Mathematics Education as a matter of cognition

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1. Introduction

The word *cognition* is defined in most dictionaries as (1) process of knowing, (2) something that is known, (3) thinking, (4) perception, (5) study of the mind. There are numerous other meanings that can be found in the domains of psychology, biology, philosophy, sociology, linguistics and phenomenology. However for mathematics education we focus primarily on psychology and secondarily on biology, philosophy, and sociology. More specifically we describe and analyze the domain of mathematics education as evolving in its notion of cognition from its roots in psychology and moving onto domains that broaden the notion of "cognition" for mathematics education researchers. This encyclopedia entry has three objectives:

a. To determine a "starting point" (if any) for research on cognition in mathematics education.

b. To unfold the development of mathematics education as a field of research based on its interaction with research in cognition.

c. To describe the different schools of thought currently found in mathematics education as they relate to research in cognition.

2. Determining a Starting Point

A watershed moment for cognition research in mathematics education in the latter part of the 20^{th} century was the publication of the book *Critical Variables in Mathematics Education* (Begle, 1979) which surveyed existent research in the field based on high standards of scientific inquiry. Inquiry defined at this point in time was empirical research based on paradigms in experimental psychology. This can be attributed to methodologies advocated by funding agencies in the U.S in the post-Sputnik era (late 1950's and 60's) where the notion of certainty of results was transposed from the physical sciences (physics, chemistry etc.) to mathematics education. As a consequence early researchers in mathematics education adapted operational definitions from cognitive and experimental psychology into their work. One sees a preponderance of Aptitude-Treatment-Interaction (ATI's) studies in the time period 1960-1985. At this point in time, study of cognition was concerned with the mental processes of perception, thinking, memory and learning. Theories that were invoked to guide research or development activities at this time were borrowed from educational psychology such as: Bloom's *taxonomy of educational objectives*, Gagne's *behavioral objectives* and learning *hierarchies*, Piaget's *stage theory*, Ausabel's *advanced organizers and meaningful verbal learning* - and later Vygotsky's *socially-mediated learning*, and finally Simon's *artificial intelligence* models for cognition (Lesh, Sriraman, English, 2014). Piaget's stage theory is examined in more detail further in this entry as it relates to mathematics education.

As a contrast to this Anglo-American view of the development of mathematics education as a matter of cognition, a different view is obtained from Europe. In the early part of the 20th century mathematicians like Felix Klein (1849-1925) became interested in the teaching and learning of school mathematics. In this era one finds increasing interest in studying the psychological development of schoolchildren and its relationship to the principles of arithmetic. The *International Commission of Mathematics Instruction* was founded in 1908 with Felix Klein as the first president. One of the founding goals of ICMI was to publish mathematics education books accessible to both teachers and their students. First published in 1908 but still printed today is the book "*Elementarmathematik vom höheren Standpunkt*'"(Elementary mathematics from an elevated viewpoint) clearly illuminate Klein´s paradigm of school mathematics: *scientific foundation of school mathematics* and *accessibility through elementarization.* The question is how did this process of elementarization of higher mathematics work? Klein borrowing from the psychological theories of that time period which claimed it was important to (1) provide an access by concentrating on the core of a mathematical topic; (2) add views from neighboring fields; (3) recognize and activate the students' previous knowledge or by changing the means of representation (Törner and Sriraman, 2006). It is interesting to note that the ideas of context and multiple representations as being essential to facilitate learning were suggested ninety years before constructivist learning theories came into vogue in the 1990s.

Another view of the influence of educational psychology in general and cognition in particular to the field of mathematics education is obtained in the *Handbook of Educational Psychology* (Alexander & Winne, 2006). In the foundations section of this book a historical case study of educational psychology (Berliner, 2006) reveals its roots in Democritus' musings in 500 B.C.E on the nature of schooling and learning, the Roman conceptions of schooling during the time of Quintilian (35-100 A.D) and its relationship to democratic ideals and individual differences in learning, onto the (European) Renaissance that led to the birth of the humanist movement and a focus on the developmental nature of human psychology, culminating with the beginnings of modern educational psychology in the foundational writings of Johann Friedrich Herbart (1776-1841), the pragmatists William James (1842-1910), G. Stanley Hall (1844-1924), John Dewey (1859-1952) and finally Edward Lee Thorndike (1874-1949). This historical lineage suggests that it is erroneous to attribute the birth of mathematics education in the experimental psychology and behaviorism of Thorndike (Sriraman, 2009) or to Begle or to Klein. Indeed it is the school of pragmatism that can be attributed as a starting point for cognition research as seen in the work of William James who unlike his student Thorndike was already aware of the dangers of over-simplifying the study of the human mind/behavior into operational and quantitatively measurable constructs (Berliner, 2006). Pragmatism is viewed as a philosophical position and one that is congruent with the shift from aptitude-treatment-interaction studies that characterized mathematics education to qualitative inquiry that recognized human beings as cognizing subjects different from each other. The former strand of mathematics education was based on cognition viewed from a behaviorist viewpoint which resulted in psychometric studies (ATI's) whereas the latter embraced methods from the human (e.g., anthropology) and social sciences.

