

# Clarification of Lapse Rate Terms and Formulas

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A “lapse rate” is the rate of change of an atmospheric variable with altitude. The implied variable is usually temperature, unless otherwise stated. This paper will deal with temperature as the variable, as it discusses thermal lapse rates such as the Environmental Dry Adiabatic and the Saturated Adiabatic Lapse Rate. I will seek to clarify the derivation of the formulas for the Environmental Lapse Rate.

Most sources approximate the composition of the atmosphere as a mixture of water vapor and dry air. In reality, the air is composed of a combination of gases and particulate matter from both natural and industrial origins. The most prevalent gases in the atmosphere (in order of abundance) are nitrogen (N<sub>2</sub>), oxygen (O<sub>2</sub>), water vapor (H<sub>2</sub>O), argon (Ar), and then carbon dioxide (CO<sub>2</sub>). Though the atmosphere contains other trace gaseous elements, for our purposes they may be disregarded, since this paper deals with air under the assumptions of an ideal gas.

A measured temperature is a snapshot in time of the larger atmospheric process of energy transference from one form to another, which requires that this paper deals in equations of state—thermodynamic equations that describe the state of matter under a given set of physical conditions. Temperature is a measure proportional to the average translational kinetic energy associated with the disordered microscopic motion of atoms and molecules. The flow of heat is from a high temperature region toward a lower temperature region. Such transformations are readily found in the cooling and heating of air, condensation of water vapor, and absorption of radiant energy (Tverskoi, 1965). Hence, temperature soundings can provide us with the information necessary to study heat flow in an atmosphere, and then the environmental lapse rate.

### Derivation of Dry and Saturated Adiabatic Lapse Rates

The first law of thermodynamics is a general statement of the fact that energy can be changed from one form to another, but never created or destroyed. The first law states that an applied quantity of heat to a system ( $\Delta Q$ ) will increase both the system’s ability to do work ( $\Delta W$ ) and its internal energy ( $\Delta U$ ). The first law can be written as

$$\Delta Q = \Delta U + \Delta W \quad (1)$$

In its differential form, the change in the quantity, signified by the delta symbol, is changed to a  $d$ , the differential, where the change in the quantity becomes infinitesimal until it provides a snapshot of a perfect instant in time.

$$dq = du + dw \quad (2)$$

Regard a parcel (our chosen unit of air), as an ideal gas, and then consider the  $\Delta W$  term from formula (1). Assuming that the following condition will apply to any

thermodynamic process under consideration: the external pressure  $\varphi$  exerted by the surrounding medium on the parcel is equal to its internal pressure  $p$ . This internal pressure of a parcel of air is determined by the molecular motion of the parcel's constituents, summarized by the ideal gas law

$$p = R\rho T \quad (3)$$

Tverskoi indicates that in the atmosphere the pressure of a gas in a small volume rapidly becomes equal to the pressure of the surrounding medium and thus the condition of external pressure being equal to internal pressure is fulfilled. The work done by the air parcel against the external pressure force is given by

$$\Delta W = p\Delta V \quad (4)$$

where  $p$  is the internal pressure and  $\Delta V$  is the change in volume.

As a differential

$$dq = pdv \quad (5)$$

Substituting formula (5) into formula (2)

$$dq = du + pdv \quad (6)$$

To logically continue our derivation of the dry and saturated adiabatic lapse rates, we must further examine enthalpy ( $h$ ), a scalar function related to the internal energy of a system at constant pressure and changing volume. Enthalpy is a measure of the total energy of a thermodynamic system, including the internal energy (the energy required to make the system), and the amount of energy required to make room for it by displacing its environment and establishing its volume and pressure. The equation for enthalpy is

$$h = U + pV$$

and given as a differential expression is

$$dh = du + pdv + vdp \quad (7)$$

Where  $du$  is the infinitesimal change in the internal energy,  $p$  is the internal pressure, and  $v$  is specific volume content.

