“Integrating” Creativity and Technology through Interpolation

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Abstract: The digital age of the 21st century is ubiquitous with easy access to information. Students of mathematics find at their fingertips (literally) immense resources such as Wolfram Math and other digital repositories where anything can be looked up in a few clicks. The purpose of this chapter is to convey to the reader that Mathematics as a discipline offers examples of how hand calculations using first principles can result in deep insights that present students with the opportunities of learning and understanding. By first principles we are referring to fundamental definitions of mathematical concepts that enable one to derive results (e.g., definition of a derivative; definition of a Taylor series etc.). We also highlight the value of integrating (pun intended) technology to understand functions that were obtained via mathematical interpolation by the likes of John Wallis (1616-1703), Lord Brouncker (1620- 1684), Johann Lambert (1728-1777) and Edward Wright (1558-1615). The interpolation techniques used by these eminent mathematicians reveals their creativity in deriving representations for functions without the aid of modern technology. Their techniques are contrasted with modern graphing techniques for the same functions.

Keywords: circular functions; π; John Wallis; quadratures (areas); conformal maps; secant function; integration; history of Calculus; history of infinite series; interpolation; mathematical creativity

1 Introduction

The title of this chapter in a book that addresses the relationship between technology and creativity in mathematics education is a pun. By integrating we mean the mathematical technique of finding anti-derivatives, and by interpolation we literally mean mathematical interpolation to find missing information. Finding a piece of knowledge or a mathematical fact is very different from actually being able to remember it or deriving it from first principles. Yet "knowing" among students is increasingly becoming associated with or even synonymous with "looking it up" as opposed to understanding first principles to be able to derive a result or understanding ways in which results are arrived at. To this end the examples presented in this chapter are deliberately chosen from the history of series and continued fractions to illustrate the value of interpolation techniques as a forgotten art of hand computation. The history of infinite series played a significant role in how functions were dealt with before modern day integration techniques were established. For instance manipulating the series representation of circular functions led to insights about their anti-derivatives; similarly the series for the logarithm function allowed better facility for calculation purposes. This is further explored in an ensuing section of the chapter.

Ironically any modern Calculus textbook in the U.S typically presents series after techniques of integration when the former actually led to the latter. While deduction is emphasized in textbooks for the sake of presenting the subject matter in a logical manner, the advent of the digital age with readily accessible results from Wolfram Math or other mathematical repositories presents the danger of a student conceiving of mathematics as a platonic and deductive subject. The examples presented will hopefully convey the inductive aspect of the mathematician's craft.

Interpolation which literally means, "inserting between other things" can be viewed as the original "cut and paste" but in mathematical parlance one that required intuitive and systematic thought as opposed to present day pastiche. Can we present hand computations requiring interpolation and *per modum inductionis* (Wallis, 2004) as a contrast to the "digital habit" of invoking a function on Mathematica or looking up the end result in Wolfram Math? The expression "digital habit" is used colloquially here to refer to observations from the authors in their classrooms where students rely on hand-held devices to access information. For instance when students are asked for the series representation of a well-known function, they rely on retrieving the representation from Wolfram Math. As a contrast to the way things are today the examples presented in this chapter reveal both a forgotten art of mathematical creativity and extol the virtues of hand computation as a necessary complement to the "looking it up" and "cutting and pasting" habits of many present day students. The definition of creativity adapted here are those from Paul Torrance and Alex Osborn. To paraphrase these two individuals who furthered the study of creativity, Torrance (1974) referred to creativity as being able to sense difficulties or gaps or missing elements or something askew when confronted with information. Osborn (1953) on the other hand suggested creativity was the process of finding a solution by first finding a mess and then finding data to explain the mess which in turn leads to defining the real problem and coming up with the ideas for a solution. The examples we present are mathematically messy and involved numerical data, but as will be evident the solutions found by the mathematicians in the past are nothing short of ingenious.

