Robert Root-Bernstein calls n-epistemological awareness, i.e., the awareness of those at the frontiers of their field of the constraints that need to be overcome to solve problems (Root-Bernstein, 1996). Mathematics provides numerous examples that such domain constraints spur creativity in the form of new tool development. For example, homological algebra was developed to answer basic questions in number theory that young students can comprehend, but solving these problems requires sophistication that is generally obtained only during one's graduate level education. Much of the early work on integrals required the likes of John Wallis, Lord Brouncker and Fermat to use interpolation techniques that overcome the constraint of not having the binomial theorem available (Sriraman & Lande, in press). Thus, we present examples of pathologies in the remainder of this paper and argue that these, along with other, mathematical pathologies can foster creativity among mathematical learners.

To begin, we look back on a smattering of historical examples to ease the reader into the notion of the mathematically pathological. In addition to the already mentioned Weierstrass function, we recall several familiar examples: the incommensurability of $\sqrt{2}$ in the time of Pythagoras, the space-filling Peano curve and the equivalent cardinality between the rationals and natural numbers, topological objects such as Mandelbrot's fractals, and Euler's Formula along with its discussion by Lakatos (1976). From Lakatos' work we segue to a discussion of the *Lakatosian heuristic* as illustrated by Davis and Hersh (1980), after which we provide a brief selection of classroom-based examples: two from our own classrooms in the United States, and two from classrooms abroad. Bearing the historical and classroom-based pathologies in mind, we utilize the subsequent section to discuss at the granular level how the Lakatosian heuristic can be effectively used to formulate non-routine mathematics problems; in this way, we come to see how the mathematically pathological connects to mathematical creativity via problem posing. We conclude by stepping back to the literature on creativity *writ large* and considering the sociocultural nature of generating mathematical pathologies.

2. Historical Examples of Mathematical Pathologies

The incommensurability of $\sqrt{2}$ for the Pythagoreans around 430 BCE can be viewed as a watershed moment in the discovery of mathematical pathologies. The Pythagoreans' attitude, mired in the superstition held about commensurable magnitudes as the building blocks of the world, however, did not prevent Theodorus and Theaetetus from studying irrational magnitudes resulting in a theory eventually covered in Book X of Euclid's *Elements* (Knorr, 1998; Roskam, 2009). Book X, although a part of the *Elements*, is the culmination of the findings of Theodorus and Theaetetus first published by Plato and then passed onto Euclid through Eudemus (Knorr, 1983).

Another classical pathology that deserves mention is the Peano curve, which was the first example of a space-filling curve. Peano's curve is dense in the unit square and used to construct a continuous function from the unit interval to the unit square. A visual demonstration of this pathological object can be given with geometric software where, for a given screen resolution, an appropriate step in the construction process of Peano fills the corresponding screen.

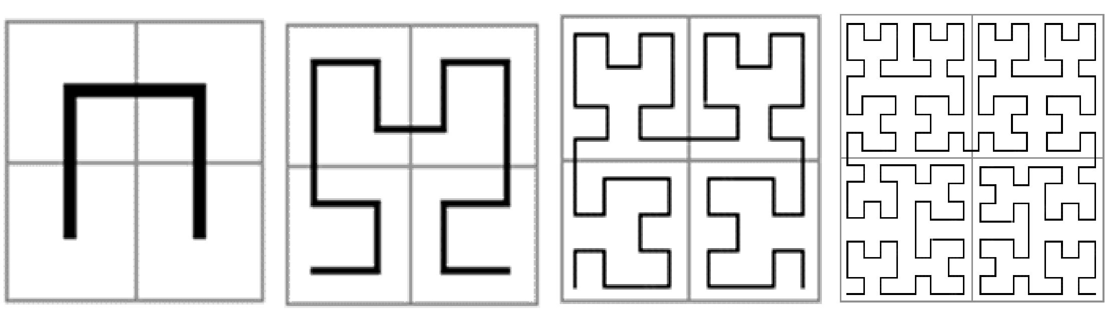

Figure 1. Peano's Space filling curve

Peano's result was built on an earlier result of Cantor that these two sets have the same cardinality and can be viewed as a fractal-like object. Cantor's development of transfinite arithmetic and his proof of the countability of the set of rational numbers, i.e., that the cardinality of **Q** is equal to the cardinality of **N** was viewed by many of his contemporaries as a pathology and one that encountered fierce resistance from the likes of Kronecker. These observations today are not viewed with the same astonishment as at the time of their discovery because they have become assimilated into the appropriate mathematical theory. From Mandelbrot's viewpoint the ensuing discovery of rough geometric objects, i.e., fractals and Hausdorff dimension led to labels such as 'monsters' and 'pathological'. In Mandelbrot's own words:

> "…a second kind of chaos became known during the half century, 1875-1925, when mathematicians who were fleeing from concerns with Nature took cognizance of the fact that a geometry shape's roughness may conceivably fail to vanish as the examination becomes more searching. It may conceivably vary endlessly, up and down. The hold of standard geometry was so powerful, however, that the shapes constructed so that they do not reduce locally to straight lines were labeled 'monsters' and 'pathological'" (Mandelbrot, 1989, p. 4).

In a major sociological study spanning the work of three generations of mathematicians on the Poincaré conjecture[3], Fisher (1973) found that "pathologies abound in mathematics. Some are objects whose existence can be proven, but for which no intuitive model can be made. Although many mathematicians look upon pathologies with distaste, they have played an important role in the development of the discipline, for example the pathologies of set theory and measure theory" (p. 1111). In a study of the different techniques employed by these mathematicians, it was found that topologists often relied on constructing objects that were twisted or pathological to reveal desired properties of related objects; a compendium of results of this nature were collected in the classic *Counterexamples in Topology* (Steen & Seebach, 1978). In this general direction, the eminent Russian mathematician Fomenko lamented the paucity of interest among undergraduates to construct and study the properties of pathological functions due to the belief that such functions did not occur in the "real" world of science (Koblitz & Koblitz, 1986). His views

[3] This famous conjecture, and its generalization as Thurston's Geometrization Conjecture, were finally proved in the positive by Grigori Perelman in a trio of papers posted to the arXiv in 2002 and 2003 (arXiv:math.DG/0211159; arXiv:math.DG/0303109; arXiv:math.DG/0307245).

stemmed from the lack of interest among students of that time period in pure, as opposed to applied, mathematics.

Lakatos (1976) popularized the term “monster-barring” to refer to the process of refining the statement of a theorem when unpleasant counterexamples or pathologies arise. The example given in his book *Proofs and Refutations* comes from non-convex polyhedra that arose in the development and proof of Euler’s formula relating the number of vertices, V, faces, F, and edges, E, with the identity: $V + F = E + 2$. Davis and Hersh (1980) described Lakatos’ method of how mathematicians go about formulating detailed conditions for the statement of a theorem and its proof (Figure 2).

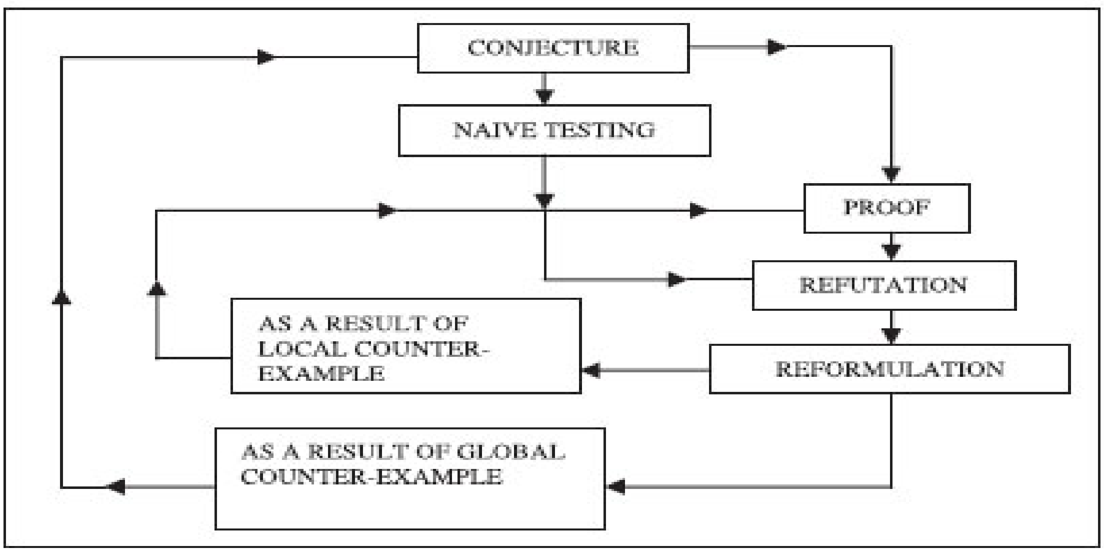

Figure 2. The Lakatosian heuristic from Davis and Hersh (1980, p. 292)