Manipulating formula (7) with information from formula (5)

$$\begin{aligned}
du &= dh - pdV - Vdp \\
dq &= dh - pdV - Vdp + pdV \quad (8) \\
dq &= dh - Vdp
\end{aligned}$$

Further definition of the differential equation for enthalpy yields

$$dh = c_p dT \quad (9)$$

where  $c_p$  is the specific heat content of a parcel of air at constant pressure and  $dT$  is the temperature differential.

Further

$$dq = c_p dT \quad (10)$$

$$du = c_v dT \quad (11)$$

$$v = \frac{1}{\rho} \quad (12)$$

where  $c_v$  is specific volume, or the ratio of a substance's volume to its mass.

Further derivation includes

$$\begin{aligned}
dh &= dq - vdp \\
dh &= c_p dT - \frac{1}{\rho} dp \quad (13)
\end{aligned}$$

$$\begin{aligned}
dq &= dh - Vdp \\
dq &= c_p dT - \frac{1}{\rho} dp \quad (14)
\end{aligned}$$

Then, dividing equation (14) by the differential height  $dz$  (the change in altitude), the result is

$$\frac{dq}{dz} = \frac{c_p dT}{dz} - \frac{1}{\rho} \frac{dp}{dz} \quad (15)$$

The differential  $\frac{dT}{dz}$  will describe the change in temperature  $T$  with the change in altitude  $z$ , the common denotation of lapse rate. Divide every term by the specific heat term, rearrange, and the equation for lapse rate becomes

$$\frac{dT}{dz} = \frac{1}{c_p \rho} \left( \frac{dp}{dz} \right) \quad (16)$$

The rate of change of the pressure with height  $\left( \frac{dp}{dz} \right)$  arises out of the fundamental hydrostatics equation.

When a model of something as complex as the atmosphere is being considered, choosing a cross section that is geometrically perfect and symmetrical simplifies the quantitative analysis. The atmosphere can be portioned into hypothetical symmetrical parcels of cylindrical form, with unit cross-section, and stacked one on top of the other in a vertical column. The three assumptions then associated with this vertical column model are: “1) the air is at rest relative to the earth’s surface; 2) it can be regarded as an ideal gas; 3) the composition of the air does not change with height” (Tverskoi, 1965). In order for equilibrium to be achieved for any given height about the earth’s surface  $z$ , the buoyance  $F_b$  of the column must balance with the weight  $W$  of the column at height  $z$ . Symbolically

$$F_b = W \quad (17)$$

As the altitude  $z$  is increased, both the buoyancy and the weight are decreased, since the relationship is directly proportional. Therefore, when we quantify the buoyancy of air, what is really being measured is “the force exerted on a unit surface by the weight of the air column standing vertically above sea level” (Tverskoi, 1965). This force is labeled as the atmospheric pressure, the force per unit area, and given the symbol  $p$ . Considering the first of Tverskoi’s assumption, listed above, it would appear that any motion within the air column would nullify the direct proportionality between the buoyant force  $F_b$  and the weight of the air column, but a detailed analysis reveals that in reality, air motions and accelerations in the atmosphere are so small that, unless there are unusually strong vertical accelerations that slightly effect pressure, their influence can be disregarded in practice (Tverskoi, 1965).

Consider a column of air at a height  $z$  in the atmosphere. The pressure at the base of the column is  $p$ , and at the top of the column the pressure is  $p - dp$ . In the absence of a pressure difference between the column and the air that borders it, the pressure drop  $-dp$  is equal to the weight of the column of air (Tverskoi, 1965). Given that  $\rho$  is the density of air at height  $z$ , and  $g$  is the acceleration due to gravity, then the result is the equation for hydrostatic equilibrium

$$-dp = \rho g dz \quad (18)$$

Dividing the equation by the differential change in altitude yields

$$\frac{dp}{dz} = -\rho g$$

Inserting this formula into equation (16) gives

$$\frac{dq}{dz} = c_p \frac{dT}{dz} + g$$

Rearranging the equation and multiplying by  $c_p$ , the specific heat content of a parcel of air at constant pressure, results in