2 Three Examples

2.1 π

Many learners of mathematics at the undergraduate level may simply think of **π** as a button on a calculator (or calculator app). The number **π** is often used in formulas for the area and circumference of circles without any real thought to where this number comes from or what it means, unless initiated by the teacher. Some early high school lessons attempt to enlighten students on its origin by having them measure the circumference and diameter of many different sized circles in order to derive **π** by using the equation: $\frac{C}{d} = \boldsymbol{\pi}$. This exercise often passes them by without any increase in understanding the intricacies and ingenuity involved in deriving this extraordinary constant. A study of past methods of derivation often results to a greater understanding and appreciation for the creativity involved in these calculations given the tools available to the mathematicians of their time.

2.1.1 Derivation of π from Tables

John Wallis's *Arithmetica Infinitorum* contained tables, which required creative interpolation to derive formulas we take for granted today. Many of these tables have served as fodder for mathematicians of today who attempt to explain the ingenuity involved in their construction. Stedall (2000) describes how problems in number theory began to capture the attention of early 17th century mathematicians in England, preceding the age of Newton. In particular focus is brought on the work of John Wallis and Lord

William Brouncker and their collaboration on the problem of calculating quadratures (areas) of circles. It should be noted that at this time period the Calculus of Newton was yet to be invented, and most of the work requiring integrals was accomplished by using infinite series or continued fractions. The infinite product formula for **π** that is attributed to John Wallis is often featured in undergraduate mathematics textbooks, more often in an honors section of the course. Honors courses in U.S universities typically offer higher-level or more academically challenging assignments. The formula in question is presented as $\frac{2}{\pi} = \left(\frac{1\cdot 3}{2\cdot 2}\right)\left(\frac{3\cdot 5}{4\cdot 4}\right)\left(\frac{5\cdot 7}{6\cdot 6}\right) \dots$

To derive the formula for $\frac{2}{\pi}$ as the quotient of infinite products, students are told to work with the integral $I_n = \int_0^{\frac{\pi}{2}} \sin^n(x)dx$ without any explanation as to how it pertains to the formula in the first place. The problem at hand concerned the quadrature of the circle, which Stedall (2000) poetically described as "Squaring the circle". To tackle the problem, Wallis began with tables that gave values of $\frac{1}{\int_0^1 \left(1-x^{\frac{1}{Q}}\right)^P dx}$ for convenient values of P and Q that resulted in integers as the answer (see Table 1). The goal was to interpolate the value of $\int_0^1 \sqrt{1-x^2}\, dx$ by generating tables for $\int_0^1 (1-x^2)^0 dx$ and $\int_0^1 (1-x^2)\, dx$, but taking the reciprocals facilitated ease of computation.

P \ Q	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
0	1		1		1		1		1
$\frac{1}{2}$			?						
1	1		2		3		4		5
$\frac{3}{2}$									
2	1		3		6		10		15
$\frac{5}{2}$									
3	1		4		10		20		35
$\frac{7}{2}$									
4	1		5		15		35		70

Table 1. From *Arithmetica Infinitorum* Proposition 169 (Stedall, 2000, p.299).

Stedall (2000) provides details of the process that Wallis went through as well as his correspondence with Lord Brouncker during his investigation of this table. In modern terms, using integration by parts and a change of variable, the integral can be transformed easily to $\frac{1}{Q\int_0^1 (1-x)^P x^{Q-1} dx}$ which can be then be evaluated to generate the table for values of P and Q. However, *not knowing* the binomial theorem called for a creative leap that resulted in the astonishing formula from the tables, namely $\frac{2}{\pi} = \left(\frac{1 \cdot 3}{2 \cdot 2}\right)\left(\frac{3 \cdot 5}{4 \cdot 4}\right)\left(\frac{5 \cdot 7}{6 \cdot 6}\right) \dots$