3. Unfolding the development of mathematics education as it relates to cognition

The notion of "operational" definitions formed an important basis for studies of mathematical cognition as seen in the extensive canon of work produced by Piaget, which influenced the field of mathematics education. In the hard sciences like physics and chemistry, theoretical definitions arise as a result of repeatedly observing invariance in operations represented by physical measurement devices. In other words operationalization means to “define something” in terms of “a process” that measures *it*. According to Dietrich (2004) in physics, terms have to be defined operationally (in terms of theories) provided experimentation can back up notions occurring within the theories. Piaget applied this to the study of cognition in children by defining operations as “internalized actions”, which were derived directly from the subject’s physical actions as enacted in sensorimotor behavior. The focus of cognition in Piaget’s work stemmed from adapting biological theories of organization, development and adaptation to study children’s understanding of number, quantity, space, time, causality and relations of invariance. Piaget's developmental theory consisted of sensorimotor, pre-operational, concrete operational and formal operational stages. One of Piaget’s claims was the posited link between mathematics and biology where cognition was characterized as a form of biological adaptation as one moved through the stages outlined in his stage model. Piaget in fact said that "the whole of mathematics be thought of in terms of creation of structures" (Beth & Piaget, 1966, p.70). These constructions are of course not physical ones, but operations carried out in the conceptual and idealized world of the mathematician. The passage from sensorimotor actions to formal thinking, in Piaget's account is one of increasing abstraction and generalization. Piaget compared his operator structures of thinking to the structures espoused by the Bourbaki. The Bourbaki identified three fundamental structures on which mathematical knowledge rests. They are (1) algebraic structures; (2) structures of order; and (3) topological structures. Piaget claimed that there existed a correspondence between the mathematical structures of the Bourbaki and the operative structures of thought. In terms of Dietrich’s (2004) analysis, Piaget took the observed regularities in children’s cognition and attempted to describe them as phylogenetically evolved mental cognitive operators. Piaget’s theory of cognitive development in children formed the cornerstone of numerous longitudinal studies in North America funded by the National Science Foundation. Based on Piaget’s body of foundational work, mathematics education researchers empirically validated models within the Rational Number Project (Lesh, Post and Behr, 1988; Behr et al., 1991) that explained how proportional reasoning develops in children that by and large cohere with Piaget’s stage theory. Some of the findings from the last two decades also suggest that when Piaget’s experiments are repeated with age appropriate materials, the stages proposed by him are not as discrete as they seem, but more porous with the possibility of children being able to reason at a more advanced level given contextual play materials (Sriraman & English, 2004). Zoltan Dienes’ six stage theory of learning mathematics bears resemblance to that of Piaget with a somewhat different conceptualization of what “operational” means (Dienes & Jieves, 1965; Sriraman & Lesh, 2007).

If mathematics education researchers pointed to one topic area where they believe theory development to be strongest, they'd point to the substantial work on mathematical cognition in the areas of (a) early number concepts, or (b) early algebraic reasoning or (c) rational numbers & proportional reasoning (Lesh, Sriraman & English, 2014). Evidence of this theory development in learning is found in the literature related to Piaget-like cognitive structures (Steffe, 1995; Steffe et al., 1996); cognitively guided instruction which focuses on task variables; the focus on counting strategies; Vygotsky's socially mediated views of development; and focus on computer-based embodiments which are in some ways similar to those used Dienes (Sriraman & Lesh, 2007).

4. Learning theories in mathematics education

Debates on different theories of learning reached a crescendo in the 1990's with mathematics education researchers arguing for and against constructivist theories of learning. Radical constructivism as proposed by von Glasersfeld took an extreme view of human cognition best captured in the phrase "every person is an island" and meaning was an individual construction whereas social constructivism as proposed by Paul Ernest argued that conversation between people was the underlying building block in creation of meaning. Theorists like Paul Cobb and Heinrich Bauersfeld attempted to bridge radical and social contructivist theories by emphasizing the role of culture and language and classroom discourse. Other researchers who valued the emphasis on culture built on Vygotsky's cultural-historical psychology in what has become known as cultural historical activity theory. Finally a theory of thinking was proposed by Anna Sfard which extended and synthesized the ideas of Vygotsky and Wittgenstein to propose commognition (cognition and communication) as a theoretical basis for analyzing discourse from the informant's point of view. A full treatment of these different theories is found in the *Theories of Mathematics Education* (Sriraman & English, 2010). This book documents a shift beyond theory-borrowing toward theory-building in mathematics education; and that relevant theories in mathematics education now draw on far more than psychology. Newer developments that address cognition in mathematics education are constantly occurring. For the sake of brevity we end this entry with two important contributions, one which addresses cognition from a historical and cultural perspective by using metaphors from biology and a remark on newer perspectives from the domain of embodied cognition