$$-\frac{dT}{dz} = \frac{g}{c_p} - \frac{1}{c_p} \frac{dq}{dz} \quad (19)$$

In a parcel of dry adiabatic air, the differential change in heat with respect to the differential change in altitude  $\frac{dq}{dz}$  will be zero. Thus, in a dry atmosphere, the dry adiabatic lapse rate (*DALR*) will be described as

$$-\frac{dT}{dZ} = \frac{g}{c_p} = -9.81 \text{ K/km} \quad (20)$$

A saturated parcel of air is still an adiabat because all heat exchange happens only within the parcel. However, this heat exchange that occurs between the condensation and evaporation of water vapor within the parcel gives the  $\frac{dq}{dz}$  term a value. A parcel's decreasing temperature with altitude is slower under conditions of saturation than under unsaturated condition (Salby, 1996). The first term in the formula *SALR* factors in the work done by the parcel due to expansion. The second term in the formula, called the saturation term, accounts for the heat exchange within the parcel during the phase changes.

The change in heat can also be defined as

$$dq = -L_v dq_{vs} \quad (21)$$

where  $L_v$  is the latent heat of vaporization and  $dq_{vs}$  is the specific humidity at saturation. The heat term in the Saturated Adiabatic Lapse Rate is a positive,

warming term due to the latent heat released during condensation. The *SALR* equation can be arranged via the definition of enthalpy in the following way

$$SALR = \frac{dT}{dz} = \frac{g}{c_p} + \frac{L_v}{c_p} \cdot \frac{dq_{v_s}}{dz} \quad (22)$$

$\frac{dr}{dz}$  is the term for the rate of change for a unitless quantity called the mixing ratio with altitude. The mixing ratio is the abundance of water vapor in proportion to the abundance of dry air. The specific humidity at saturation  $dq_{v_s}$  can be defined in terms of the mixing ratio as

$$q_{v_s} = \frac{r_{v_s}}{1 + r_{v_s}}$$

where  $r_{v_s}$  is the mixing ratio. In terms of partial pressures, the mixing ratio can be defined as

$$r_{v_s} = \frac{0.622e_s}{p - e_s}$$

where  $p_{tot} = e_s + e_{dry}$

where  $e_s$  is the saturated vapor pressure of the air over liquid water and  $e_{dry}$  is the dry vapor pressure of the air over liquid water.

Where  $h_i$  is the initial altitude,  $h$  is another measured altitude value.  $T_d(h)$  is the measured dew point temperature at that  $h$ , and  $T_d(h_i)$  is the initial dew point temperature coupled with the initial altitude.

The *SALR* and measured *Dew Point Lapse Rate* are associated because they describe processes taking place within the parcel, but the Environmental Lapse Rate, ELR, is different in that it is a direct measurement of the air surrounding the parcel. The formula for ELR is

$$ELR = \lim \frac{T(h) - T(h_i)}{h - h_i} = \frac{dT}{dz}$$

where  $h_i$  is the initial altitude,  $h$  is another measured altitude value.  $T(h)$  is the measured temperature at that specific  $h$ , and  $T(h_i)$  is the initial temperature associated with the initial altitude.

Data collection of atmospheric lapse rates using radiosondes is an important tool in weather and climate analysis. Particularly in regards to climate, since the early

1990s, scientists have carried-on an extensive debate as to whether radiosonde data can be treated as reliable or not. The concern emerged from apparent discrepancies between observed (radiosonde and satellite) data and model predictions on account that observations did not show the warming in the troposphere predicted by models (Spencer and Christy, 1990, Hansen et al., 1995). These differences have been attributed to a variety of causes over the years and have recently been narrowed down to the “temporal inhomogeneity” of observed radiosonde data (Gaffen, 1994, Parker and Cox, 1995). These time-varying biases are generally induced by the change of instrumentation, the scarce quantification of related uncertainties, and the lack of documentation. Recent research supports the current conclusion that no strong evidence exists of discrepancies between observed and modeled tropospheric trends when correcting for these biases (Trenberth et al., 2007, Thorne et al., 2011). Thus understanding and continued monitoring of atmospheric lapse rates is fundamental to weather and climate research.