The actual value calculated by Wallis was $\frac{4}{\pi} = \left(\frac{3}{2}\right)\left(\frac{3 \cdot 5}{4 \cdot 4}\right)\left(\frac{5 \cdot 7}{6 \cdot 6}\right) \dots$ which meant the quadrature was the reciprocal, namely $\frac{\pi}{4}$. The mathematical constraints of the time period called for ingenuity in methods. Haught and Stokes (2016) argue that domains are defined by constraints, which in turn lead to specific goal setting. For them creativity is mastering the basic constraints and the competency to achieve the goals despite the constraints. In the same vein, having access to the table produced by Wallis led Lord Brouncker to an entirely different interpretation to solve the quadrature problem. Lord Brouncker's work on the same tables led to a continued fraction for **π**, namely $\frac{4}{\pi} = 1 + \frac{1^2}{2 +} \frac{3^2}{2 +} \frac{5^2}{2 +} \cdots$

Again, for lack of a better descriptor, this is another astonishing formula for **π**, which is often found in books without any reference to the context of the problem. In modern terms, this interpolation can be understood by looking at the integral $I_n = \int_0^1 \frac{x^n}{1+x^2} dx$ and setting up a recursive formula for I_n to arrive at $\frac{4}{\pi}$. Graphing the integral for various values of n is the advantage that technology offers today's student (see Figure 1). The graphs can be used to generate a discussion on how the value of $\frac{4}{\pi}$ relates to the integral.

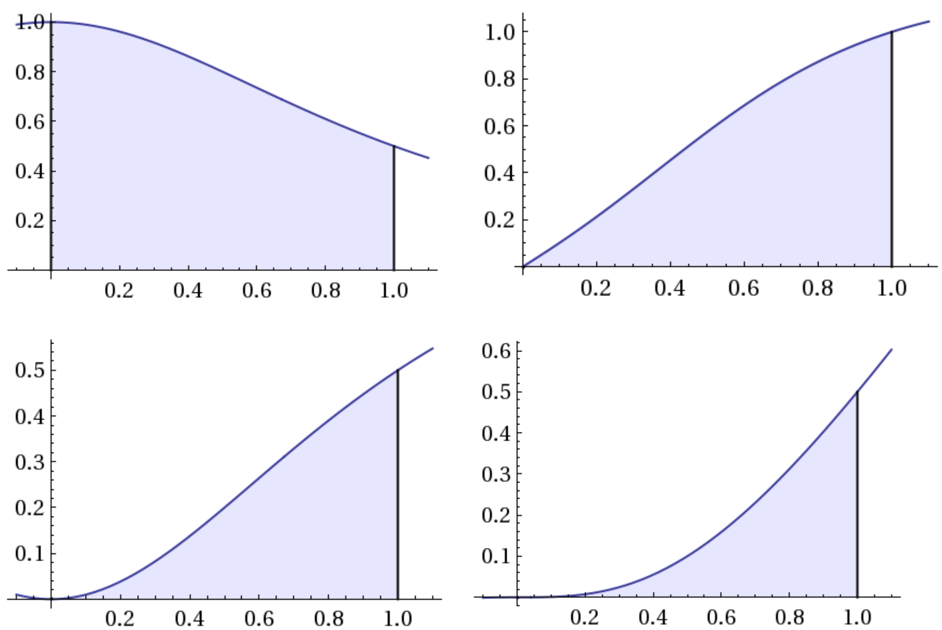

Figure 1: Graphs of $I_n = \int_0^1 \frac{x^n}{1+x^2} dx$ *for* $I_0, I_1, I_2,$ *and* I_3

Similarly, as mentioned earlier, Wallis's infinite product is evaluated by starting with the integral $I_n = \int_0^{\frac{\pi}{2}} \sin^n(x) dx$, which involves a circular function, and evaluating $\lim_{n\to\infty} \frac{I_{2n}}{I_{2n+1}}$ after arriving at the fact that $\frac{I_{2n-1}}{I_{2n+1}} = \frac{2n+1}{2n} \geq \frac{I_{2n}}{I_{2n+1}} \geq 1$. Again, there is a recursive formula involved and the necessity to integrate by parts; tools, which were not available to John Wallis! However technology again helps us understand the nature of the integrals and the value that is obtained by the infinite product. Graphing the integral (see Figure 2) is an advantage that was not available to the 17th century mathematician, but available to us today.