The Lakatosian heuristic as summarized by Davis and Hersh (1980) does not presume that all mathematicians employ these particular stratagems; it is useful, however, in instances where mathematical objects that violate given properties need to be constructed (Hersh, 2014). Although Lakatos (1976) devotes considerable time to discussing the evolution of $V + F = E + 2$

for convex polyhedra, the main argument of *Proofs and Refutations* is to show how the domain of validity of this result has changed over time in this particular context (Sriraman & Mousoulides, 2014).

3. Contemporary Examples of Mathematical Pathologies in the Classroom

In this section we briefly consider pathologies that can be understood and unpacked by school-age students, and which have the capacity to foster deep structural understandings. By structural understandings we mean the ability to discern the basic property or general rule within an axiomatic system (e.g., Sriraman, 2004, 2006). The first example involves anomalous fraction cancellations that lead to the correct answer. Consider the following:

$$\frac{19}{95} = \frac{1\not{9}}{\not{9}5} = \frac{1}{5} \qquad\qquad \frac{26}{65} = \frac{2\not{6}}{\not{6}5} = \frac{2}{5}$$

Figure 3. Two-digit anomalous fractions

The rule for cancellation in the simplification of fractions is misinterpreted, purposefully or not, in Figure 3. The phenomenon that differentiates these examples as pathologies, rather than simple errors, is that the erroneous method at hand produces correct simplifications. The natural follow-up question is whether there are other two digit fractions with this property (cf. Borasi, 1996; Johnson, 1985). Examining this question necessitates an understanding of place value and arithmetic manipulations; furthermore, the search can also be extended to fractions with more than two digits. Osler (2007) provides a more detailed exposition of this type of cancellation in what he terms as “lucky fractions” (see Figure 4) and ways to generalize the results to fractions with an arbitrary number of digits in the numerator and denominator. The first author of this paper used the results of Osler in an honors Calculus seminar with undergraduate students to develop algorithms that could generate these fractions using the method of exhaustion[4].

$$\frac{13}{325} = \frac{1}{25}, \quad \frac{83}{332} = \frac{8}{32} = \frac{1}{4}, \quad \frac{27}{756} = \frac{2}{56} = \frac{1}{28}$$

$$\frac{39}{975} = \frac{3}{75} = \frac{1}{25}.$$

Figure 4. Higher digit lucky fractions (Osler, 2007, p. 162)

The second example comes from a mathematical content course taught by the second author to pre-service and in-service elementary school teachers. The course emphasized multiple approaches to whole number multiplication, including a pictorial representation in which two digits numbers are decomposed at their tens place and ones place, and the four partial products are located within a corresponding area model diagram. The course also emphasized the fundamental theorem of arithmetic, and, in particular, the uniqueness of representation (up to order) of a whole number's prime factorization. Despite the uniqueness component with respect to one form of a whole number's multiplicative representation (i.e., its prime factorization), the instructor noted that that the partial products representation, somewhat counterintuitively, does *not* uniquely define whole numbers. Prompted by the existence of a single set of partial products corresponding to two different factorizations, a student in the second author's course wrote a program in the Python programming language to generate all products of two digit whole number pairs for which the partial products are the same; multiple examples for which two or three factorizations corresponded to a single set of partial products were found. An example

[4] The following URL gives a calculator developed by one of the students to generate anomalous fractions http://www.chasemaier.com/project/Anomalous-Cancellation-Calculator

decomposition of 1,008 such that its partial products[5] correspond to three different factorizations is illustrated in Figure 5.