5. Two overlooked metaphors from biology

Furinghetti & Radford (2002) traced the evolution of *Haeckel's (1874) law of recapitulation* from the point of view that parallelism is inherent in how mathematical ideas evolve and the cognitive growth of an individual (Piaget & Garcia, 1989). In other words the difficulties or reactions of those who encounter a mathematical problem can invariably be traced to the historical difficulties during the development of the underlying mathematical concepts. The final theoretical product (namely the mathematical theorem or object), is the result of the historical interplay between phylogenetic and ontogenetic developments of mathematics, where phylogeny is recapitulated by ontogeny. Beth and Piaget's (1966) claim that there was a correspondence between the Bourbakian structures of mathematics and operator structures of thought were conjectural at best. However when analyzed from the perspective of Haeckel's law the correspondence can be conceived of as Bourbakian ontogeny recapitulating individual mathematicians' phylogenetic contributions against the backdrop of history. Furinghetti & Radford (2002) argue that psychological constructs as well as the study and formation of intellectual mechanisms are not as tenable as the clearly dated and archived transformations of mathematics in its historical development. We suggest that the use of Haeckel's recapitulation theory as a link between the psychological and historical domains offers better insights into the evolution of cognition as opposed to the one-to-one correspondence conjectured by Beth and Piaget (1966). It is important to note that Haeckel's law in its original form was rejected by the community of biologists but has been transformed numerous times by some, over the last 100 years to better explain the relationship between phylogeny and ontogeny in different species. However in mathematics education unlike biology, we are referring to psychological recapitulation. In a similar vein, a neo-Lamarckian perspective of recapitulation is also available to mathematics education in any discussion of biological metaphors that capture mathematical cognition.

Jean-Baptiste Lamarck's recapitulation cannot be applied or transposed directly to the study of didactical problems in mathematics education because it does not take into account the influence of experience (or more broadly culture). However just as Lamarck proposed in vain to his peers in 1803, that hereditary characteristics may be influenced by culture, mathematics education increasingly takes into account how culture influences the mutation of historical ideas. The idea however is not new. In fact Gould (1979) wrote that

> *Cultural evolution has progressed at rates that Darwinian processes cannot begin to approach…[t]his crux in the Earth's history has been reached because Lamarckian processes have finally been unleashed upon it. Human cultural evolution, in strong opposition to our biological history, is Lamarckian in character. What we learn in one generation , we transmit directly by teaching and writing. Acquired characters are inherited in technology and culture. Lamarckian evolution is rapid and accumulative…*

The teaching and learning of mathematics bears strong evidence to this Lamarckian nature. Indeed, what took Fermat, Leibniz and Newton collectively a hundred years to develop is taught and often digested by students in one year of university Calculus. Any higher level mathematics textbook is a cultural artifact which testifies to rapid accumulation and transmission of hundreds (if not thousands of years) of knowledge development. So, evolutionary epistemologists have now begun to accept the fact that for humans, cultural evolution in a manner of speaking is neo-Lamarckian (Callebaut, 1987, Gould, 1979).

6. Embodied Cognition

Embodied cognition came to define a new frontier for cognition research in mathematics education. Simply put cognitive psychology began to focus on the interaction between an individual and the environment with implications for mathematics. In other words mathematical thinking and learning could be viewed within social, cultural and contextual factors (Lave 1988; Nunez, Edwards & Matos, 1999). Lakoff and Nunez (2000) discussed the cognitive science of mathematics based on the key concept of embodied cognition. The basic assumption was that mathematics is not mind free and mathematical cognition often occurs unconsciously (Lakoff & Nunez, 2000). Two types of conceptual metaphors are suggested to play an important role in the development of mathematical ideas, namely grounding metaphors and linking metaphors. The interested reader can pursue this further in the entry on mathematics education as a matter of the body.

7. Conclusion

To summarize, in this entry we have described different “starting points” for research on cognition in mathematics education and then traced the development of the field by analyzing the role of operational definitions and theories of learning that were developed. The entry also examines the work of Piaget, Haeckel and Lamarck in the use of metaphors from biology to study and understand cognition from an individual to a collective historical perspective. In doing so, we have narrated the development of mathematics education as a matter of cognition.

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