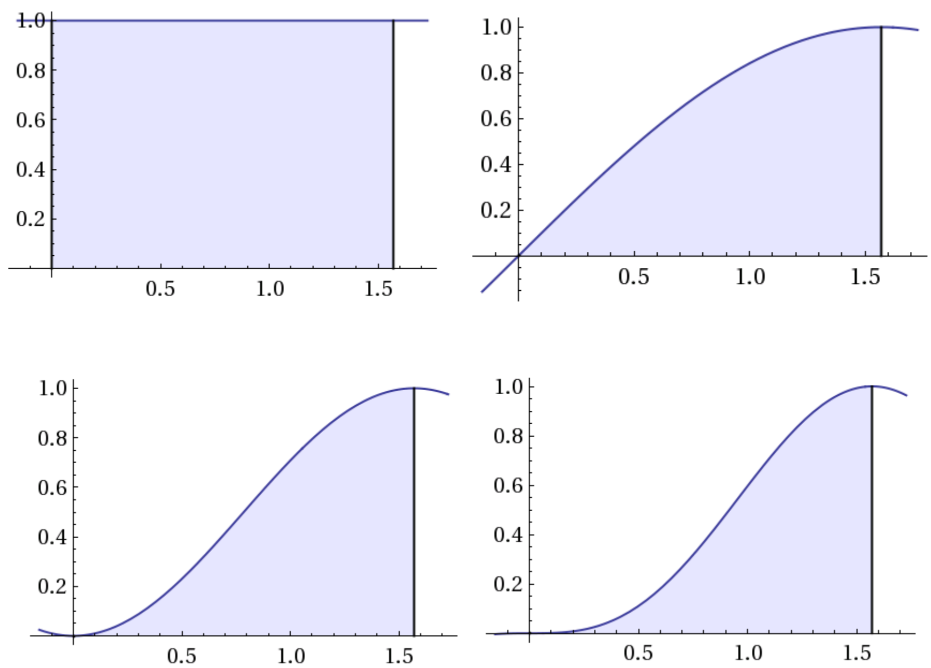

Figure 2: Graphs of $\int_0^{\frac{\pi}{2}} sin^n(x)dx$ *for* $I_0, I_1, I_2,$ *and* I_3

In both these examples, integrals involving circular functions need to be invoked by a 21st century student. However, the fact remains that interpolating tables played a significant part in arriving at the astonishing closed forms seen earlier. Dutka (1990) in his exposition of the history of the factorial function presents a different route available to us today since closed forms for the integral that Wallis had to tackle are now available to us.

He suggested that Wallis's computations are understood better by integrating $(x - x^2)^n dx$ between the limits of 0 and 1. By knowing the closed form for this integral, namely

$$\int_0^1 (x - x^2)^n dx = \frac{1}{2n+1} \frac{n!\,n!}{(2n)!} \text{ for } n = 0, 1, 2, 3, \ldots$$

we can calculate the following values for n=0,1,2,3,...

For $n = 0$

$$\int_0^1 (x - x^2)^0 dx = 1$$

For $n = 1$

$$\int_0^1 (x - x^2)^1 dx = \frac{x^2}{2} - \frac{x^3}{3}\bigg|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} = \frac{1}{2(1)+1} \frac{1!\,1!}{2!} = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

For $n = 2$

$$\int_0^1 (x - x^2)^2 = \int_0^1 (x^2 - 2x^3 + x^4) dx$$

$$= \left(\int_0^1 x^2 - 2\int_0^1 x^3 + \int_0^1 x^4\right) dx$$

$$= \frac{x^3}{3}\bigg|_0^1 - 2\frac{x^4}{4}\bigg|_0^1 + \frac{x^5}{5}\bigg|_0^1$$

$$= \frac{1}{3} - \frac{2}{4} + \frac{1}{5}$$

$$= \frac{1}{30} = \frac{1}{2(2)+1} \frac{2!\,2!}{4!} = \frac{1}{5} \cdot \frac{4}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{30}$$