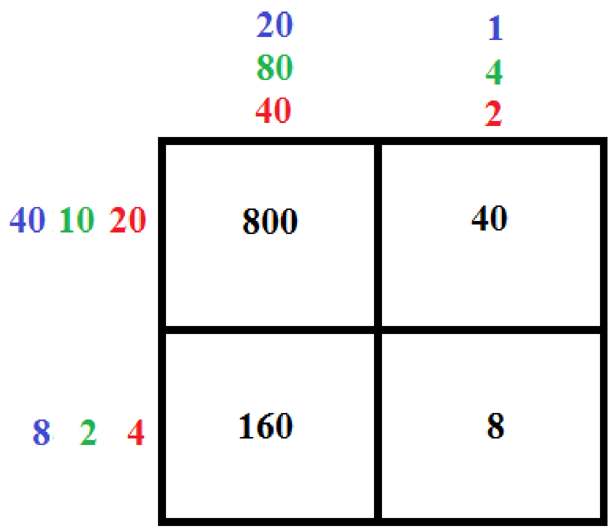

Figure 5. Partial products for 1,008 that correspond to 21x48, 84x12, and 42x24

The third example comes from the realm of geometry in a replication of Lakatos' thought experiment with gifted fifth and sixth grade students in South Korea. This study built on the pedagogical value of using Lakatosian heuristics to foster insight into the structure of combinatorial problems with ninth grade students in the United States (Sriraman, 2006). Yim, Song and Kim (2008) explored the geometric constructions of 11 elementary students who were asked to validate Euler's formula as well as find examples that conflicted with the formula. The researchers classified pathologies constructed by the students into four types: solids with curved surfaces; solids made of multiple polyhedra sharing points, lines, or faces; polyhedra with holes; and polyhedra containing polyhedra (p. 125).

[5] Note that for the product of a pair of two digit whole numbers, any three partial products uniquely determine the fourth partial product; thus, for example, any one of the four entries 800, 40, 160, or 8 could be removed from the interior of Figure 5 without yielding additional solutions.

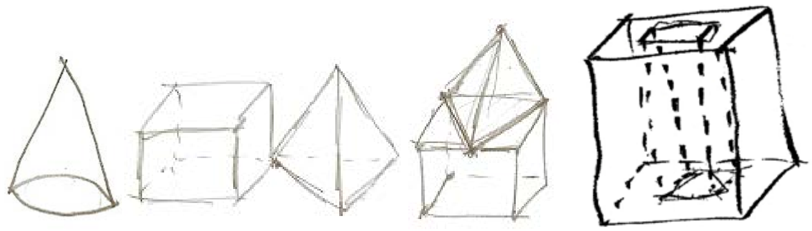

Figure 6. Pathologies constructed by fifth and sixth grade students (Yim, Song, Kim, 2008)

Students in the study were also quite successful in resolving conflicts between pathologies and the formula by proposing "monster-barring" methods. In other words, these pathologies needed to be eliminated to arrive at the criterion of *convex* polyhedra being the objects for which the formula applied.

Finally, the fourth example comes from recent work by Hadjichristou and Ogbonnaya (2015) in which they reported on the effectiveness of using the Lakatosian heuristic with Cypriot secondary school students who were learning the surface area of a cone. In this study the experimental group was taught using the Lakatosian heuristic of conjecture-proof-refutation whereas the control group was taught using the traditional Euclidean (deductive) method. These researchers concluded that the experimental group, in addition to scoring higher than the control group on the post-test, also showed higher-order thinking skills hypothesized in the literature (Sriraman, 2006).

These four examples, although limited to small clusters in different countries, nevertheless illustrate the pedagogical value of using pathologies to develop the mathematical thinking and curiosity of students. Why (and when) does an incorrect ("pathological") method of fraction cancellation yield correct solutions? Why (and how) is one multiplicative representation of whole numbers unique, whereas another can, pathologically, yield multiple solutions? Is Lakatos' thought experiment realizable with actual students, even at the elementary age? Can the

Lakatosian heuristic be applied to other questions of solid geometry? In an effort to understand the Lakatosian heuristic as connecting to mathematical creativity via the pathological, we illustrate in detail its role in an individual example of problem formulation.

4. Formulating Pathologies with the Lakatosian Heuristic: A Detailed Example

In this section, we aim to weave together several of the strands already introduced. Our work thus far has centered on mathematical pathologies for their ability to challenge our perception and intuition, and the resulting advances as we equilibrate. Whether dealing with a space-filling Peano curve in the plane or a misapplied cancellation law sporadically holding among fractions, we are discontent to remain stationary in response to these phenomena. Instead, we mobilize and investigate how to reconcile our worldviews with the novel pathologies now recognized as co-occupying the same mathematical spaces in which we had long operated.