We arrive at the sequence of values $1, \frac{1}{6}, \frac{1}{30}$, etc. but we want the value at $n = \frac{1}{2}$. Using trigonometric substitution we can show:

$\int_0^1 (x-x^2)^{\frac{1}{2}}\,dx = \frac{\sin^{-1}(2x-1)+(4x-2)\left(x-x^2\right)^{\frac{1}{2}}}{8} = \frac{\pi}{8}$; when evaluated between the limits of 0 and 1. So, we now have the sequence of values $1, \frac{\pi}{8}, \frac{1}{6}, \frac{1}{30}$, etc.

Knowing

$$\int_0^1 (x-x^2)^{\frac{1}{2}}\,dx = \frac{\pi}{8} = \frac{\left(\frac{1}{2}\right)!\left(\frac{1}{2}\right)!}{\left(2\cdot\frac{1}{2}\right)!}\cdot\frac{1}{2\left(\frac{1}{2}\right)+1}$$

We arrive at the astonishing fact that

$$\frac{\pi}{8} = \frac{\left(\frac{1}{2}\right)!\left(\frac{1}{2}\right)!}{2}$$

or

$$\frac{\pi}{4} = \left(\frac{1}{2}\right)!\left(\frac{1}{2}\right)!$$

which means:

$$\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$$

While Wallis interpolated this value between 1 and $\frac{1}{6}$ by “unknowingly” constructing values for P and Q, which gave binomial coefficients, we have today at our disposal both the binomial theorem, methods of integration, and the ability to graph the said integral to arrive at value. However, hand computation leads to the surprising discovery that $\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$ which is the value of the gamma function at $\frac{1}{2}$. The pedagogical point is that even today a student would be hard pressed to find a calculator that gives the value of $\left(\frac{1}{2}\right)!$, and in the event a numerical answer is obtained from a calculator, it conceals the fact that it is connected to π.

2.2 The irrationality of $\boldsymbol{\pi}$

Our second example involving old-fashioned calculations that can be illuminating to students is establishing irrationality of transcendental numbers like $\boldsymbol{\pi}$. Establishing irrationality of $\boldsymbol{\pi}$ is relegated to being a difficult problem even in undergraduate mathematics courses because it is assumed that students do not have the mathematical tools necessary to construct a proof. Ivan Niven(1947) started with $\int_{\mathbf{0}}^{\boldsymbol{\pi}} \frac{\boldsymbol{x^n(\pi-x)^a}}{\boldsymbol{n}!} \boldsymbol{sinx\, dx}$ to give a simple proof by contradiction that $\boldsymbol{\pi}$ is irrational. However, irrationality was established much earlier by the polymath Johann Lambert in 1761 by starting with the tangent function, namely $\boldsymbol{nx} = \frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-+\cdots}{1-\frac{x^2}{2!}+\frac{x^4}{4!}-+\cdots}$, and using long division along with the Euclidean Algorithm to calculate a sequence of remainders that could produce a continued fraction for the tangent function.

In other words if we write out the remainders at each stage of the long division, we get :

$$R_1 = sinx - xcosx = \frac{x^3}{3} - \frac{x^5}{2\cdot 3\cdot 5} + \cdots$$

$$R_2 = (3-x^2)sinx - 3xcosx = \frac{x^5}{3\cdot 5} - \frac{x^7}{2\cdot 3\cdot 5\cdot 7} + \cdots$$

$$R_3 = (15-6x^2)sinx - (15-x^3)cosx = \frac{x^7}{3\cdot 5\cdot 7} - \frac{x^9}{2\cdot 3\cdot 5\cdot 7\cdot 9} + \cdots$$

.
.
.