The Lakatosian heuristic is empowering in its presentation as conjecture-proof-refutation; one may reasonably expect instead an approach of conjecture-proof-generalization or conjecture-proof-corollary or some such thing. After all, how does one follow a proof by *refuting* it? In the case of Euler's formula, the issue at hand was insufficient precision with respect to the initial proposition's formulation, specifically, the polyhedra under discussion. We carry over a similar technique into the world of mathematical problem posing, long connected with creativity in mathematics (e.g., Silver, 1994, 1997; Yuan & Sriraman, 2011). To illustrate by way of example, consider the following problem of number theory: Denoting the *least common multiple* as 'LCM', suppose LCM(A, B, C) = 210; LCM(A, B, D) = 330; LCM(A, C, D) = 462; and LCM(B, C, D) = 770. How many possible solutions are there for whole numbers A, B, C, D?

Before delving deeper, we emphasize that the purpose of the example in this section is to indicate how the Lakatosian heuristic can be used to formulate a problem. The means by which we arrive at a problem's formulation should not be taken to suggest an ideal approach to resolving the problem ultimately posed (nor do we suggest that an 'ideal approach' is well-defined). Indeed, requiring multiple solutions for a single problem allows for additional explorations of mathematical creativity (Leikin, 2009, 2011), and extending this requirement alongside attempts to generalize the posed problem, and the conditions under which a generalized proposition holds, yields another approach to enriching creative mathematical problems and tasks (Dickman, in press).

In the LCM problem above, we eschew routine exercises in which whole numbers are given and the "solver" is asked to find their least common multiple; instead, the problem has been inverted such that the least common multiples are first given, and the whole numbers are unknown. Moreover, we avoid even asking for the precise solutions, and choose instead to ask for the total solution count. How does one formulate a problem of this nature? We illustrate below how the Lakatosian heuristic can provide such a service.

An initial observation is the use of Venn diagrams to illustrate the LCM of whole numbers; specifically, given whole numbers A and B, we construct a standard Venn diagram with one circle corresponding to A, the other to B, and the overlap to the *greatest common factor* (GCF) of A and B. The product across all three regions is then LCM(A, B). We aim from the outset to produce something novel, hence ask not for the LCM of a given pair of whole numbers, but rather supply it: Drawing first the Venn diagram with 2 in the overlapping region, 3 in the remaining portion of A, and 5 in the remaining portion of B, we find that LCM(A, B) = 30. Yet a moment's thought indicates a problem that provides only the LCM in this case is insufficient to determine a unique solution; rather than producing the answers of A = 6 and B = 10 as intended by our pictorial representation, ‘LCM(A, B) = 30’ alone allows for numerous additional solutions (e.g., A = 2 and B = 15). Our quest for uniqueness has been refuted with a local counterexample; no matter, we may specify that GCF(A, B) = 2 to recover a unique (again, up to symmetry) solution. We now conjecture that by extending the problem to allow for three whole numbers, providing LCMs alone will suffice; and so let us draw a 3-Venn diagram for whole numbers A, B, and C for which the three singletons contain prime factors of 3, 5, and 7, respectively, and place a 2 in the center as their GCF. Articulating our 3-Venn in LCM terms, we have conditions: LCM(A, B) = 30; LCM(A, C) = 42; and LCM(B, C) = 70. Yet back-solving from these three LCM conditions allows us only an opportunity to place the 3, 5 and 7 in unique regions of the 3-Venn; the 2 may be placed in any one of the remaining four unoccupied regions (the overlap of A and B; of A and C; of B and C; or of A, B, and C). To this end, we find four unique solution sets, and begin to realize that our seemingly local counterexample has reared its head again. No matter; we may modify the original conditions to produce a well-defined problem: Given LCM(A, B) = 30; LCM(A, C) = 42; and LCM(B, C) = 70, find all four of the solutions sets for (A, B, C).