Each of these remainders "measures" the tangent function in terms of truncations of an infinite contained function. The representation constructed by Lambert (1761) was $tanx = \cfrac{x}{1-\cfrac{x^2}{3-\cfrac{x^2}{5-\cfrac{x^2}{\cdots}}}}$ and now using the result that an infinite continued fraction is irrational, we quickly arrive at a contradiction when we assume $\boldsymbol{\pi}$ is rational and substitute the value of $\frac{\pi}{4}$ for x in the expression. This method of establishing irrationality of π is quite different from Niven's proof which also relies on contradiction. The pedagogical point here is that Lambert's method requires one to perform long division of infinite series, which is rarely done in mathematics courses. Also, hand computation results in some astonishing results that interpolate intermediate values of functions by truncating the series representation of a function. Lambert also played an important role in the field of map projections and write extensively on the topics of conformal projections. Our third and final example shows a fascinating integral that relates to interpolation and conformal map making.

2.3 Integrating the Secant Function

An interesting application of interpolation involves integrating the secant function using the series representation of $\log(1+x)$. A practical application for this calculation resulted in the need to construct navigational charts. Deriving the integral of the secant function is often a difficult task for calculus students. It involves making a trigonometric substitution that seems contrived and logically convenient. However understanding how this integral was approximated before there was even an understanding of calculus can be illuminating to students and give them a full appreciation for the creativity that can arise through hand calculations and approximations. Again the notion of domain constraints leads to unexpected results when required to solve a pressing problem, which in this case was conformal maps needed for navigation in the 16th century.

2.3.1 Mercator Projection Map

In 1569, the Flemish cartographer Gerhardus Mercator created a map now known as the Mercator Projection Map. This map allowed navigation using lines of constant course also known as rhumb lines. These maps allow a navigator to draw a line between two points on a map, find a bearing, and then follow a compass reading to their destination. This is allowed through the scaling of the space between latitudes on Mercator projection maps. This scaling caused distortion in sizes of landmasses on the map but provided a great advance in navigational abilities. Mercator did not document his method of construction, but it is thought that it was constructed using a compass and straight edge (Carslaw, 1924).

The English mathematician Edward Wright mathematically derived Mercator's projection in 1599 by creating a table that provided the scale factor as a function of the latitude. This table allowed for the accurate construction of Mercator projection maps by converting latitudes into distances from the equator. Wright's table was constructed using approximate sums, what would now be known as Riemann sums. He constructed his table with an interval of one minute of arc, or $\frac{1}{60}$ degree for all latitudes to 75°. His table was later found be an accurate table of the integral of secants. The creation of his table is even more astounding given the lack of understanding of logarithms or calculus that existed in his time.

2.3.2 Derivation of Wright's Table

Wright realized that in order to preserve angles on the Mercator projection, i.e., to keep conformality, the vertical and horizontal direction on the map needs to be stretched by the same factor. This allowed the meridians to be parallel and intersect the equator at a right angle. By simultaneously scaling in the vertical direction, the Mercator map can be constructed to allow the constant course navigation that so greatly aided sailors of the time.