Our upward climb in admissible whole numbers has not been thwarted by the local issue of non-uniqueness; we have simply folded this feature into our problem posing. The next rung up brings us closer to the initial query: Draw a “4-Venn” diagram with circles for A, B, C, and D, and place 3, 5, 7, and 11 in the respective singletons as illustrated in Figure 7. A problem formulated

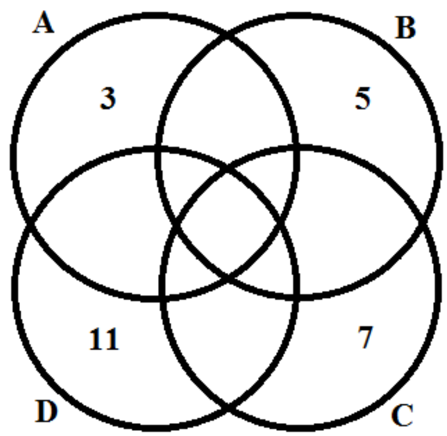

Figure 7. “4-Venn” diagram for 3, 5, 7, and 11

based solely on Figure 7 would, at this point of refinement, essentially amount to an exercise. Instead, we consider the corresponding diagram if GCF(A, B, C, D) = 2. The result is that LCM(A, B, C) = 210; LCM(A, B, D) = 330; LCM(A, C, D) = 462; and LCM(B, C, D) = 770, as described earlier. However, providing only the LCMs leads to additional solutions; in the previous case, using only the LCMs of three whole numbers, we simply counted the empty regions in the 3-Venn diagram to establish the total number of solutions as 4. In this case, we may similarly count the empty regions to arrive at an answer of 9. Unfortunately, this method of solution is *refuted* once again: In Figure 7, we have used scare quotation marks around the term “4-Venn” because the diagram does not actually depict all possible regions of overlap. Specifically, neither the intersection of A and C, nor the intersection of B and D, is illustrated; this is the consequence of a sort of global counterexample, namely, that a true 4-Venn diagram cannot be illustrated with four circles[6]. In this particular case, we may hand count the 9 empty regions, note the missing {A, C} and {B, D} regions, and arrive at the correct answer to our earlier query: There are a total of 11 solutions to the four number LCM problem, which can be

[6] For completeness, we note that it *is* possible to construct a 4-Venn diagram using ellipses. However, as we move into N-Venn diagrams for N > 5 the constructions are no longer viable.

found by using prime factorization to place the first four odd primes in unique singletons, and observing that the initial conditions are satisfied by a 2 placed in any other nonempty region that is not a singleton (including the "invisible" {A, C} and {B, D} regions).

Rather than explicitly outlining the natural extension to an LCM problem with five whole numbers, we simply note that the three and four whole number LCM problems considered here can be modified to the general case of N>1 whole numbers, for which the total number of solutions is $2^N - N - 1$; indeed, with N = 3 and N = 4 we recover the answers of 4 and 11, respectively. To find such a formula, observe that the regions can be interpreted as nonempty elements of the power set on N elements, which has cardinality $2^N - 1$; given the already described constructions, the remaining factor of 2 can be placed in any of these regions except for the N singletons, whence the general formula follows.[7]

The reason for delving so deeply into the method through which the four number LCM non-routine problem was formulated is that it demonstrates in granular detail how the Lakatosian heuristic can be used to pose a non-routine mathematics problem: We conjectured that LCMs could be used to determine two whole numbers uniquely; when this was refuted through naïve testing, we reformulated as a consequence of a local counterexample; our new conjecture dropped the uniqueness feature and adopted a method of Venn diagrams to solve for each of the solutions; however, even after we "proved" the new conjecture (i.e., found all four solutions) we refuted its generalization to a four whole number LCM problem as we encountered a global counterexample around planar graphs (i.e., circles cannot be used to construct a 4-Venn diagram). Although the depicted diagram in Figure 7 allowed us to finagle a solution, we

[7] As mentioned earlier, one may ask for an alternative solution, or, at least, articulation of a solution. In this case, it is possible to observe that each of the N primes can be multiplied by 2, or not; this gives 2^N scenarios. N of these scenarios involve multiplication of just one prime by a 2, each of which will not yield a solution, and 1 of these scenarios will involve no 2s, which will not yield a solution; thus, we find a total of $2^N - N - 1$ possible solutions.

ultimately reformulated the problem to indicate how we could, with the right “power tools” (i.e., a counting technique using the power set on N elements) subsume the family of problems with a simple formula. Each of the aforementioned steps: conjecture; refute through naïve testing; reformulate due to a local counterexample; conjecture again; prove; reformulate due to a global counterexample; conjecture yet again; and prove yet again using power tools can be characterized through techniques drawn more generally from the Lakatosian heuristic. This pedagogical method as imagined by Lakatos has the potential to lead students through ways in which professional mathematicians “play” with the conditions of a problem to arrive at a valid statement that can then be proved (Sriraman, 2006).