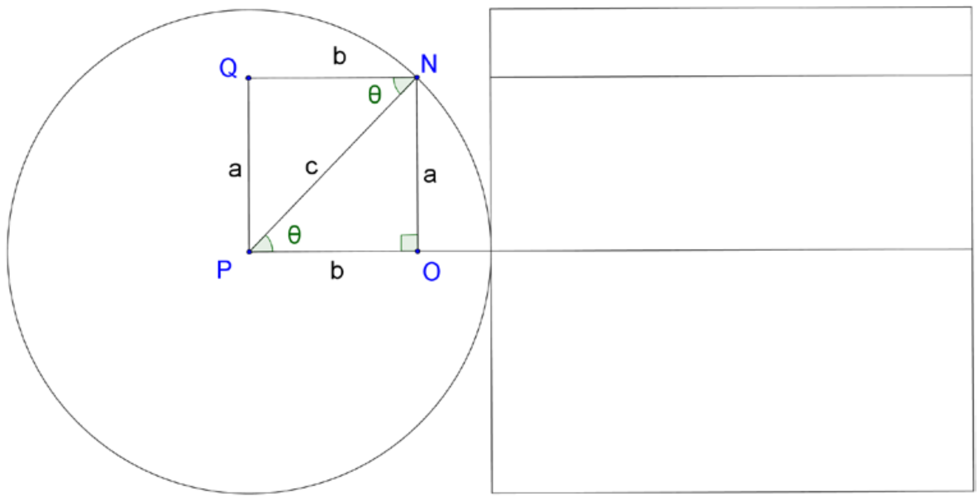

Figure 3: $f(\theta) = sec(\theta)$

Wright recognized that by choosing a common interval he could determine the position of a line of latitude on the Mercator projection by summing the distance of all of the lines separated by the previous intervals. The smaller the interval that is used, the more accurate is the approximation. In order to build his table, Wright used a trigonometric table of secants available during his time. For example, if we want to find the distance from the equator to the 30th line of latitude with an interval of 5°, we would use the following table of secants:

Table of Secants	
Secant 5°	1.0038
Secant 10°	1.0154
Secant 15°	1.0353
Secant 20°	1.0642
Secant 25°	1.1034
Secant 30°	1.1547
Secant 35°	1.2208
Secant 40°	1.3054
Secant 45°	1.4142
Secant 50°	1.5557
Secant 55°	1.7434
Secant 60°	2.0000
Total	15.6163

Table 2: Table of secants approximated at an interval of 5°

Taking the total of these secants multiplied by the interval results in $15.6163 * 5° = 78.0815°$. The 60th line of latitude should be placed on the Mercator projection map at 78.0815°.

2.3.3 Modern Derivation

Wright determined an approximation of the integral of the secant function using numerical summation with an interval of one minute ($1'$) or $\frac{1}{16}°$ for all lines of latitudes up to 75°. This results in $16 * 75 = 1{,}200$ entries. These calculations would be very tedious without the use of calculus (or modern technology). Wright's method is not exact, but provides a very reasonable approximation. The method could be improved by looking at even smaller interval widths, though again this would require a very tedious number of calculations. The real improvement comes with the development of the logarithmic function and calculus. The following modern proof demonstrates the closed form of the integral of the secant.

$$\int \sec\theta d\theta = \int \frac{1}{\cos\theta} d\theta$$

$$= \int \frac{\cos\theta}{\cos^2\theta} d\theta$$

$$= \int \frac{\cos\theta}{1-\sin^2\theta} d\theta$$

$$= \int \frac{\cos\theta}{(1-\sin\theta)(1+\sin\theta)} d\theta$$

$$= \int \frac{1}{2}\left(\frac{\cos\theta}{1-\sin\theta} + \frac{\cos\theta}{1+\sin\theta}\right) d\theta$$

$$= \frac{1}{2}\int \frac{\cos\theta}{1-\sin\theta} + \frac{\cos\theta}{1+\sin\theta} d\theta$$

$$= \frac{1}{2}[-\ln|1-\sin\theta| + \ln|1+\sin\theta|] + c$$

$$= \frac{1}{2}\ln\left|\frac{\sin\theta+1}{\sin\theta-1}\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{(1+\sin\theta)^2}{(\cos\theta)^2}\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin\theta}{\cos\theta}\right| + c$$

$$= \ln|\sec\theta + \tan\theta|$$

This closed form of the integral of the secant function is assumes a radian measure. If θ is measured in degrees the result would be the following equation:

$$\int_0^\theta \sec\theta \; d\theta = \frac{180}{\pi}\ln|\sec\theta + \tan\theta|$$

Performing this calculation for 60° results in a value of $\frac{180}{\pi}\ln|\sec 60 + \tan 60| = 75.4651°$. Using Wright's method of approximation with a large interval of 5° results in a value that is only off by 2.6164°. Using an Excel spreadsheet to calculate Wright's approximation at an interval of $1'$ would result in a value equal to 75.4874°, a difference of only 0.0223°. An approximation of this accuracy likely approaches the limits of the measurement and mapmaking tools of Wright's time. A tool such Wolfram-Alpha can readily provide the value of the secant function, but insight into the meaning and origin of this calculation is lost. There is a beauty and elegance in Wright's approximation that students have to experience to appreciate. Another point to note is that many Calculus textbooks evaluate the integral by using the trick of multiplying and dividing the given function by $\sec(\theta) + \tan(\theta)$ to convert the integral $\int \sec\theta d\theta = \int \frac{1}{u} du$ where $u = \sec(\theta) + \tan(\theta)$.

Such a trick though resulting in the correct answer is devoid of any mathematical insight whatsoever and as far removed from Wright's interpolation as the secant of a right angle!

3 Concluding Remarks

The three examples provided in this chapter illustrate the complementary nature of hand calculations to the affordances provided by the digital age. As argued by the editors of this book, the connection between technology and creativity in mathematics education is still unexplored territory. If the past serves as a reminder, it is important to remember that graphing calculators and computer algebra systems did not really change the nature of college Calculus much other than the fact that instructors needed to actually think about the type of questions that could be asked of students if the hand held utility more or less did everything that was taught in a traditional course. Unlike the resistance to technology in the 1990's, present day learner's experience in any classroom is ubiquitous with the use of ICT. The advent of big data as the next frontier for computing to reveal patterns and trends relating to human behavior requires "interpolation" to fill in incomplete data sets no different from the gaps in information encountered by mathematicians in the past.

Learning environments that utilize the wealth of information provided by mathematical repositories in addition to computing tools like Mathematica and Wolfram Alpha provide a much richer experience than the classrooms of yore. However simply using multimodal software to show multiple representations does not necessarily mean that it results in any deeper understanding or insight unless the complementary and "beautiful" nature of hand calculations are also incorporated into the environment. Paper and pencil mathematics has long been the shibboleth of mainstream mathematics[1] and is unlikely to change even with the advent of the digital age. One may think of such orthodoxy as simply the motivation for teachers to show the ingenuity and beauty of hand calculations with historical examples such as those described in this chapter. These examples also illustrate the creative nature of mathematical interpolation to complement other modes of representation and serve as useful lessons to those that think "big data" is a recent product of the information age!

References

Carslaw, H. S. (1924). The story of Mercator's map. *The Mathematical Gazette*, *12*(168), 1-7.

Dutka, J. (1990). The early history of the factorial function. *Archive for History of Exact Sciences 41*, 225-249.

Haught, C., & Stokes, P. (2016, in press). Constraints, competency and creativity in the class

[1] Personal communication with Reuben Hersh, October 8, 2015

room. In R. Leikin and B. Sriraman (Eds), *Creativity and Giftedness: Interdisciplinary perspectives from mathematics and beyond*. The Netherlands:.Springer Science and Business, Dodrecht.

Lewi, P.J. (2006). *Speaking of Graphics, An Essay on Graphicacy in Science, Technology and Business*. Retrieved on Feb 29,2016 from http://www.datascope.be/sog.htm

Niven, I. (1947). A simple proof $\boldsymbol{\pi}$ is irrational. *Bulletin of the American Mathematical Society*,*53*(6), 509.

Osborn, A. (1953). *Applied imagination: Principles and procedures of creative problem solving.* New York: Charles Scribner's Sons.

Torrance, E.P. (1974). *Norms-technical manual: Torrance Tests of Creative Thinking*.

Lexington, MA: Ginn and Company

Stedall, J.A. (2000). Catching Proteus: The collaborations of Wallis and Brouncker I Squaring the circle. *Notes and Records of the Royal Society of London, 54*, 293-316.

Wallis, J. (2004). *The Arithmetic of Infinitesimals* (translated from Latin by J, A, Stedall), New York: Springer.