5. Creativity and Mathematical Pathologies: Implications for Mathematics Education

In reviewing the literature, we find a sizable body of work suggesting that learners do not typically experience mathematics as a creative subject (Burton, 2004), despite the fact that research mathematicians often describe their field as a highly creative endeavor (Sriraman, 2009). In a similar vein, educators may feel that content standards stifle their students and their own creativity, yet creativity researchers have argued that such standards can serve as the context for classroom creativity (Beghetto & Kaufman, 2010). These contradictions place educators in a difficult situation.

Creativity as divergent production was originally proposed by Guilford and Torrance and is grounded in both associative theory and Guilford’s theory of the Structure of Intellect (SOI) (Runco, 1999). Guilford (1959) considered creative thinking as involving divergent thinking, in which fluency, flexibility, originality and elaboration were central features. Building on Guilford’s work, Torrance (1966) developed the Torrance Tests of Creative Thinking, which in turn has inspired the use of different divergent production tests in numerous different contexts, including mathematics education (Leikin & Lev, 2013; Pitta-Panatzi, Sophocleous & Christou, 2013). As opposed to convergent thinking, divergent thinking tasks open up for many possible solutions (Haylock, 1987). However, divergent thinking alone constitutes only an "estimate" of creative "potential" (Runco, 2010, p. 416). In the world of mathematics, convergent thinking forms the basis of reasoning required to discover invariant principles or properties, as well as to formulate generalizations from seemingly different situations by focusing on structural properties during abstraction (Sriraman, 2004a, 2004b). In identifying mathematical invariants, one powerful approach is to attempt to construct or locate objects that violate perceived principles or properties; that is, to pursue mathematical pathologies. For example, Euler's formula $V + F = E + 2$ can be explored using the Lakatosian heuristic to pin down the precise sort of polyhedra under consideration, and perhaps even to identify additional invariants (e.g., genus, Euler characteristic).

In the world of mathematics, convergent thinking forms the basis of reasoning required to discover invariant principles or properties, as well as to formulate generalizations from seemingly different situations by focusing on structural properties during abstraction. Convergent thinking is a vital aspect of moving the field forward, as patterns from seemingly different sets of examples suggest abstractions that reveal structural invariances; when this occurs, seemingly disparate ideas cohere to form the basis of deep theorems. But with every sequence of ideas that seems to converge, and every instance of apparent coherence, the creative mathematical thinker seeks to gauge the depth of abstraction: What is (co)incidental, what was tacitly assumed, and how can apparent patterns and properties be challenged by mathematical pathologies?

We close our discussion on fostering creativity through mathematical pathologies by adopting a meta-perspective: Mathematical pathologies were associated at the outset of this paper with interesting and counterintuitive behavior. As Hanchett Hanson (2015) notes, behaviorists and cognitive researchers such as Weisberg (e.g., 2006) argue against the notion that creativity corresponds to *out of the box thinking*. In engaging with mathematical pathologies, where does the learner operate? Does the term ‘pathological’ arise simply because an object lies outside of our area of expertise – that is, are we seeking objects “outside of our boxes”? We contend that generating pathologies is fundamentally a sociocultural act: For an object to be ‘counterintuitive’ presupposes intuition, which, in turn, suggests familiarity with a domain – a building-block of cultures (Csikszentmihalyi, 1999) – to which the object can be attributed, as well as sufficient knowledge of the domain’s corresponding norms, expectations, sign systems, and societal gatekeepers. The formula $P(n) = n^2 + n + 41$ is pathological only insofar as we recognize the meaning of these symbols, possess a knowledge of prime numbers in terms of definition and distinction, and hold some sort of intuition whereby returning a prime number on forty consecutive inputs, $n = 0$ to $n = 39$, seems unlikely: whether unlikely in the sense that the pattern “suddenly” breaks at $n = 40$, or unlikely in the sense that the expression for $P(n)$ is “surprisingly” simple given the deeper mathematical fact that no polynomial can generate primes at all integer values. To this end, we engage with mathematical pathologies in an attempt to grow our box of mathematical expertise and conceptions: we push the edges to see what will hold, but also to see when we can stretch the box further, perhaps puncturing it with a counterintuitive refutation in the process.

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